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On Antimagic Labeling for Some Families of Graphs

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Abstract

Antimagic labeling of a graph G with p vertices and q edges is assigned the labels for its edges by some integers from the set $\{1,2,\dots,q\}$, such that no two edges received the same label, and the weights of vertices of a graph G are pairwise distinct. Where the vertex-weights of a vertex v under this labeling is the sum of labels of all edges incident to this vertex, in this paper, we deal with the problem of finding vertex antimagic edge labeling for some special families of graphs called strong face graphs. We prove that vertex antimagic, edge labeling for strong face ladder graph L_n^*, strong face wheel graph W_n^*, strong face fan graph F_n^*, strong face prism graph $(C_n \times P_2)^*$ and finally strong face friendship graph $(T_n)^*$.

Keywords: Antimagic graph, vertex antimagic graph, edge labeling, strong face graph.

1. Introduction

Let a graph $G = (V, E)$ be a finite, simple and undirected graph, where $V(G)$ and $E(G)$ are the vertex set and edge set respectively. An antimagic labeling graphs had first been introduced by [1]. If a graph G with p vertices and q edges can have its edges labeled without repetition and the sums of the labels of the edges incident to each vertex are pairwise distinct, the graph is said to be antimagic [2]. In addition [3] show that If all vertex weights are distinct, an edge labeling of a graph G is said to have a vertex antimagic edge labeling (VAE labeling). Hartsfield and Ringel proved that the path graphs P_n, complete graph $K_n, n \geq 3$, wheels and cycles graphs are antimagic, moreover, they conjectured that every tree except P_2 is antimagic, and every connected graph except P_2 is antimagic, both these conjectures are still open.

In this paper, we will prove several graphs derived from a plane graph are edge labeling vertex antinagic, these graphs are called strong face graphs. The strong face graph, first introduced by [4]. Where they proved that face antimagic total labeling for some families of these graphs. In our study we address the problem of finding vertex antimagic edge labeling for this family of graphs.

The strong face plane graphs are obtained from a plane graph, by adding a new vertex to every face, in such way that, the results graphs are all three-sided faces, moreover if the faces of original plane graphs are three sided faces, then the number of faces will be increasing. We address the problem of finding vertex antimagic edge labeling, for short (*VAEL*), for the strong face ladder graph L_n^*, the strong face wheel graph W_n^*, the strong face fan graph F_n^*, The strong face prism graph $(C_n \times P_2)^*$, and finally the strong face friendship graph $(T_n)^*$.

2. Main Results

Theorem1

For every $n \geq 3, n \not\equiv 2,$ (mod 4) the strong face ladder graph L_n^* is *VAEL*.

Proof: we define the vertex and edge sets of L_n^* graph as follows:

$V(L_n^*) = \{v_i : i = 1, 2, \ldots, 3n-1\}$,

$E(L_n^*) = \{v_i v_{i+1}, v_{n+i} v_{n+i+1}, v_i v_{2n+i}, v_{i+1} v_{2n+i}, v_{n+i} v_{2n+i}, \; v_{n+i+1} v_{2n+i} : i = 1, 2, \ldots, n-1\} \cup \{v_i v_{n+i} : i = 1, 2 \ldots, n\}$.

For $n \geq 3, n \not\equiv 2$ (mod 4) we define the labeling of L_n^* as:

$\mu_1 : E(L_n^*) \to \{1, 2, \ldots, 7n-6\}$.

Such that;

$\mu_1(v_i v_{n+i}) = 7i - 6$ for $i = 1, 2, \ldots, n$.

For $i = 1, 2, \ldots, n-,1$, we have:

$\mu_1(v_i v_{i+1}) = 7i - 5$,

$\mu_1(v_{n+i} v_{n+i+1}) = 7i - 2$,

$\mu_1(v_i v_{2n+i}) = 7i - 4$,

$\mu_1(v_{i+1} v_{2n+i}) = 7i - 1$,

$\mu_1(v_{n+i} v_{2n+i}) = 7i - 3$,

$\mu_1(v_{n+i+1} v_{2n+i}) = 7i$.

For the vertex-weights we get:

$wt_{\mu_1}(v_1) = \mu_1(v_1 v_2) + \mu_1(v_1 v_{n+1}) + \mu_1(v_1 v_{2n+1})$
$= 6$,

$wt_{\mu_1}(v_i) = \mu_1(v_i v_{i+1}) + \mu_1(v_i v_{i-1}) + \mu_1(v_1 v_{2n+i}) + \mu_1(v_i v_{2n+i-1}) + \mu_1(v_i v_{n+i})$ for $i = 2, 3 \ldots, n-1$
$= (7i-5) + (7i-12) + (7i-4) + (7i-8) + (7i-6)$
$= 35i - 35$ for$i = 2, 3 \ldots, n-1$,

$wt_{\mu_1}(v_n) = \mu_1(v_n v_{n-1}) + \mu_1(v_n v_{2n}) + \mu_1(v_n v_{3n-1})$
$= (7n-12) + (7n-6) + (7n-8)$
$= 21n - 26$,

$wt_{\mu_1}(v_{n+1}) = \mu_1(v_{n+1} v_1) + \mu_1(v_{n+1} v_{n+2}) + \mu_1(v_{n+1} v_{2n+1})$
$= 1 + 5 + 4 = 10$,

$wt_{\mu_1}(v_{n+i}) = \mu_1(v_{n+i} v_{n+i+1}) + \mu_1(v_{n+i} v_{n+i-1}) + \mu_1(v_{n+i} v_{2n+i}) + \mu_1(v_{n+i} v_{2n+i-1})$
$+\mu_1(v_{n+i} v_i)$ for $i = 2, 3 \ldots n, -1$,
$= (7i-2) + (7i-9) + (7i-3) + (7i-7) + (7i-6)$ for$i = 2, 3 \ldots, n-1$
$= 35i - 27$ for $i = 2, 3 \ldots, n-1$,

$wt_{\mu_1}(v_{2n}) = \mu_1(v_{2n} v_{2n-1}) + \mu_1(v_{2n} v_n) + \mu_1(v_{2n} v_{3n-1})$
$= (7n-9) + (7n-6) + (7n-7)$
$= 21n - 22$.

Finally:

$wt_{\mu_1}(v_{2n+i}) = \mu_1(v_{2n+i} v_i) + \mu_1(v_{2n+i} v_{i+1}) + \mu_1(v_{2n+i} v_{n+i}) + \mu_1(v_{2n+i} v_{n+i+1})$ for $i = 1,2, \ldots, n-1$,
$= (7i-4) + (7i-1) + (7i-3) + (7i)$ for $i = 1,2, \ldots, n-1$,
$= 28i - 8$ for $i = 1, 2, \ldots, n-1$.

Based on the vertex weight of the previous we conclude that they are all distinct and the following notes are observed:

1-$wt_{\mu_1}(v_1) < wt_{\mu_1}(v_{n+1}) < wt_{\mu_1}(v_n) < wt_{\mu_1}(v_{2n})$,

2- $wt_{\mu_1}(v_{2n+i}) < wt_{\mu_1}(v_{2n+i+1})$ for $i = 1,2, \dots, n-2$,

3- $wt_{\mu_1}(v_i) < wt_{\mu_1}(v_{n+i}) < wt_{\mu_1}(v_{i+1}) < wt_{\mu_1}(v_{n+i+1})$ for $i = 2,3, \dots, n-2$.

Thus, the strong face ladder graph L_n^* is vertex antimagic edge labeling for every $n \geq 3, n \not\equiv 2$ (mod 4). However, when $n = 6$, we can easily show that $wt_{\mu_1}(v_{12})$ = $wt_{\mu_1}(v_{16})$ and similarly when $n = 10$, then $wt_{\mu_1}(v_{20})$ = $wt_{\mu_1}(v_{27})$ and so on.

Theorem 2

The strong face wheel graph W_n^* is *VAEL*, for every $n \geq 3$, $n \not\equiv 0$ (mod 3).

Proof: we define the vertex and edge sets of W_n graph as follows:

$V(W_n^*) = \{v_i: i = 1, 2, \dots, 2n+1\}$,

$E(W_n^*) = \{v_i v_{2n+1}, v_{2n+1} v_{n+i}, v_{n+i} v_i : i = 1, 2, \dots, n\} \cup \{v_{n+i} v_{i+1}, v_i v_{i+1} : i = 1, 2 \dots n-1\} \cup \{v_1 v_n, v_1 v_{2n}\}$.

For $n \geq 3, n \not\equiv 0$ (mod 3) we define the labeling of W_n^* as:

$\mu_2: E(W_n^*) \to \{1, 2, \dots, 5n\}$.

Such that:

For $i = 1,2, \dots, n$,

$\mu_2(v_{2n+1} v_{n+i}) = 4i$,

$\mu_2(v_{2n+1} v_i) = 4i - 2$,

$\mu_2(v_i v_{n+i}) = 4i - 1$.

And for $i = 1,2, \dots, n-1$,

$\mu_2(v_i v_{i+1}) = 4n + i$,

$\mu_2(v_{n+i} v_{i+1}) = 4i + 1$.

Finally,

$\mu_2(v_n v_1) = 5n$,

$\mu_2(v_{2n} v_1) = 1$.

For the vertex-weights we get:

$$wt_{\mu_2}(v_1) = \mu_2(v_1 v_2) + \mu_2(v_1 v_n) + \mu_2(v_1 v_{n+1}) + \mu_2(v_1 v_{2n+1}) + \mu_2(v_1 v_{2n})$$
$$= (4n + 1) + (5n) + 3 + 2 + 1$$
$$= 9n + 7,$$

$$wt_{\mu_2}(v_i) = \mu_2(v_i v_{i+1}) + \mu_2(v_i v_{i-1}) + \mu_2(v_i v_{n+i}) + \mu_2(v_i v_{2n+1}) + \mu_2(v_i v_{n+i-1}) \quad \text{for } i = 2, 3 \dots n-1,$$
$$= (4n + i) + (4n - 1 + i) + (4i - 1) + (4i - 2) + (4i - 3)$$
$$= 8n + 14i - 7 \quad \text{for } i = 2, 3, \dots, n-1,$$

$$wt_{\mu_2}(v_n) = \mu_2(v_n v_1) + \mu_2(v_n v_{n-1}) + \mu_2(v_n v_{2n}) + \mu_2(v_n v_{2n+1}) + \mu_2(v_n v_{2n-1})$$
$$= (5n) + (5n - 1) + (4n - 1) + (4n - 2) + (4n - 3)$$
$$= 22n - 7,$$

$$wt_{\mu_2}(v_{n+i}) = \mu_2(v_{n+i} v_i) + \mu_2(v_{n+i} v_{i+1}) + \mu_2(v_{n+i} v_{2n+1}) \quad \text{for } i = 1, 2, \dots n-1,$$
$$= (4i - 1) + (4i + 1) + (4i)$$
$$= 12i \quad \text{for } i = 1, 2, \dots, n-1,$$

$$wt_{\mu_2}(v_{2n}) = \mu_2(v_{2n} v_1) + \mu_2(v_{2n} v_n) + \mu_2(v_{2n} v_{2n+1})$$
$$= 1 + (4n - 1) + (4n)$$
$$= 8n.$$

Finally:

$$wt_{\mu_2}(v_{2n+1}) = \sum_{i=1}^{n} \mu_2(v_{2n+1} v_i) + \sum_{i=1}^{n} \mu_2(v_{2n+1} v_{n+i})$$

$$= \sum_{i=1}^{n}(4i-2) + \sum_{i=1}^{n}(4i)$$
$$= 2n(n+1) - 2n + 2n(n+1)$$
$$= 4n^2 + 2n.$$

Which means that the vertex weights are all distinct and the following notes are observed:

1- $wt_{\mu_2}(v_i) < wt_{\mu_2}(v_{i+1})$ for $i = 1,2,\dots,n-1$,
2- $wt_{\mu_2}(v_{n+i}) < wt_{\mu_2}(v_{n+i+1})$ for $i = 1,2,\dots,n-1$,
3- the weight of vertex (v_{2n+1}) is greater than the weight of any other vertex in the strong face wheel graph W_n^*.

Thus, the strong face wheel graph W_n^* is vertex antimagic edge labeling, for every $n \geq 3, n \not\equiv 0$ (mod 3). However, when $n = 3$, we can easily show that $wt_{\mu_2}(v_5) = wt_{\mu_2}(v_6)$ and similarly when $n = 6$, $then\ wt_{\mu_2}(v_{10}) = wt_{\mu_2}(v_{12})$ and so on.

Theorem 3

The strong face fan graph F_n^*, is *VAEL*, for every $n \geq 4$, $n \not\equiv 3$ (mod 5).

Proof: we define the vertex and edge sets of F_n^* graph as follows:

$V(F_n^*) = \{v, v_i : i = 1, 2, \dots 2n-1\}$,

$E(F_n^*) = \{vv_i : i = 1, 2, \dots, n\} \cup \{v_iv_{i+1}, v\,v_{n+i}, v_iv_{n+i}, v_{i+1}v_{n+i} : i = 1, 2 \dots, n-1\}$.

For $n \geq 4$, $n \not\equiv 3$ (mod 5) we define the labeling of F_n^*, as:

$\mu_3: E(F_n^*) \to \{1, 2, \dots, 5n-4\}$.

Such that:

$\mu_3(vv_i) = 5i - 4$ for $i = 1, 2, \dots, n$, and for $i = 1, 2, \dots, n-1$,

$\mu_3(v_iv_{i+1}) = 5i - 3$,

$\mu_3(v_iv_{n+i}) = 5i - 2$,

$\mu_3(v_{i+1}v_{n+i}) = 5i$,

$\mu_3(vv_{n+i}) = 5i - 1$.

For the vertex-weights we get:

$$wt_{\mu_3}(v_1) = \mu_3(v_1v) + \mu_3(v_1v_2) + \mu_3(v_1v_{n+1})$$
$$= 1 + 2 + 3 = 6,$$
$$wt_{\mu_3}(v_i) = \mu_3(v_iv) + \mu_3(v_iv_{i+1}) + \mu_3(v_iv_{i-1}) + \mu_3(v_iv_{n+i-1}) + \mu_3(v_iv_{n+i}) \quad \text{for } i = 2, 3 \dots, n-1,$$
$$= (5i-4) + (5i-3) + (5i-8) + (5i-5) + (5i-2)$$
$$= 25i - 22 \quad \text{for } i = 2, 3, \dots, n-1,$$
$$wt_{\mu_3}(v_n) = \mu_3(v_nv) + \mu_3(v_nv_{n-1}) + \mu_3(v_nv_{2n-1})$$
$$= (5n-4) + (5n-8) + (5n-5)$$
$$= 15n - 17,$$
$$wt_{\mu_3}(v_{n+i}) = \mu_3(v_{n+i}v_i) + \mu_3(v_{n+i}v_{i+1}) + \mu_3(v_{n+i}v) \quad \text{for } i = 1,2,\dots,n-1,$$
$$= (5i-2) + (5i) + (5i-1) \quad \text{for } i = 1,2,\dots,n-1,$$
$$= 15i - 3 \quad \text{for } i = 1,2,\dots,n-1.$$

Finally:

$$wt_{\mu_3}(v) = \sum_{i=1}^{n}\mu_3(vv_i) + \sum_{i=1}^{n-1}\mu_3(vv_{n+i})$$
$$= \sum_{i=1}^{n}(5i-4) + \sum_{i=1}^{n-1}(5i-1)$$
$$= 5\left(\frac{n^2+n}{2}\right) - 4n + 5\left(\frac{n^2-n}{2}\right) - n + 1$$

$= 5n^2 - 5n + 1.$

So, the weights of all vertices are distinct and the following notes are observed:

1- $wt_{\mu_3}(v_i) < wt_{\mu_3}(v_{i+1})$ for $i = 1,2, \dots, n-1$,

2- $wt_{\mu_3}(v_{n+i}) < wt_{\mu_3}(v_{n+i+1})$ for $i = 1,2, \dots, n-2$.

Thus, the strong face fan graph F_n^* is vertex antimagic edge labeling, for every $n, n \geq 4, n \not\equiv 3 \pmod 5$. However, when $n = 8$, we can easily show that $wt_{\mu_3}(v_5) = wt_{\mu_3}(v_8)$ and similarly when $n = 13$, $then\ wt_{\mu_3}(v_8) = wt_{\mu_3}(v_{13})$ and so on.

Theorem4

The strong face prism graph $(C_n \times P_2)^*$ is $VAEL$, for every $n \geq 3$, $n \not\equiv 0 \pmod 5$.

Proof: we define the vertex and edge sets of $(C_n \times P_2)^*$graph as follows;

$V((C_n \times P_2)^*) = \{v_i : i = 1, 2, \dots, 3n+1\}$,

$E((C_n \times P_2)^*) = \{v_i v_{i+1}, v_{n+i} v_{n+i+1} : i = 1, 2, \dots n-1\}$

$\cup \{v_n v_1, v_{2n} v_{n+1}, v_{3n} v_1, v_{3n} v_{n+1}\}$

$\cup \{v_{3n+1} v_{n+i}, v_{n+i} v_i : i = 1, 2 \dots n\}$

$\cup \{v_{2n+i} v_i, v_{2n+i} v_{n+i} : i = 1, 2 \dots n\}$

$\cup \{v_{2n+i} v_{i+1}, v_{2n+i} v_{n+i+1} : i = 1, 2 \dots n-1\}$.

For $n \geq 3$, $n \not\equiv 0 \pmod 5$ we define the labeling of $(C_n \times P_2)^*$ graph as:

$\mu_4 : E((C_n \times P_2)^*) \to \{1, 2, \dots, 8n\}$.

Such that:

For $i = 1, 2, \dots, n$,

$\mu_4(v_{3n+1} v_{n+i}) = i$

$\mu_4(v_{n+i} v_i) = 4n + 3i - 1$,

$\mu_4(v_{2n+i} v_i) = 4n + 3i$,

$\mu_4(v_{2n+i} v_{n+i}) = 2n + 2i - 1$.

And for $i = 1, 2, \dots n-1$,

$\mu_4(v_i v_{i+1}) = 7n + i$,

$\mu_4(v_{n+i} v_{n+i+1}) = n + i$,

$\mu_4(v_{2n+i} v_{i+1}) = 4n + 3i + 1$,

$\mu_4(v_{2n+i} v_{n+i+1}) = 2n + 2i$.

Finally,

$\mu_4(v_{2n} v_{n+1}) = 2n$,

$\mu_4(v_n v_1) = 8n$,

$\mu_4(v_{3n} v_1) = 4n + 1$,

$\mu_4(v_{3n} v_{n+1}) = 4n$.

For the vertex-weights we get:

$wt_{\mu_4}(v_1) = \mu_4(v_1\, v_n) + \mu_4(v_1 v_2) + \mu_4(v_1 v_{n+1}) + \mu_4(v_1 v_{3n}) + \mu_4(v_1 v_{2n+1})$

$= (8n) + (7n + 1) + (4n + 2) + (4n + 1) + (4n + 3)$

$= 27n + 7$,

$wt_{\mu_4}(v_i) = \mu_4(v_i\, v_{i+1}) + \mu_4(v_i v_{i-1}) + \mu_4(v_i v_{2n+i}) + \mu_4(v_i v_{2n+i-1}) + \mu_4(v_i v_{n+i})$ for $i = 2, 3, \dots, n-1$,

$= (7n + i) + (7n + i - 1) + (4n + 3i) + (4n + 3i - 2) + (4n + 3i - 1)$

$= 26n + 11i - 4$ for $i = 2, 3, \dots, n-1$,

$wt_{\mu_4}(v_n) = \mu_4(v_n v_1) + \mu_4(v_n v_{n-1}) + \mu_4(v_n v_{3n}) + \mu_4(v_n v_{2n}) + \mu_4(v_n v_{3n-1})$

$= (8n) + (8n - 1) + (7n) + (7n - 1) + (7n - 2)$

$= 37n - 4$,

$wt_{\mu_4}(v_{n+1}) = \mu_4(v_{n+1} v_1) + \mu_4(v_{n+1} v_{3n}) + \mu_4(v_{n+1} v_{2n+1}) + \mu_4(v_{n+1} v_{n+2}) + \mu_4(v_{n+1} v_{2n}) + \mu_4(v_{n+1} v_{3n+1})$

$= (4n + 2) + (4n) + (2n + 1) + (n + 1) + (2n) + 1$

$= 13n + 5$,

$wt_{\mu_4}(v_{n+i}) = \mu_4(v_{n+i}\, v_i) + \mu_4(v_{n+i}v_{n+i+1}) + \mu_4(v_{n+i}v_{n+i-1}) + \mu_4(v_{n+i}v_{2n+i-1})$
$+\mu_4(v_{n+i}v_{2n+i}) + \mu_4(v_{n+i}v_{3n+1})$ for $i = 2, 3, \ldots, n-1$
$= (4n + 3i - 1) + (n + i) + (n + i - 1) + (2n + 2i - 2) + (2n + 2i - 1) + i$
for $i = 2, 3, \ldots, n-1$,
$= 10n + 10i - 5$ for $i = 2, 3, \ldots n-1$,
$wt_{\mu_4}(v_{2n}) = \mu_4(v_{2n}v_{n+1}) + \mu_4(v_{2n}v_{2n-1}) + \mu_4(v_{2n}v_{3n+1}) + \mu_4(v_{2n}v_{3n})$
$+\mu_4(v_{2n}v_n) + \mu_4(v_{2n}v_{3n-1})$
$= (2n) + (2n - 1) + (n) + (4n - 1) + (7n - 1) + (4n - 2)$
$= 20n - 5$,
$wt_{\mu_4}(v_{2n+i}) = \mu_4(v_{2n+i}\, v_i) + \mu_4(v_{2n+i}v_{i+1}) + \mu_4(v_{2n+i}v_{n+i}) + \mu_4(v_{2n+i}v_{n+i+1})$
for $i = 1, 2, \ldots, n-1$,
$= (4n + 3i) + (4n + 3i + 1) + (2n + 2i - 1) + (2n + 2i)$ for $i = 1, 2, \ldots, n-1$,
$= 12n + 10i$ for $i = 1, 2, \ldots, n-1$,
$wt_{\mu_4}(v_{3n}) = \mu_4(v_{3n}v_1) + \mu_4(v_{3n}v_n) + \mu_4(v_{3n}v_{2n}) + \mu_4(v_{3n}v_{n+1})$
$= (4n + 1) + (7n) + (4n - 1) + (4n)$
$= 19n.$

Finally:

$$wt_{\mu_4}(v_{3n+1}) = \sum_{i=1}^{n} \mu_3(v_{3n+1}\, v_{n+i})$$
$$= \sum_{i=1}^{n}(i) = \frac{n(n+1)}{2}.$$

So, the weights of all vertices are distinct and the following notes are observed:

1- $wt_{\mu_4}(v_i) < wt_{\mu_4}(v_{i+1})$ for $i = 1,2, \ldots,, n-1$.

2- The weight of vertex (v_n) is greater than the weight of any other vertex in the $(C_n \times P_2)^*$ graph.

Thus, the strong face prism graph $(C_n \times P_2)^*$ is vertex antimagic edge labeling, for every $n, n \geq 3, n \not\equiv 0$ (mod 5). However when $n = 5$, we can easily show that $wt_{\mu_4}(v_6) = wt_{\mu_4}(v_{11})$ and similarly when $n = 10$, then $wt_{\mu_4}(v_9) = wt_{\mu_4}(v_{11})$ and so on.

Theorem 5

The strong face friendship graph $(T_n)^*$ is *VAEL* for every $n \geq 3$.

Proof: we define the vertex and edge sets of $(T_n)^*$ graph as follows;

$V((T_n)^*) = \{v, v_i : i = 1, 2, \ldots 2n\} \cup \{u_i : i = 1, 2, \ldots n\}$,

$E((T_n)^*) = \{vv_i : i = 1, 2, \ldots 2n\} \cup \{vu_i : i = 1, 2, \ldots n\} \cup \{v_i\, v_{i+1} : i = 1, 3, \ldots 2n-1\} \cup \{u_i v_{2i-1}, u_i\, v_{2i} : i = 1, 2, \ldots, n\}$.

For every $n \geq 3$, we define the labeling of $(T_n)^*$ graph as:

$\mu_5((T_n)^*) \to \{1, 2, \ldots, 6n\}$.

Such that:

1- $\mu_5(vv_i) = \begin{cases} 3i - 2 & \text{for } i = 1, 3, \ldots, 2n-1, \\ 3i - 3 & \text{for } i = 2, 4, \ldots, 2n, \end{cases}$

2- $\mu_5(v_i v_{i+1}) = 3i - 1$ for $i = 1, 3, \ldots, 2n-1$,

3- $\mu_5(vu_i) = 6i - 2$ for $i = 1, 2, \ldots, n$,

4- $\mu_5(u_i v_{2i-1}) = 6i - 1$ for $i = 1, 2, \ldots, n$,

5- $\mu_5(u_i\, v_{2i}) = 6i$ for $i = 1, 2, \ldots, n$.

For the vertex-weights we get:

$$wt_{\mu_5}(v_i) = \begin{cases} \mu_5(v_i v) + \mu_5(v_i v_{i+1}) + \mu_5\left(v_i\, u_{\frac{i+1}{2}}\right) & \text{for } i = 1, 3, \dots, 2n-1, \\ \mu_5(v_i\, v) + \mu_5(v_i v_{i-1}) + \mu_5\left(v_i\, u_{\frac{i}{2}}\right) & \text{for } i = 2, 4, \dots, 2n, \end{cases}$$

$$= \begin{cases} (3i-2) + (3i-1) + (3i+2) & \text{for } i = 1, 3, \dots, 2n-1, \\ (3i-3) + (3i-4) + 3i & \text{for } i = 2, 4, \dots, 2n, \end{cases}$$

$$= \begin{cases} 9i-1 & \text{for i} = 1, 3, \dots, 2\text{n}-1, \\ 9i-7 & \text{for i} = 2, 4, \dots, 2\text{n}, \end{cases}$$

$$wt_{\mu_5}(u_i) = \mu_5(u_i v_{2i-1}) + \mu_5(u_i v_{2i}) + \mu_5(u_i v) \quad \text{for } i = 1, 2, \dots, n,$$

$$= (6i-1) + (6i) + (6i-2)$$

$$= 18i - 3 \quad \text{for } i = 1, 2, \dots, n.$$

Finally:

$$wt_{\mu_5}(v) = \sum_{i=1}^{2n} \mu_5(vv_i) + \sum_{i=1}^{n} \mu_5(vu_i)$$

$$= (3n^2 - 2n) + (3n^2) + (3n^2 + n)$$

$$= 9n^2 - n.$$

Which implies that:

$$wt_{\mu_5}(v_1) < wt_{\mu_5}(v_2) < wt_{\mu_5}(u_1) < wt_{\mu_5}(v_3) < wt_{\mu_5}(v_4) < wt_{\mu_5}(u_2) < \cdots < wt_{\mu_5}(v_{2n-1}) < wt_{\mu_5}(v_{2n}) < wt_{\mu_5}(u_n) < wt_{\mu_5}(v).$$

Based on vertex weights, the vertices weights are all distinct. Thus, the strong face friendship graph $(T_n)^*$ is vertex antimagic edge labeling, for every $n \geq 3$.

3. Conclusion

In this paper, we proved that some families of graphs related to strong face graphs, admit vertex antimagic edge labeling. Our future work will involve finding another family of strong face graphs that admits antimagic labeling.

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