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ɷ −Semi-p Open Set

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Abstract

Csaszar introduced the concept of generalized topological space and a new open set in a generalized topological space called ɷ-preopen in 2002 and 2005, respectively. Definitions of ɷ-preinterior and ɷ-preclosuer were given. Successively, several studies have appeared to give many generalizations for an open set. The object of our paper is to give a new type of generalization of an open set in a generalized topological space called ɷ-semi-p-open set. We present the definition of this set with its equivalent. We give definition of ɷ-semi-p-interior and ɷ-semi-p-closure of a set and discuss their properties. Also the properties of ɷ-preinterior and ɷ-preclosuer are discussed. In addition, we give a new type of continuous function in a generalized topological space as $(ɷ_1, ɷ_2)$-semi-p-continuous function and $(ɷ_1, ɷ_2)$-semi-p-irresolute function. The relationship between them is showed. We prove that every ɷ-open (ɷ-preopen) set is an ɷ-semi-p-open set, but not conversely. Every $(ɷ_1, ɷ_2)$-semi-p-irresolute function is an $(ɷ_1, ɷ_2)$-semi-p-continuous function, but not conversely. Also we show that the union of any family of ɷ-semi-p-open sets is an ɷ-semi-p-open set, but the intersection of two ɷ-semi-p-open sets need not to be an ɷ-semi-p-open set.

Keywords: ɷ-semi-p-open , ɷ-semi-p-interior , ɷ-semi-p-closure, $(ɷ_1, ɷ_2)$-semi-p-irresolute and $(ɷ_1, ɷ_2)$-semi-p-continuous.

1.Introduction and Preliminaries

In this paper, we denote a topological space by $(Z, Ӽ)$ and the closure (interior) of a subset $Ħ$ of Z by $cl(Ħ)(int(Ħ))$, respectively.

1. The interior of $Ħ$ is the set $int(Ħ) = \cup\{Ш : Ш \in Ӽ \; and \; Ш \subseteq Ħ\}$.

2. The closure of H is the set $\mathrm{cl}(\mathrm{H}) = \bigcap\{F: F \in \mathfrak{X}' \; and \; \mathrm{H} \subseteq F\}$ [1], where $\mathfrak{X}'$ symbolizes the family of closed subsets of Z.

The term "preopen" was introduced for the first time in 1984 [2]. A subset A of a topological space $(\mathrm{Z}, \mathfrak{X})$ is called a preopen set if $\mathrm{A} \subseteq \mathrm{Int}(\mathrm{clA})$. The complement of a preopen set is called a preclosed set. The family of all preopen sets of Z is denoted by $\mathrm{PO}(\mathrm{Z})$. The family of all preclosed sets of Z is denoted by $\mathrm{PC}(\mathrm{Z})$. In 2000, Navalagi used "preopen" term to define a "Semi-p-open set" [3]. A subset A of a topological space $(\mathrm{Z}, \mathfrak{X})$ is said to be semi-p-open set if there exists a preopen set U in Z such that $\mathrm{U} \subseteq \mathrm{A} \subseteq$ pre-cl U. The family of all semi-p-open sets of Z is denoted by S-PO(Z). The complement of a semi-p-open set is called semi-p-closed set. The family of all semi-p-closed sets of Z is denoted by S-PC(Z). A function $\mathrm{f}: (\mathrm{Z}_1, \mathfrak{X}_1) \to (\mathrm{Z}_2, \mathfrak{X}_2)$ is said to be a continuous function if the inverse image of any open set in Z_2 is an open set in Z_1 [4]. Navalagi used the term "preopen" to introduce new types of a continuous function "pre-irresolute function" and "pre-continuous function". A function $\mathrm{f}: (\mathrm{Z}_1, \mathfrak{X}_1) \to (\mathrm{Z}_2, \mathfrak{X}_2)$ is called pre-irresolute(pre-continuous) function if the inverse image of any pre-open set in Z_2 is a pre-open set inZ_1 (the inverse image of any open set in Z_2 is a pre-open set Z_1). In [5], Al-Khazraji used the term of "Semi-p-open set" to define new types of continuous functions "semi-p-irresolute" and "semi-p-continuous" function. A function $\mathrm{f}: (\mathrm{Z}_1, \mathfrak{X}_1) \to (\mathrm{Z}_2, \mathfrak{X}_2)$ is called a semi-p-irresolute (semi-p-continuous) function if the inverse image of any semi-p-open set in Z_2 is a semi-p-open set in Z_1(the inverse image of any open set in Z_2 is a semi-p- open set in Z_1). Let Z be a nonempty set, a collection ω of subsets of Z is called a generalized topology (in brief, GT) on Z if $\emptyset$ belongs to ω and the arbitrary unions of elements of ω is an element in ω, (Z, ω) is called generalized topological space (in brief, GTS) [6]. Every set in ω is called ω-open, while the complement of ω-open is called ω-closed; the family of all ω-closed sets is denoted by ω'. The union of all ω-open set contained in a set H is called the ω- interior of H and is denoted by $\mathrm{int}_{\omega}(\mathrm{H})$, whereas the intersection of all ω-closed set containing H is called the ω-closure of H and is denoted by $\mathrm{cl}_{\omega}(\mathrm{H})$[7].

2. ω-Pre-Open Set

Definition 2.1 [8]

In a GTS (Z, ω) by an ω-pre-open (in brief, $\omega - \mathrm{p} - \mathrm{o}$) set, we mean a subset H of Z with $\mathrm{H} \subseteq \mathrm{int}_{\omega}\mathrm{cl}_{\omega}\,\mathrm{H}$. An ω-pre-closed (in brief, $\omega - \mathrm{p} - \mathrm{c}$) set is the complement of an ω-pre-open set. The collection of all $\omega - \mathrm{p} - \mathrm{o}$ ($\omega - \mathrm{p} - \mathrm{c}$) subsets of Z will be denoted by ω-PO(Z) (ω-PC(Z), respectively).

Proposition 2.2

For a subset H of a (Z, ω) , we have $\bigcup_{\alpha\in\Lambda} \mathrm{int}_{\omega}\, \mathrm{cl}_{\omega}\, \mathrm{H}_{\alpha} \subseteq \mathrm{int}_{\omega}\mathrm{cl}_{\omega} \bigcup_{\alpha\in\Lambda} \mathrm{H}_{\alpha}$.

Proof:

$\mathrm{H}_{\alpha} \subseteq \bigcup_{\alpha\in\Lambda} \mathrm{H}_{\alpha}$, for every $\alpha \in \Lambda$, so $\mathrm{cl}_{\omega}\mathrm{H}_{\alpha} \subseteq \mathrm{cl}_{\omega} \bigcup_{\alpha\in\Lambda} \mathrm{H}_{\alpha}$ for every $\alpha \in \Lambda$, it follows that $\mathrm{int}_{\omega}\mathrm{cl}_{\omega}\mathrm{H}_{\alpha} \subseteq \mathrm{int}_{\omega}\mathrm{cl}_{\omega} \bigcup_{\alpha\in\Lambda} \mathrm{H}_{\alpha}$ $\forall\alpha \in \Lambda$.

Hence $\bigcup_{\alpha\in\Lambda} \mathrm{int}_{\omega}\, \mathrm{cl}_{\omega}\, \mathrm{H}_{\alpha} \subseteq \mathrm{int}_{\omega}\mathrm{cl}_{\omega} \bigcup_{\alpha\in\Lambda} \mathrm{H}_{\alpha}$.

Proposition 2.3

The union of any collection of $\omega - p - o$ sets is an $\omega - p - o$ set.

Proof:

Let $\{ H_\alpha : \alpha \in \Lambda \}$ be a family of $\omega - p - o$ sets, so $H_\alpha \subseteq int_\omega \, cl_\omega \, H_\alpha, \forall \alpha \in \Lambda$. Which means $\bigcup_{\alpha \in \Lambda} H_\alpha \subseteq \bigcup_{\alpha \in \Lambda} int_\omega \, cl_\omega \, H_\alpha$, but $\bigcup_{\alpha \in \Lambda} int_\omega \, cl_\omega \, H_\alpha \subseteq int_\omega cl_\omega \bigcup_{\alpha \in \Lambda} H_\alpha$ (by Proposition 2.2), therefore, we obtain $\bigcup_{\alpha \in \Lambda} H_\alpha \subseteq int_\omega \, cl_\omega \bigcup_{\alpha \in \Lambda} H_\alpha$, hence $\bigcup_{\alpha \in \Lambda} H_\alpha$ is an $\omega - p - o$ set .

Corollary 2.4

The intersection of any collection of $\omega - p - c$ sets is an $\omega - p - c$ set.

Definition 2.5: [6]

Let (Z, ω) be a GTS, and H be a subset of Z

1. The union of all $\omega - p - o$ sets contained in H is called the ω-preinterior of H and denoted by pre-$int_\omega H$.
2. The intersection of all $\omega - p - c$ sets containing H is called the ω-preclosuer of H and denoted by pre-$cl_\omega H$.

Theorem 2.6

Let $\mathcal{H}$ and $\mathcal{T}$ be subsets of (Z, ω). Then, the following properties are true:

1. $\mathcal{H} \subseteq$ pre-$cl_\omega \mathcal{H}$.
2. pre-$int_\omega \mathcal{H} \subseteq \mathcal{H}$.
3. If $\mathcal{H} \subseteq \mathcal{T}$, then pre-$int_\omega \mathcal{H} \subseteq$ pre-$int_\omega \mathcal{T}$.
4. If $\mathcal{H} \subseteq \mathcal{T}$, then pre-$cl_\omega \mathcal{H} \subseteq$ pre-$cl_\omega \mathcal{T}$.

Proof:

1. From Definition of pre-$cl_\omega \mathcal{H}$.
2. From Definition of pre-$int_\omega \mathcal{H}$.
3. Let $\mathcal{H} \subseteq \mathcal{T}$, we have from 2, pre-$int_\omega \mathcal{H} \subseteq \mathcal{H}$, so $pre - int_\omega \mathcal{H} \subseteq \mathcal{T}$, but $pre - int_\omega \mathcal{T}$ is the largest $\omega - p - o$ set contained in $\mathcal{T}$. So $pre - int_\omega \mathcal{H} \subseteq pre - int_\omega \mathcal{T}$.
4. Let $\mathcal{H} \subseteq \mathcal{T}$, we have from 1, $\mathcal{T} \subseteq$ pre-$cl_\omega \mathcal{T}$, so $\mathcal{H} \subseteq$ pre-$cl_\omega \mathcal{T}$, but pre-$cl_\omega \mathcal{H}$ is the smallest $\omega - p - c$ set containing $\mathcal{H}$. So pre-$cl_\omega \mathcal{H} \subseteq$ pre-$cl_\omega \mathcal{T}$.

Proposition 2.7

Let (Z, ω) be a GTS let H be a subset of Z. Then:

1. H is an $\omega - p - c$ set, if and only if $H =$ pre-$cl_\omega H$.
2. H is an $\omega - p - o$set, if and only if H = pre-$int_\omega H$.

Proposition 2.8

$$\bigcup_{\alpha \in \Lambda} pre - cl_\omega H_\alpha \subseteq pre - cl_\omega \bigcup_{\alpha \in \Lambda} H_\alpha$$

Proof:

$Ⱨ_α ⊆ ⋃_{α∈Λ} Ⱨ_α$, for every $α ∈ Λ$, so pre-cl$_{ῳ}$$Ⱨ_α$ ⊆ pre-cl$_{ῳ}$ $⋃_{α∈Λ} Ⱨ_α$ for every $α ∈ Λ$, therefore, $⋃_{α∈Λ} \text{pre} - \text{cl}_{ῳ} Ⱨ_α ⊆ \text{pre} - \text{cl}_{ῳ} ⋃_{α∈Λ} Ⱨ_α$.

Remark 2.9

The reverse of Proposition 2.8 is not correct in general, as we show in the following example:

For example

$Z = \{a, b, c\}$, $ῳ = \{Z, Ø, \{a, b\}\}$, and $ῳ' = \{Z, Ø, \{c\}\}$, then:

ῳ-PO(Z) = $\{Z, Ø, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

ῳ-PC(Z) = $\{Ø, Z, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$,

let $Ƕ = \{b\}$ and $Ƭ = \{a\}$, so pre-cl$_{ῳ}$ Ƕ = {b} and pre-cl$_{ῳ}$ Ƭ = {a}, note that $Ƕ ∪ Ƭ = \{a, b\}$, and pre-cl$_{ῳ}$ (Ƕ ∪ Ƭ) = Z, while pre-cl$_{ῳ}$Ƕ ⋃pre-cl$_{ῳ}$Ƭ = {a, b}.

Hence, pre-cl$_{ῳ}$(Ƕ ∪ Ƭ) ⊈ pre-cl$_{ῳ}$Ƕ ⋃ pre-cl$_{ῳ}$Ƭ.

Proposition: 2.10

If Ɥ is any subset of a topological space $(Z, Ӿ)$, then:

1. $[\text{pre} - \text{int}\ (Ⱨ)]^c = \text{pre} - \text{cl}\ (Ⱨ^c)$.
2. $\text{pre} - \text{int}(Ⱨ^c) = [\text{pre} - \text{cl}\ (Ⱨ)]^c$.

3. ῳ-Semi-P-Open Set

Definition 3.1

A subset G of a GTS $(Z, ῳ)$ is said to be ῳ-semi-p-open(in brief, $ῳ - sp - o$) set if there exists an $ῳ - p - o$ set Ⱨ in Z such that Ⱨ ⊆ G ⊆ pre-cl$_{ῳ}$Ⱨ. Any subset of Z is called ῳ-semi-p-closed(in brief, $ῳ - sp - c$) set if its complement is ῳ-semi-p-open set .The collection of all $ῳ - sp - o$ subsets of Z will be denoted by ῳ-SPO(Z). The collection of all $ῳ - sp - c$ subsets of Z will be denoted by ῳ-SPC(Z).

Theorem 3.2

Let $(Z, ῳ)$ be a GTS and G ⊆ Z. Then G is an $ῳ - sp - o$set ⇔ G ⊆ pre-cl$_{ῳ}$pre-int$_{ῳ}$G .

Proof:

The "if" part

Assume that G is an $ῳ - sp - o$set, then there exists a $ῳ - p - o$ subset Ⱨ of Z such that Ⱨ ⊆ G ⊆ pre-cl$_{ῳ}$Ⱨ, it follows by Theorem 2.6 (4) that pre-int$_{ῳ}$Ⱨ ⊆pre-int$_{ῳ}$G, but pre-int$_{ῳ}$Ⱨ = Ⱨ, therefore Ⱨ ⊆ pre-int$_{ῳ}$G. It follows by Theorem 2.6 (3) that pre-cl$_{ῳ}$Ⱨ ⊆ pre-cl$_{ῳ}$pre-int$_{ῳ}$G. Now, we get G ⊆pre-cl$_{ῳ}$Ⱨ ⊆ pre-cl$_{ῳ}$pre-int$_{ῳ}$G. Thus G ⊆ pre-cl$_{ῳ}$pre-int$_{ῳ}$G.

The "only if" part

Assume that G ⊆ pre-cl$_{ῳ}$pre-int$_{ῳ}$G, we have to show that G is a $ῳ - sp - o$set. Take pre-int$_{ῳ}$G = Ⱨ , then Ⱨ is a $ῳ - p - o$ set and Ⱨ ⊆ G ⊆ pre-cl$_{ῳ}$Ⱨ. Hence G is an $ῳ - sp - o$set.

Corollary 3.3

Let (Z, ω) be a GTS and $F \subseteq Z$. Then Ƕ is $\omega - sp - c$if and only if pre- int_{ω}(pre-cl_{ω}H) ⊆ H.

Proof:

The "if" part

Let F be an $\omega - sp - c$ subset of Z, then pre-cl_{ω} H = H (by Proposition 2.9(1)) which implies pre-int_{ω}(pre-cl_{ω}H) ⊆ H, since pre-int_{ω}Ƕ ⊆ Ƕ (by Theorem 2.3(2).

The "only if" part

Assume that pre- int_{ω}pre-cl_{ω}H ⊆ H. We have to show Ƕ is an $\omega - sp -$ cset. Since pre-int_{ω}pre-cl_{ω}H ⊆ H, then $H^c \subseteq$ [pre- int_{ω}(pre-cl_{ω} H)$]^c$, so we obtain from Proposition 2.10 $H^c \subseteq$ pre-cl_{ω}(pre-cl_{ω}H$)^c$ and $H^c \subseteq$ pre- cl_{ω}pre-$int_{\omega}H^c$. Hence H^c is an $\omega - sp - o$ set by Theorem (2.2.2) which means H is an $\omega - sp - c$.

Proposition 3.4

The union of any collection of $\omega - sp - o$ sets is an $\omega - sp - o$ set.

Proof:

Let $\{G_\alpha, \alpha \in \Lambda\}$ be any family of $\omega - sp - o$ sets. Then there exists an $\omega - p - o$ set H_α for each G_α , $\alpha \in \Lambda$ such that $H_\alpha \subseteq G_\alpha \subseteq$ pre-cl_{ω} H_α, so $\bigcup_{\alpha\in\Lambda} H_\alpha \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha \subseteq \bigcup_{\alpha\in\Lambda} pre - cl_{\omega} H_\alpha$, but $\bigcup_{\alpha\in\Lambda} H_\alpha$ is an $\omega - p - o$ set by Theorem 2.3, and $\bigcup_{\alpha\in\Lambda} pre - cl_{\omega} H_\alpha \subseteq$ pre-$cl_{\omega} \bigcup_{\alpha\in\Lambda} H_\alpha$ by Proposition 2.8. Now we get $\bigcup_{\alpha\in\Lambda} H_\alpha \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha \subseteq$ pre-$cl_{\omega} \bigcup_{\alpha\in\Lambda} H_\alpha$. Hence $\bigcup_{\alpha\in\Lambda} G_\alpha$ is an $\omega - sp - o$ set.

Corollary 3.5

The intersection of any collection of $\omega - sp - c$ sets is an $\omega - sp - c$ set.

Proof:

Let $\{ F_\alpha : \alpha \in \Lambda \}$ be any family of $\omega - sp - c$ subsets of Z. we have to show that $\bigcap_{\alpha\in\Lambda} F_\alpha$ is an $\omega - sp - c$ set, we know that $Z - \bigcap_{\alpha\in\Lambda} F_\alpha = \bigcup_{\alpha\in\Lambda}(Z - F_\alpha)$ (De Morgan's laws). But $\bigcup_{\alpha\in\Lambda}(Z - F_\alpha)$ is an $\omega - sp - c$ set, so $Z - \bigcap_{\alpha\in\Lambda} F_\alpha$ is an $\omega - sp - o$ set. Hence $\bigcap_{\alpha\in\Lambda} F_\alpha$ is an $\omega - sp - c$.

Remark 3.6

The intersection of two $\omega - sp - o$ sets need not to be an $\omega - sp - o$ set, as we show in the following example:

Example

Let $Z = \{a, b, c, d\}$, $\omega = \{Z, \emptyset, \{a\}, \{d\}, \{a, d\}\}$,

ω-PO(Z) $= \{Z, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$, and

ωSPO(Z)=
$\{Z, \emptyset, \{a\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}\}$. Let $\mathcal{H} = \{a,b,c\}$ and $\mathcal{T} = \{b,d\}$, $\mathcal{H}$ and $\mathcal{T}$ are $\omega - sp - o$ sets, but $\mathcal{H} \cap \mathcal{T} = \{b\}$ which is not an $\omega - sp - o$ set because there is not $\omega - p - o$ set V_{ω}, therefore $V \subseteq \{b\} \subseteq$ pre- $cl_{\omega}V$.

Remark 3.7

If $\mathcal{H}$ and $\mathcal{T}$ are two $\omega - sp - c$ sets, then $\mathcal{H} \cup \mathcal{T}$ need not be $\omega - sp - c$ as we show in the following example:

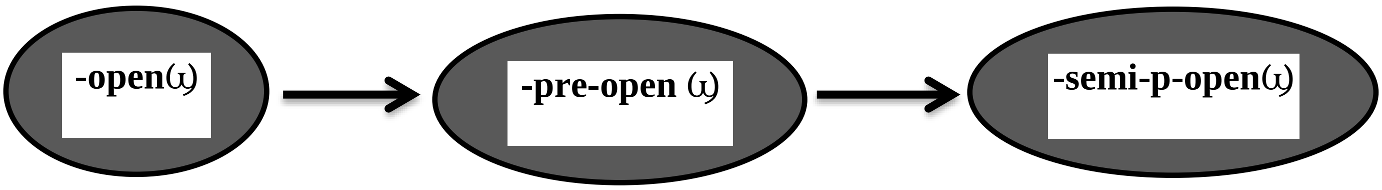
Example

From the example of Remark 3.7 Let $\mathcal{H} = \{d\}$ and $\mathcal{T} = \{a,c\}$.

$\mathcal{H}$ and $\mathcal{T}$ are $\omega - sp - c$ set, but $\mathcal{H} \cup \mathcal{T} = \{a,c,d\}$ is not an $\omega - sp - c$ set because

Z - $\{a,c,d\} = \{b\}$ is not an $\omega - sp - o$ set.

The following diagram illustrates the relation among ω-open, ω-pre-open, and ω-semi-p-open set



Definition 3.8

1. The union of all $\omega - sp - o$ sets contained in $\mathcal{H}$ is called the ω-semi-p-interior of $\mathcal{H}$, denoted by s-p-int$_{\omega}(\mathcal{H})$.
2. The intersection of all $\omega - sp - c$ sets containing $\mathcal{H}$ is called the ω-semi-p-closure of $\mathcal{H}$, denoted by s-p-cl$_{\omega}(\mathcal{H})$.

Proposition 3.9

Let $\mathcal{H}$ and $\mathcal{T}$ be two subsets of (Z, ω). Then, the following properties are true:

1. $\mathcal{H} \subseteq s - p - cl_{\omega}\, \mathcal{H}$.
2. If $\mathcal{H} \subseteq \mathcal{T}$, then $s - p - cl_{\omega}\mathcal{H} \subseteq s - p - cl_{\omega}\mathcal{T}$.
3. $s - p - cl_{\omega}\mathcal{H} \cup s - p - cl_{\omega}\mathcal{T} \subseteq s - p - cl_{\omega}\, (\mathcal{H} \cup \mathcal{T})$.
4. $s - p - cl_{\omega}\, (\mathcal{H} \cap \mathcal{T}) \subseteq s - p - cl_{\omega}\, \mathcal{H} \cap s - p - cl_{\omega}\, \mathcal{T}$.

Proof:

1. It is clear from Definition 3.14(2).

2. Let $\mathcal{H} \subseteq \mathcal{T}$, from (1) we have $\mathcal{T} \subseteq s - p - cl_{\omega}\, \mathcal{T}$, so $\mathcal{H} \subseteq s - p - cl_{\omega}\, \mathcal{T}$ which is $\omega - sp - c$ set, but $s - p - cl_{\omega}\mathcal{H}$ is the smallest $\omega - sp - c$ set containing $\mathcal{H}$, thus $s - p - cl_{\omega}\mathcal{H} \subseteq s - p - cl_{\omega}\mathcal{T}$.

3. Since Ƕ ⊆ Ƕ ∪ Ƭ and Ƭ ⊆ Ƕ ∪ Ƭ, it follows from (1) that $s-p-\mathrm{cl}_{\omega}$Ƕ ⊆ $s-p-\mathrm{cl}_{\omega}$(Ƕ ∪ Ƭ) and $s-p-\mathrm{cl}_{\omega}$Ƭ ⊆ $s-p-\mathrm{cl}_{\omega}$ (Ƕ ∪ Ƭ), therefore $s-p-\mathrm{cl}_{\omega}$Ƕ ∪ $s-p-\mathrm{cl}_{\omega}$ Ƭ ⊆ $s-p-\mathrm{cl}_{\omega}$ (Ƕ ∪ Ƭ).

4. Since (Ƕ ∩ Ƭ) ⊆ Ӈ and (Ƕ ∩ Ƭ) ⊆ Ƭ, so semi-p-cl$_{\omega}$(Ƕ ∩ Ƭ) ⊆ semi-cl$_{\omega}$Ƕ and $s-p-\mathrm{cl}_{\omega}$(Ƕ ∩ Ƭ) ⊆ $s-p-\mathrm{cl}_{\omega}$Ƭ, thus $s-p-\mathrm{cl}_{\omega}$(Ƕ ∩ Ƭ) ⊆ $s-p-\mathrm{cl}_{\omega}$Ƕ ∩ $s-p-\mathrm{cl}_{\omega}$Ƭ.

Theorem 3.10
Ӈ is $\omega-\mathrm{sp}-\mathrm{c}$ set ⇔ H $= s-p-\mathrm{cl}_{\omega}$H.
Proof: Is clear.

Corollary 3.11

$s-p-\mathrm{cl}_{\omega}Z = Z$.

Theorem 3.12
Let Ӈ and Ƭ be two subsets of (Z,ω). Then the following properties are true:

1. $s-p-\mathrm{int}_{\omega}$Ƕ ⊆ Ƕ.
2. If Ƕ ⊆ Ƭ, then $s-p-\mathrm{int}_{\omega}$Ƕ ⊆ $s-p-\mathrm{int}_{\omega}$ Ƭ.
3. $s-p-\mathrm{int}_{\omega}$ (Ƕ ∩ Ƭ) ⊆ $s-p-\mathrm{int}_{\omega}$ Ƕ ∩ $s-p-\mathrm{int}_{\omega}$ Ƭ
4. $s-p-\mathrm{int}_{\omega}$ Ƕ ∪ $s-p-\mathrm{int}_{\omega}$ Ƭ ⊆ $s-p-\mathrm{int}_{\omega}$ Ƕ ∪ Ƭ).

Proof:

1. Clear.
2. Let Ƕ ⊆ Ƭ, from (1) we have $s-p-\mathrm{int}_{\omega}$Ƕ ⊆ Ƕ, so $s-p-\mathrm{int}_{\omega}$Ƕ ⊆ Ƭ where $s-p-\mathrm{int}_{\omega}$Ƕ is $\omega-\mathrm{sp}-\mathrm{o}$ set, but $s-p-\mathrm{int}_{\omega}$Ƭ is the largest $\omega-\mathrm{sp}-\mathrm{o}$ set contained in Ƭ, hence $s-p-\mathrm{int}_{\omega}$ Ƕ ⊆ $s-p-\mathrm{int}_{\omega}$ Ƭ.
3. Since (Ƕ ∩ Ƭ) ⊆ Ӈ and (Ƕ ∩ Ƭ) ⊆ Ƭ, so $s-p-\mathrm{int}_{\omega}$(Ƕ ∩ Ƭ) ⊆ $s-p-\mathrm{int}_{\omega}$Ƕ and $s-p-\mathrm{int}_{\omega}$(Ƕ ∩ Ƭ) ⊆ $s-p-\mathrm{int}_{\omega}$Ƭ, so $s-p-\mathrm{int}_{\omega}$(Ƕ ∩ Ƭ) ⊆ $s-p-\mathrm{int}_{\omega}$Ƕ ∩ $s-p-\mathrm{int}_{\omega}$Ƭ.
4. Since Ƕ ⊆ Ƕ ∪ Ƭ and Ƭ ⊆ Ƕ ∪ Ƭ, then $s-p-\mathrm{int}_{\omega}$Ƕ ⊆ $s-p-\mathrm{int}_{\omega}$ (Ƕ ∪ Ƭ) and $s-p-\mathrm{int}_{\omega}$Ƭ ⊆ $s-p-\mathrm{int}_{\omega}$ (Ƕ ∪ Ƭ). Thus $s-p-\mathrm{int}_{\omega}$Ƕ ∪ $s-p-\mathrm{int}_{\omega}$ Ƭ ⊆ $s-p-\mathrm{int}_{\omega}$ (Ƕ ∪ Ƭ).

Theorem 3.13
H is an $\omega-\mathrm{sp}-\mathrm{o}$ set⇔H $= s-p-\mathrm{int}_{\omega}$H .
Proof: Is Clear.

Corollary 3.14

$s-p-\mathrm{int}_{\omega}\emptyset = \emptyset$

4. (ω_1, ω_2)-semi-p-continuous function

Definition 4.1:[8]

Let (Z, ω_1) and (Y, ω_2) be two GTS's. A function $f: Z \to Y$ is said to be (ω_1, ω_2)-continuous function if the inverse image of any ω_2-open subset of Y is an ω_1-open set in Z.

Definition 4.2:[9]

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is called (ω_1, ω_2)-M- pre-open function if the direct image of any ω_1- pre-open set in Z is an ω_2- pre-open set in Y.

Definition 4.3:

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is called (ω_1, ω_2)-M-semi-p-open ((ω_1, ω_2)-M-semi-p-closed) function if the direct image of any ω_1-semi-p-open (ω_1-semi-p-closed) set in Z is an ω_2-semi-p-open (ω_2-semi-p-closed) set in Y.

Definition 4.4

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is said to be (ω_1, ω_2)-semi-p-continuous function if the inverse image of any ω_2-open set in Y is an ω_1-semi-p-open set in Z.

Theorem 4.5

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is an (ω_1, ω_2)-semi-p-continuous function ⇔ the inverse image of any ω_2-closed set in Y is an ω_1-semi-p-closed set in Z .

Proof:

The "if" part. Let F be any ω_2-closed set in Y, thus $(Y - F)$ is an ω_2-open set in Y, then $f^{-1}(Y - F)$ is an ω_1-semi-p-open set in Z (since f is an (ω_1, ω_2)-semi-p-continuous function), but $f^{-1}(Y - F) = Z - f^{-1}(F)$, then $f^{-1}(F)$ is an ω_1-semi-p-closed set.

The "only if" part. Let H be any ω_2-open set in Y, thus $(Y - H)$ is an ω_2-closed set in Y, then $f^{-1}(Y - H)$ is an ω_1-semi-p-closed set in Z (by hypothesis) but $f^{-1}(Y - H) = Z - f^{-1}(H)$, then $f^{-1}(H)$ is an ω_1-semi-p-open set in Z , therefore f is an (ω_1, ω_2)-semi-p-continuous function.

Definition 4.6

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is said to be (ω_1, ω_2)-semi-p-irresolute function if the inverse image of any ω_2-semi-p-open set in Y is an ω_1-semi-p-open set in Z

Theorem 4.7

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is an (ω_1, ω_2)-semi-p-irresolute function ⇔the inverse image of each ω_2-semi-p-closed set in Y is an ω_1-semi-p-closed set in Z .

Proof:

The "if" part. Let F be any ω_2-semi-p-closed set in Y, thus $(Y- F)$ is an ω_2-semi-p-open set in Y, then $f^{-1}(Y-F)$ is an ω_1-semi-p-open set in Z (since f is an (ω_1,ω_2)-semi-p-irresolute function) , but $f^{-1}(Y-F) = Z - f^{-1}(F)$, therefore $f^{-1}(F)$ is an ω_1-semi-p-closed set.

The "only if" part . Let H be any ω_2-semi-p-open set in Y, thus $(Y- H)$ is an ω_1-semi-p-closed set in Y then $f^{-1}(Y-H)$ is an ω_1-semi-p-closed set in Z (by hypothesis) , but $f^{-1}(Y-H) = Z - f^{-1}(H)$, then $f^{-1}(H)$ is an ω_1-semi-p-open set in Z, therefore f is an (ω_1,ω_2)-semi-p-irresolute function.

Proposition 4.8

Every (ω_1,ω_2)-semi-p-irresolute function is an (ω_1,ω_2)-semi-p-continuous function.

Proof:

Let f be any (ω_1,ω_2)-semi-p-irresolute function from (Z,ω_1) into (Y,ω_2) . Let H by any ω_2-open in Y , thus H is an ω_2-semi-p- open set (Corollary 3.11), then $f^{-1}(H)$ is an ω_1-semi-p-open set in Z(since f is (ω_1,ω_2)-semi-p-irresolute function), therefore f is an (ω_1,ω_2)-semi-p-continuous function.

Remark 4.9

The reverse of Proposition 4.7 is not correct in general as we show in the following example:

Example

Let $Z = \{1,2,3,4\}$, $\omega_1 = \{Z, \emptyset, \{1\}, \{4\}, \{1,4\}\}$,

$\omega_1 - PO(Z) = \{Z, \emptyset, \{1\}, \{4\}, \{1,4\}, \{1,2,4\}, \{1,3,4\}\}$, and

$\omega_1 - SPO(Z) = \omega_1 - PO(Z) \cup \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \}$.

Let $Y = \{a, b, c, d\}$, $\omega_2 = \{\emptyset, \{b, d\}\}$, $\omega_2 - PO(Y) = \{\emptyset, \{b, d\}, \{b\}, \{d\}\}$,

$\omega_2 - SPO(Y) = \mathbb{P}(Y)$ (The power set of Y).

Define $f : (Z,\omega_1) \rightarrow (Y,\omega_2)$ such that $f(1) = f(2) = \{d\}, f(3) = \{b\}$

f is an (ω_1,ω_2)-semi-p-continuous function. But not (ω_1,ω_2)-semi-p-irresolute function, since $\{b\}$ is an ω_2-semi-p-open set in Y, but $f^{-1}(\{b\}) = \{3\}$ is not an ω_1-semi-p-open set in Z.

Proposition 4.10

Every (ω_1,ω_2)-continuous function is an (ω_1,ω_2)-semi-p-continuous function.

Proof:

Let f be any (ω_1,ω_2)- continuous function from (Z,ω_1) into (Y,ω_2) . Let H by any ω_2-open in Y , it follows from Definition 4.1 that $f^{-1}(H)$ is an ω_1- open set in Z, but every ω_1- open set is an ω_1-semi-p –open. Therefore f is an (ω_1,ω_2)-semi-p-continuous function.

Remark 4.11

The reverse of Remark 4.9 is not correct in general as we show in the following example:

Example

Let $Z = \{1,2,3\}$, $\omega_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, and $\omega_1 - PO(Z) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$,

$\omega_1 - SPO(Z) = \mathbb{P}(Z)$ (The power set of Z).

Let $Y = \{a, b, c, d\}$, $\omega_2 = \{\emptyset, \{b, d\}\}$, $\omega_2 - PO(Y) = \{\emptyset, \{b, d\}, \{b\}, \{d\}\}$,

$\omega_2 - SPO(Y) = \mathbb{P}(Y)$ (The power set of Y).

Define $f : (Z, \omega_1) \rightarrow (Y, \omega_2)$ such that $f(1) = f(2) = \{a\}, f(3) = \{b\}$,

f is an (ω_1, ω_2)-semi-p-continuous function, but it is not an (ω_1, ω_2)-continuous function, since $\{b, d\}$ is an ω_2-open set in Y, but $f^{-1}(\{b, d\}) = \{3\}$ is not an ω_1-open set in Z .

Proposition 4.12

The composition of (ω_1, ω_2)-semi-p-irresolute function and (ω_2, ω_3)-semi-p-irresolute function is an (ω_1, ω_3)-semi-p- irresolute function.

Proof

Let f: $(Z, \omega_1) \rightarrow (Y, \omega_2)$ be (ω_1, ω_2)-semi-p-irresolute function and $g : (Y, \omega_2) \rightarrow (W, \omega_3)$ be (ω_2, ω_3)-semi-p-irresolute functions, we have to show that $g \circ f : (Z, \omega_1) \rightarrow (W, \omega_3)$ is an (ω_1, ω_3)-semi-p-irresolute function. Let Ḩ be any ω_3-semi-p-open set in W, then $(g \circ f)^{-1}(H) = f^{-1} \circ g^{-1}(H) = f^{-1}(g^{-1}(H))$, but $g^{-1}(H)$ is an ω_2-semi-p-open set in Y (since g is an (ω_2, ω_3)-semi-p-irresolute function), and $f^{-1}(g^{-1}(H))$ is an ω_1-semi-p-open set in Z (since f is an (ω_1, ω_2)-semi-p-irresolute functions) , therefore $g \circ f$ is an (ω_1, ω_3)-semi-p-irresolute functions .

Remark 4.13

The composition of (ω_1, ω_2)-semi-p-continuous function and (ω_2, ω_3)-semi-p-continuous function need not to be (ω_1, ω_3)-semi-p-continuous function as we show in the following example:

Example

Let $Z = \{1, 2, 3\}$, $\omega_1 = \{Z, \emptyset, \{1, 2\}\}$,

$Y = \{a, b, c\}$, $\omega_2 = \{Y, \emptyset, \{a, b\}\}$,

$W = \{i, j, k\}$, $\omega_3 = \{W, \emptyset, \{i, k\}\}$,

ω_1-PO(Z)= $\{Z, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} = \omega_1$-SPO(Z)

ω_2-PO(Y)= $\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} = \omega_2$-SPO(Y), and

ω_3-PO(W)= $\{W, \emptyset, \{i\}, \{k\}, \{i, j\}, \{i, k\}, \{j, k\}\} = \omega_3$-SPO(W)

Define $f : (Z, \omega_1) \rightarrow (Y, \omega_2)$ by $f(1) = f(3) = \{b\}$, $f(2) = \{c\}$.

And $g: (Y, \omega_2) \to (W, \omega_3)$ by $g(a) = g(c) = \{j\}$, $g(b) = \{k\}$.

Then $g \circ f: (Z, \omega_1) \to (W, \omega_3)$ is defined by:

$g \circ f\,(1) = g(f(1)) = g(b) = \{k\}$,

$g \circ f\,(2) = g(f(2)) = g(c) = \{j\}$,

$g \circ f\,(3) = g(f(3)) = g(b) = \{k\}$,

f is an (ω_1, ω_2)-semi-p-continuous function and g is an (ω_2, ω_3)-semi-p-continuous function. But $g \circ f$ is not an (ω_1, ω_3)-semi-p-continuous function, since $\{i, k\}$ is an ω_3-semi-p-open set in W, but $f^{-1}(\{i, k\}) = \{3\}$ is not ω_1-semi-p-open set in Z.

Proposition 4.14

The composition of an (ω_1, ω_2)-semi-p-continuous function and (ω_2, ω_3)- continuous function is an (ω_1, ω_3)-semi-p-continuous function.

Proof:
Let $f: (Z, \omega_1) \to (Y, \omega_2)$ be any (ω_1, ω_2)-semi-p-continuous function and $g: (Y, \omega_2) \to (W, \omega_3)$ be any (ω_2, ω_3)- continuous function. We have to show that $g \circ f: (Z, \omega_1) \to (W, \omega_3)$ is an (ω_1, ω_3)-semi-p- continuous function. Let H be any ω_3- open set in W. Then, $g^{-1}(H)$ is an ω_2-open set in Y (since g is an (ω_2, ω_3)-continuous function), so $f^{-1}(g^{-1}(H)$ is an ω_1-semi-p-open set in Z (since f is an (ω_1, ω_2)-semi-continuous function), but $(g \circ f)^{-1}(H) = f^{-1} \circ g^{-1}(H) = f^{-1}(g^{-1}(H)$. Hence $g \circ f$ is an (ω_1, ω_3)-semi-p-continuous function.

Theorem 4.15

Let $f: (Z, \omega_1) \to (Y, \omega_2)$ be an onto function, then f is an (ω_1, ω_2)-M-semi-p-open function if and only if it is an (ω_1, ω_2)-M-semi-p-closed function.

Proof:

The "if" part. Let F be any ω_1-semi-p-closed set, so $(Z - F)$ is an ω_1-semi-p-open set, then $f(Z - F)$ is an ω_2-semi-p-open set (since f is an (ω_1, ω_2)-M-semi-p-open function), but $f(Z - F) = Y - f(F)$, therefore $f(F)$ is an ω_2-semi-p-closed. Hence f an (ω_1, ω_2)-M-semi-p-closed function.

The "only if" part. Let H be any ω_1-semi-p-open set , so $(Z - H)$ is an ω_1-semi-p-closed set, then $f(Z - H)$ is an ω_2-semi-p-closed set (since f is an (ω_1, ω_2)-M-semi-p-closed function), but $f(Z - H) = Y - f(H)$, therefore $f(H)$ is an ω_2-semi-p-open. Hence f an (ω_1, ω_2)-M-semi-p-closed function.

Theorem 4.16

Let $f: (Z, \omega_1) \to (Y, \omega_2)$ be a bijective function, then f is an (ω_1, ω_2)-M-semi-p-open function, $\Leftrightarrow$ $f^{-1}: (Y, \omega_2) \to (Z, \omega_1)$ is an (ω_1, ω_2)-semi-p-irresolute function.

Proof

The "if" part. Suppose that f is an (ω_1, ω_2)-M-semi-p-open function, to show that f^{-1} is an (ω_1, ω_2)-semi-p-irresolute function. Let H be any ω_1-semi-p-open set in Z, then $(f^{-1})^{-1}(H) =$

f(H) is an ω_2-semi-p-open set in Y (since f is an (ω_1, ω_2)-M-semi-p-open function), so f^{-1} is an (ω_1, ω_2)-semi-p- irresolute function.

The "only if" part. Suppose that f^{-1} is an (ω_1, ω_2)-semi-p-irresolute function, to show that f is an (ω_1, ω_2)-M-semi-p-open function. Let H be any ω_1-semi-p-open set in Z, then $(f^{-1})^{-1}(H) = f(H)$ is an ω_2-semi-p-open set in Y(since f^{-1} is an (ω_1, ω_2)-semi-p-irresolute function), so f is an (ω_1, ω_2)-M-semi-p-open function.

Definition 4.17

A bijection function $f : (Z, \omega_1) \to (Y, \omega_2)$ is called (ω_1, ω_2)-semi-p-homeomorphism function if f is both (ω_1, ω_2)-semi-p-irresolute function and (ω_1, ω_2)-M-semi-p-open function.

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