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A Review of the Some Fixed Point Theorems for Different Kinds of Maps

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Abstract

The focus of this paper reviewed generalized contraction mapping and nonexpansive maps and recall some theorems about the existence and uniqueness of common fixed points and coincidence fixed-point for such maps under some conditions. Moreover, some schemes of different types as one-step schemes, two-step schemes, and three-step schemes (Mann scheme algorithm, Ishukawa scheme algorithm, Noor scheme algorithm, $SP-$.scheme algorithm, $CR-$ scheme algorithm Modified SP scheme algorithm Karahan scheme algorithm, and others. The convergence of these schemes has been studied. On the other hand, we also reviewed the convergence, valence, and stability theories of different types of near-plots in convex metric space.

Keywords: Convergence, Fixed Point, Nonexpansive Map, Pseudocontractive Map and Iterative Methods.

Introduction

Fixed point theory is an important topic, and it has many applications in branches of mathematics various. In the year 1970, introduced Takahashi the idea of convexity in m-spaces and studied it as well as common f-point theorems for nonexpansive mappings. The convex m-space is a public, important, and, expansive space with a convex structure, where the Banach cone space is convex m-space. The principle of the Banach contraction states that they can approximate the contraction maps f-point by Picard proximal scheme. The seq $\langle x_n \rangle$ of this scheme can be defined as follows: Let $\emptyset \neq \mathcal{M}$ be a closed-convex lies in $\mathcal{H}$and $\mathcal{J}\colon \mathcal{M} \to \mathcal{M}$ be a mapping:

$$a_0 \in \mathcal{M}, \quad a_{n+1} = \mathcal{J}a_n , n \in N \tag{1}$$

Picard's proximal scheme for nonexpansive mappings does not converge to a f-point. Hence, to approximate the f-points of the non'expansion maps, a proximal scheme is introduced as:

$$a_0 \in \mathcal{M}, \quad a_{n+1} = (1-\alpha_n)a_n + \alpha_n \mathcal{J} a_n, \; n \in N \tag{2}$$

Because the iterative Mann proximal scheme [1], fails to converge to the f-points of the spurious systolic maps, and for spurious systolic maps introduced Ishikawa proximal scheme to f-points. The sequence $\langle x_n \rangle$ of the Ishikawa proximal scheme[2], defined as:

$$a_0 \in \mathcal{M}, \quad a_{n+1} = (1-\alpha_n)a_n + \alpha_n \mathcal{J} b_n, \quad b_n = (1-\beta_n)a_n + \beta_n \mathcal{J} a_n, \; n \in N \tag{3}$$

Noor,in 2000[3] introduced proximal scheme as:

$$w_0 \in \mathcal{M}, w_{n+1} = (1-\alpha_n)w_n + \alpha_n \mathcal{J} u_n, u_n = (1-\beta_n)w_n + \beta_n \mathcal{J} v_n,$$

$$v_n = (1-\gamma_n)w_n + \gamma_n \mathcal{J} w_n, \; n \in N \tag{4}$$

In[4],Agrawal introduced for nearly non'expansive maps,two steps as:

$$x_0 \in \mathcal{M}, \quad x_{n+1} = (1-\alpha_n)\mathcal{J}x_n + \alpha_n \mathcal{J} t_n, \quad t_n = (1-\beta_n)x_n + \beta_n \mathcal{J} x_n, \quad n \in \mathrm{N} \tag{5}$$

SP **−iteration** [5]:

$$x_0 \in \mathcal{M}, \quad x_{n+1} = \alpha_n \mathcal{J} y_n + (1-\alpha_n)y_n, \quad y_n = \beta_n \mathcal{J} z_n + (1-\beta_n)z_n,$$

$$z_n = \gamma_n \mathcal{J} x_n + (1-\gamma_n)x_n \qquad n \in N \tag{6}$$

CR −iteration [6]:

$$w_0 \in \mathcal{M}, \quad w_{n+1} = (1-\alpha_n)u_n + \alpha_n \mathcal{J} u_n, \; u_n = (1-\beta_n)\mathcal{J} w_n + \beta_n \mathcal{J} v_n,$$
$$v_n = (1-\gamma_n)w_n + \gamma_n \mathcal{J} w_n, \quad n \in \mathrm{N} \tag{7}$$

Modified SP iteration [7]:

$$x_0 \in \mathcal{M}, \tag{8}$$

$$x_{n+1} = \mathcal{J} y_n,$$

$$y_n = (1-\alpha_n)z_n + \alpha_n \mathcal{J} z_n,$$

$$z_n = (1-\beta_n)x_n + \beta_n \mathcal{J} x_n$$

Karahan iteration [8]:

$$w_0 \in \mathcal{M}, \; w_{n+1} = (1-\alpha_n)\mathcal{J} w_n + \alpha_n \mathcal{J} u_n, \; u_n = (1-\beta_n)w_n + \beta_n \mathcal{J} v_n,,$$

$$v_n = (1-\gamma_n)w_n + \gamma_n \mathcal{J} w_n, \qquad n \in \mathrm{N} \tag{9}$$

Finally, [9] studied the existence of a f- point for type of contraction-maps and the convergence of a common f-point for Noor iteration in complete convex metric spaces(Com Con M-S). Then a lot of studies were carried out on this topic,see[10-18].

Preliminaries

In this part,we introduce some concepts which is need in this work,see[8,11and12].

1. A mapping$\mathcal{J}$ is called non'expansive if:

$$\|\mathcal{J}a - d\| \leq \| a - d \| \text{ for all } a,\ d \in \mathcal{M}$$

2. A mapping $\mathcal{J}$ is called quasi'nonexpansive if:

$F(\mathcal{J}) \neq \emptyset$ and $\|\mathcal{J}a - \mathcal{J}b\| \leq \|a - b\|$ for all $a, b \in \mathcal{M}$ and $y \in F(\mathcal{J})$.

3. It is easy to see that if $\mathcal{J}$ is non'expansive with $F(\mathcal{J}) \neq \emptyset$, then it is quasi'nonexpansive.

4. A mapping $\mathcal{J}$ is said to be e pseudocontractive if the inequality

$\|a - b\| \leq \|a - b + t[(I - \mathcal{J})_a - (I - \mathcal{J})_b]\|$

Hold for each $a, b \in \mathcal{M}$ and all $t > 0$.

Some proximal scheme are used to approximate a f- point of Zamfirescu maps are the most general contractive maps satisfying the condition: $\forall a,\ b\ lies\ in\ \mathcal{M}$ at least one of the conditions is true:

$(i)\ d\,(\mathcal{J}a,\ \mathcal{J}b) \leq \mathcal{P}d\,(a, b),$

$(ii)\ d(\mathcal{J}a,\ \mathcal{J}b) \leq \mathcal{Q}\,[d\,(a,\ \mathcal{J}a) + d(b, \mathcal{J}b)],$

$(iii)\ d\,(\mathcal{J}a,\ \mathcal{J}b) \leq \mathcal{R}\,[d\,(a,\ \mathcal{J}b) + d\,(b,\ \mathcal{J}a)].$

Where $0 \leq \mathcal{P} \leq 1$, $0 \leq \mathcal{Q}$, and $\mathcal{R} \leq 1/2$

Definition :Let ʄ , ց : $\mathcal{H} \to \mathcal{H}$ be a two mappings. A point $a \in \mathcal{H}$ is called f- point of ʄ if ʄ $(a) = a$, a common f-point of a pair (ʄ, ց) if ʄ $(a) =$ ց$(a) = a$ an a coincidence point of (ʄ, ց) if ʄ $(a) =$ ց(a).

Remarks :**A**mapping- Zamfirescu is equivalent to the condition:

$d\,(Ta,\ Tb) \leq e\, max\,\{d\,(a,\ b)\,, \frac{\{d\,(a, \mathrm{T}a) + d(b, \mathrm{T}b)\}}{2}\,, \frac{\{d\,(a, Tb) + d\,(b, Ta)\}}{2}\,\}$
$\forall a,\ b \in \mathcal{H},\ 0 < e < 1.$

Definition[12]: A mapping $\mathcal{R} : \mathcal{H} \times \mathcal{H} \times [0,1] \to \mathcal{H}$ is called convex structure on m-space,if for each $(a, b, \lambda) \in \mathcal{H} \times \mathcal{H} \times [0,1]$ and

$u \in \mathcal{H},\ d(u, \mathcal{R}(a, b, \lambda)) \leq \lambda d(u, a) + (1 - \lambda) d(u, b).$

Definition [13]**:** Letց : $\mathcal{H} \to \mathcal{H}$ be a mappings, $\{\mathcal{K}_n\}_{n=0}^{\infty} \subset \mathcal{H}$,and $\varepsilon_n = d\big(\mathcal{K}_{n+1}, ʄ\,(ց, \mathcal{K}_n)\big),\ n = 0,1,2, \ldots$ .Then $\mathcal{K}_{n+1} = ʄ\,(ց, \mathcal{K}_n)$ is said to be T-stable or stable with respect to ց if and only if

$\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} \mathcal{K}_n = \mathcal{P}$.

Definition [14]**:** Let$\{ a_n \}_0^\infty$, $\{ b_n \}_0^\infty \in R$ and converge to a and b a,respectively, and

$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = s$, if $s = 0$, then $\{ a_n \}_0^\infty \to a$ faster than$\{ b_n \}_0^\infty \to b$and if $0 < s < \infty$, then it can be said that a_n and $\{ b_n \}_0^\infty$ have the same rate of convergence.

Lemma : [15]. If $0 \le \mathcal{Q} < 1$ and $\{\mathcal{N}_n\}_n^\infty=0$ is a positive R-sequence such that $\lim_{n\to\infty} \mathcal{N}_n = 0$, then for any positive R-sequence $\{\hbar_n\}_n^\infty=0$ satisfying

$\hbar_{n+1} \le \mathcal{Q}\hbar_n + \mathcal{N}_n$, n $= 0, 1, 2, \ldots \Longrightarrow \lim_{n\to\infty} \hbar_n = 0$.

There are many studies on the iterations in other spaces see[18-21]

Previous Results

One of the most important previous results on this topic

Theorem: In any metric space if $\mathcal{J}$ satify the condition

$$d(a, \mathcal{J}b) + d(b, \mathcal{J}b) \le qd(a, b), \quad (1)$$

for all $a, b \in \mathcal{M}$, where $2 \le q < 4$. Then,$\mathcal{J}$has at least one fixed point.

Theorem: Let $\mathcal{J}$ be a mappingsatisfy the condition

$$d(\mathcal{J}a, \mathcal{J}b) + d(a, \mathcal{J}b) + d(b, \mathcal{J}b) \le rd(a, b) \quad \forall\, a, b \in \mathcal{M} \quad (2)$$

Then, $\mathcal{J}$ has at least one f-point.

Theorem: Consider a Com Con M-S. Suppose that ʄ , ǥ are mappings of $\mathcal{M}$, and there exist ą, ɓ, ç, m as:

2ɓ − |ç| ≤ m < 2(ą + ɓ + ç) − |ç|,

ąd(ǥ(x), ʄ (x)) + ɓd(ǥ(y), ʄ(y)) + çd(ʄ (x), ʄ(y)) ≤ md(ǥ(x), ǥ(y))

then f has at least one f-point.

In appreciably, a f-point iteration is useful for applications if it satisfies the following requirements:

(a) study data dependence results.

(b) it converges to f- point.

(c) it is $\mathfrak{J}$ -stable.

Theorem: Consider each of proximal processes Noor, Karhan and ModifiedSP.scheme converge to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ contraction map.Then the ModifiedSP.iteration converges faster than Noor and Karhan scheme.

Theorem: Consider each of proximal processes Mann, Ishikawa and Modified SP-scheme converge to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ contraction map.Then the ModifiedSP.iteration converges faster than Mann, Ishikawa scheme .

Theorem: In a Com Con M-S consider the Mann proximal processes , converge to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ contraction map.Then the Mann scheme is T - stable scheme .

Theorem: In a Com Con M-S consider the Ishikawa proximal processes , converge to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ contraction map.Then the Ishikawa scheme is T - stable scheme

Theorem: In a H-S consider the Ishikawa and Mann proximal processes such that converge it to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ quasi δ -contraction map.Then the Ishikawa scheme⇢ a iff Mann scheme ⟶ a.

Theorem: In a H-S consider the Ishikawa and Modified$\mathcal{SP}$.proximal processes such that converge it to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ quasi δ -contraction map.Then Modified$\mathcal{SP}$. iteration scheme⟶ a iff Mann scheme ⇢ a.

Theorem: In a H-S consider the CR and Mann proximal processes such that converge it to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ quasi δ -contraction map.Then the CR a scheme⇢ a iff Mann scheme ⇢ a.
Theorem: In a H-S consider the Noor and Mann proximal processes such that converge it to $\mathcal{b} \in \mathfrak{J}$ where $\mathfrak{J}$ quasi δ -contraction map.Then the Noor a scheme⟶ a iff Mann scheme⟶ a.

Conclusion

A generalized review of contractionary mapping and non-expansion maps has been reviewed and some theories are recalled about the existence and uniqueness of the common fixed point and congruent fixed point of such maps under some conditions. Moreover, we also inferred the convergence and acceleration range of some schemes of different types such as one-step schemes, two-step schemes, and three-step schemes in convex metric space.

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