Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq

Extend Nearly Pseudo Quasi-2-Absorbing submodules(I)

Omar A. Abdullah
Department of Mathematics College of Computer Science and Mathematics
Tikrit University / Iraq.
omer.a.abdullah35383@st.tu.edu.iq

Haibat K. Mohammadali
Department of Mathematics College of Computer Science and Mathematics
Tikrit University / Iraq.
H.mohammadali@tu.edu.iq

Article history: Received 16 September 2022, Accepted 6 November 2022, Published in April 2023.

doi.org/10.30526/36.2.3019

Abstract

The concept of a 2-Absorbing submodule is considered as an essential feature in the field of module theory and has many generalizations. This articale discusses the concept of the Extend Nearly Pseudo Quasi-2-Absorbing submodules and their relationship to the 2-Absorbing submodule, Quasi-2-Absorbing submodule, Nearly-2-Absorbing submodule, Pseudo-2-Absorbing submodule, and the rest of the other concepts previously studied. The relationship between them has been studied, explaining that the opposite is not true and that under certain conditions the opposite becomes true. This article aims to study this concept and gives the most important propositions, characterizations, remarks, examples, lemmas, and observations related to it. In the end, we will present a very important equivalent of our concept with the rest of the concepts presented previously.

Keywords: 2-Absorbing submodule, Quasi-2-Absorbing submodule, socal of module, Jacobson of module, cyclic and multiplication modules.

1. Introduction

As a generalization of the 2-Absorbing Ideal, the 2-Absorbing Submodule notion was originally introduced in 2011 by Darani A. and Sohelinia F. Badawi A. first proposed the 2-Absorbing Ideal in 2007. In recent years, various generalizations of the 2-Absorbing submodule, including the Quasi-2-Absorbing submodule, have been introduced. In this article, we presented the Extend Nearly Pseudo Quasi-2-Absorbing submodule, a new generalization on previously studied concepts, particularly the 2-Absorbing submodule and Quasi-2-Absorbing submodule. It is worth noting that Ʀ is a commutative ring with a $nonzero$ $i$dentity and Ɯ be a unitary Ʀ-$module$. In the end, we presented the most important propositions in this research and studied all possible relationships with this concept within the best conditions that helped us reach the best solutions.

2. Preliminaries

In the following, we mention some basic definitions and notations in module that will be used in this paper.

Definition 2.1[1].
A $proper$ submodule V of an Ɽ-$module$ Ꮃ is called 2-Absorbing submodule if whenever $ɑɓx \in V$ for ɑ, ɓ ∈ Ɽ, $x \in$ Ꮃ $implies$ that either $ɑx \in V$ or $ɓx \in V$ or $ɑɓ \in [V :_{Ɽ} Ꮃ]$. Where $[V :_{Ɽ} Ꮃ] = \{ɑ \in Ɽ : ɑᎳ \subseteq V\}$**[2]**.

Definition 2.2[3].
soc(Ꮃ) is the $intersection$ of all $essential$ submodules of Ꮃand a $nonzero$ submodule V of Ꮃ is $essential$ in Ꮃ if $V \cap G \neq (0)$ for any $nonzero$ submodule G of Ꮃ.

Definition 2.3[4].
A submodule V of an Ɽ-$module$ Ꮃ is called a $maximal$ submodule of Ꮃ, if V is a $proper$ submodule of Ꮃ and for all $B \subseteq$ Ꮃ with $V \subset B$, then $B =$ Ꮃ, J(Ꮃ) is the intersection of all $maximal$ submodules of Ꮃ.

Definition 2.4[5].
A $proper$ submodule V of an Ɽ-$module$ Ꮃ is called Nearly-2-Absorbing submodule if whenever $rsm \in V$, for $r, s \in$ Ɽ, $m \in$ Ꮃ, implies that either $rm \in V + J$(Ꮃ) or $sm \in V + J$(Ꮃ) or rsᎳ $\subseteq V + J$(Ꮃ).

Definition 2.5[6].
A $proper$ submodule V of an Ɽ-$module$ Ꮃ is called Quasi-2-Absorbing submodule if whenever $rstm \in V$ for $r, s, t \in$ Ɽ, $m \in$ Ꮃ, implies that either $rtm \in V$ or $stm \in V$ or rsᎳ $\subseteq V$.

Definition 2.6[7].
A $proper$ submodule V of an Ɽ-$module$ Ꮃ is called Pseudo-2-Absorbing submodule if whenever $rsm \in V$ for $r, s \in$ Ɽ, $m \in$ Ꮃ, $implies$ that either $rm \in V + soc$(Ꮃ) or $sm \in V + soc$(Ꮃ) or rsᎳ $\subseteq V + soc$(Ꮃ). And in the same paper **[7]** the concept of Pseudo Quasi-2-Absorbing submodule is introduced, where a $proper$ submodule V of an Ɽ-$module$ Ꮃ is called Pseudo Quasi-2-Absorbing submodule if for any $rstm \in V$ for $r, s, t \in R$, $m \in$ Ꮃ, $implies$ that either $rsm \in V + soc$(Ꮃ) or $stm \in V + soc$(Ꮃ) or $rtm \in V + soc$(Ꮃ).

Definition 2.7[8].
A $proper$ submodule V of an Ɽ-$module$ Ꮃ is called Nearly Quasi-2-Absorbing submodule if whenever $rstm \in V$ for $r, s, t \in$ Ɽ, $m \in$ Ꮃ, $implies$ that either $rtm \in V + J$(Ꮃ) or $stm \in V + J$(Ꮃ) or rsᎳ $\subseteq V + J$(Ꮃ).

Definition 2.8[9].
An Ɽ-$module$ Ꮃ is a $semisimple$, if every submodule of Ꮃ is a direct summand.

Definition 2.9[4].
An Ɽ-$module$ Ꮃ is cyclic if Ꮃ $=$ Ɽ$x = \langle x \rangle$.

Lemma 2.10[4, Ex(12). P 239].

1) Let V is a submodule of an Ɽ-$module$ Ꮃ with V as a direct summand of Ꮃ, then $J\left(\frac{Ꮃ}{V}\right) = \frac{J(Ꮃ)+V}{V}$.

2) An Ɽ-$module$ Ꮃ is a $semisimple$ if and only if for each submodule V of Ꮃ $soc\left(\frac{Ꮃ}{V}\right) = \frac{soc(Ꮃ)+V}{V}$.

Lemma 2.11[10, Ex12(5). P 242].
A submodule V of an Ɽ-$module$ Ꮃ is $maximal$ and $essential$ if and only if soc(Ꮃ) $\subseteq V$.

Lemma 2.12[4, Lemma (2.3.15)].
"Let L, V and D are submodules of an Ɽ-$module$ Ꮃ with $V \subseteq D$, then $(L + V) \cap D = (L \cap D) + V = (L \cap D) + (V \cap D)$."

Definition 2.13[11].
An Ɽ-$module$ Ꮃ is multiplicatiion, if every submodule V of Ꮃ is of the form $V = I$Ꮃ for some $ideal$ I of Ɽ. Equivalently, Ꮃ is a multiplicatiion Ɽ-$module$ if every submodule V of Ꮃ of the form $V = [V :_{Ɽ} Ꮃ]$Ꮃ.

Lemma 2.14[12, Prop. (2.3)].
Let W be a $m$ultiplication Ɽ-$module$. Then a submodule V of W is a 2-Absorbing if and only if V Quasi-2-Absorbing submodule of W.

3. The Results

In this part, we define Extend Nearly Pseudo Quasi-2-Absorbing submodule and characterize some of its fundamental properties using examples:

Definition 3.1
A $proper$ submodule V of an Ɽ-$module$ W is said to be Extend Nearly Pseudo Quasi-2-Absorbing (for short EXNPQ2AB) submodule of W if whenever $\alpha б c ꭓ \in V$, where α,б, $c \in$ Ɽ, $ꭓ \in \text{W}$ implies that either $\alpha c ꭓ \in V + soc(\text{W}) + J(\text{W})$ or $б c ꭓ \in V + soc(\text{W}) + J(\text{W})$ or $\alpha б ꭓ \in V + soc(\text{W}) + J(\text{W})$.
And an $ideal$ I of a ring Ɽ is called EXNPQ2AB ideal of Ɽ, if I is an EXNPQ2AB Ɽ-$submodule$ of an Ɽ-$module$ Ɽ.

Remarks and Examples 3.2

1. Every 2-Absorbing submodule of an Ɽ-$module$$\text{W}$ is EXNPQ2AB submodule however, the opposite is not true.

Proof.

Let V be a 2-Absorbing submodule of an Ɽ-$module$ W and $\alpha б c ꭓ \in V$, for α,б, $c \in$ Ɽ, $ꭓ \in \text{W}$. That is $\alpha б (c ꭓ) \in V$. But V is 2-Absorbing submodule of W, then either $\alpha (c ꭓ) \in V$ or $б(c ꭓ) \in V$ or $\alpha б \text{W} \subseteq V$. Thus, either $\alpha c ꭓ \in V \subseteq V + soc(\text{W}) + J(\text{W})$ or $б c ꭓ \in V \subseteq V + soc(\text{W}) + J(\text{W})$ or $\alpha б ꭓ \in V \subseteq V + soc(\text{W}) + J(\text{W})$ for all $ꭓ \in \text{W}$, then either $\alpha c ꭓ \in V + soc(\text{W}) + J(\text{W})$ or $б c ꭓ \in V + soc(\text{W}) + J(\text{W})$ or $\alpha б ꭓ \in V + soc(\text{W}) + J(\text{W})$. Hence, V is EXNPQ2AB submodule of W.

For the opposite, think about the following illustration:

Let $\text{W} = Z_{48}$, Ɽ $= Z$ and the submodule $V = \langle \overline{16} \rangle$ is EXNPQ2AB submodule of W, since $soc(Z_{48}) = \langle \bar{8} \rangle$ and $J(Z_{48}) = \langle \bar{6} \rangle$. That is for all α, б, $c \in Z$ and $m \in Z_{48}$ such that $\alpha б c m \in \langle \overline{16} \rangle$, implies that either $\alpha c m \in \langle \overline{16} \rangle + soc(Z_{48}) + J(Z_{48}) = \langle \overline{16} \rangle + \langle \bar{8} \rangle + \langle \bar{6} \rangle = \langle \bar{2} \rangle$ or $б c m \in \langle \bar{8} \rangle + soc(Z_{48}) + J(Z_{48}) = \langle \overline{16} \rangle + \langle \bar{8} \rangle + \langle \bar{6} \rangle = \langle \bar{2} \rangle$ or $\alpha б m \in \langle \overline{16} \rangle + \langle \bar{8} \rangle + \langle \bar{6} \rangle = \langle \bar{2} \rangle$. But V is not 2-Absorbing, since $2.4.\bar{2} \in \langle \overline{16} \rangle$, for $4,2 \in Z$ and $\bar{2} \in Z_{48}$, implies that $4.\bar{2} = \bar{8} \notin \langle \overline{16} \rangle$ and $2.\bar{2} = \bar{4} \notin \langle \overline{16} \rangle$ and $2.4 = 8 \notin 16Z$.

2. Every Quasi-2-Absorbing submodule of an Ɽ-$module$ W is EXNPQ2AB submodule however, the opposite is not true.

Proof.

Let V be a Quasi-2-Absorbing submodule of an Ɽ-$module$ W and $\alpha б c ꭓ \in V$, for α,б, $c \in$ Ɽ, $ꭓ \in \text{W}$. Since V is Quasi-2-Absorbing submodule of W, then either $\alpha c ꭓ \in V \subseteq V + soc(\text{W}) + J(\text{W})$ or $б c ꭓ \in V \subseteq V + soc(\text{W}) + J(\text{W})$ or $\alpha б ꭓ \in V \subseteq V + soc(\text{W}) + J(\text{W})$, hence either $\alpha c ꭓ \in V + soc(\text{W}) + J(\text{W})$ or $б c ꭓ \in V + soc(\text{W}) + J(\text{W})$ or $\alpha б ꭓ \in V + soc(\text{W}) + J(\text{W})$. Therefore V is EXNPQ2AB submodule of W.

For the opposite, think about the following illustration:

Let $\text{W} = Z_{48}$, Ɽ $= Z$ and the submodule $V = \langle \overline{12} \rangle$ is EXNPQ2AB submodule of W, since $soc(Z_{48}) = \langle \bar{8} \rangle$ and $J(Z_{48}) = \langle \bar{6} \rangle$. That is for all α, б, $c \in Z$ and $m \in Z_{48}$ such that $\alpha б c m \in$

$\langle\overline{12}\rangle$, implies that either $\alpha cm \in \langle\overline{12}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\overline{12}\rangle + \langle\overline{8}\rangle + \langle\overline{6}\rangle = \langle\overline{2}\rangle$ or $\beta cm \in \langle\overline{12}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\overline{12}\rangle + \langle\overline{8}\rangle + \langle\overline{6}\rangle = \langle\overline{2}\rangle$ or $\alpha\beta m \in \langle\overline{12}\rangle + \langle\overline{8}\rangle + \langle\overline{6}\rangle = \langle\overline{2}\rangle$. But V is not Quasi-2-Absorbing, since $2.3.2.\overline{1} \in \langle\overline{12}\rangle$, for $3,2 \in Z$ and $\overline{1} \in Z_{48}$, implies that $2.2.\overline{1} = \overline{4} \notin \langle\overline{12}\rangle$ and $2.3.\overline{1} = \overline{6} \notin \langle\overline{12}\rangle$.

3. Every Nearly-2-Absorbing submodule of an Ʀ-$module$ Ш is EXNPQ2AB submodule however, the opposite is not true.

Proof.

Let V be a Nearly-2-Absorbing submodule of an Ʀ-$module$ Ш and $\alpha\beta c$ꭓ $\in V$, for $\alpha,\beta, c \in$ Ʀ, ꭓ $\in$ Ш. That is $\alpha\beta(c$ꭓ$) \in V$. But V is Nearly-2-Absorbing submodule of Ш, then either $\alpha(c$ꭓ$) \in V + J($Ш$)$ or $\beta(c$ꭓ$) \in V + J($Ш$)$ or $\alpha\beta$Ш $\subseteq V + J($Ш$)$. Thus either αcꭓ $\in V + J($Ш$) \subseteq V + soc($Ш$) + J($Ш$)$ or βcꭓ $\in V + J($Ш$) \subseteq V + soc($Ш$) + J($Ш$)$ or $\alpha\beta$ꭓ $\in V + J($Ш$) \subseteq V + soc($Ш$) + J($Ш$)$ for all ꭓ $\in$ Ш, then either αcꭓ $\in V + soc($Ш$) + J($Ш$)$ or βcꭓ $\in V + soc($Ш$) + J($Ш$)$ or $\alpha\beta$ꭓ $\in V + soc($Ш$) + J($Ш$)$. Hence V is EXNPQ2AB submodule of Ш.

For the opposite, think about the following illustration:

Take a look at the Z-$module$ Z_{60} and the submodule $V = \langle\overline{30}\rangle$, we see that the only $essential$ submodules of Z_{60} are Z_{60} itself and the submodule $\langle\overline{2}\rangle$, so that $soc(Z_{60}) = Z_{60} \cap \langle\overline{2}\rangle = \langle\overline{2}\rangle$. And the only $maximal$ submodules $\langle\overline{2}\rangle$, $\langle\overline{3}\rangle$ and $\langle\overline{5}\rangle$. So that $J(Z_{60}) = \langle\overline{30}\rangle$, hence $\langle\overline{30}\rangle$ is EXNP-2-Absorbing submodule of Z_{60}, however Nearly-2-Absorbing submodule of Z_{60}, because $2.3.\overline{5} \in V$, for $2,3 \in Z$, $\overline{5} \in Z_{60}$, implies that $2.\overline{5} \notin V + J(Z_{60}) = \langle\overline{30}\rangle + \langle\overline{30}\rangle = \langle\overline{30}\rangle$ and $3.\overline{5} \notin \langle\overline{30}\rangle$ and $2.3 \notin 30Z$.

4. Every Nearly Quasi-2-Absorbing submodule of an Ʀ-$module$ Ш is EXNPQ2AB submodule however, the opposite is not true.

Proof.

Clear.

For the opposite, think about the following illustration:

See the submodule $\langle\overline{30}\rangle$ as the Z-module Z_{60}.

5. Every Pseudo-2-Absorbing submodule of an Ʀ-$module$ Ш is EXNPQ2AB submodule however, the opposite is not true.

Proof.

Let $\alpha\beta c$ꭓ $\in V$, for $\alpha,\beta, c \in$ Ʀ, ꭓ $\in$ Ш. That is $\alpha\beta(c$ꭓ$) \in V$. But V is Pseudo-2-Absorbing submodule of Ш, then either $\alpha(c$ꭓ$) \in V + soc($Ш$)$ or $\beta(c$ꭓ$) \in V + soc($Ш$)$ or $\alpha\beta$Ш $\subseteq V + soc($Ш$)$. Thus either αcꭓ $\in V + soc($Ш$) \subseteq V + soc($Ш$) + J($Ш$)$ or βcꭓ $\in V + soc($Ш$) \subseteq V + soc($Ш$) + J($Ш$)$ or $\alpha\beta$ꭓ $\in V + soc($Ш$) \subseteq V + soc($Ш$) + J($Ш$)$ for all ꭓ $\in$ Ш, then either αcꭓ $\in V + soc($Ш$) + J($Ш$)$ or βcꭓ $\in V + soc($Ш$) + J($Ш$)$ or $\alpha\beta$ꭓ $\in V + soc($Ш$) + J($Ш$)$. Hence V is EXNPQ2AB submodule of Ш.

For the opposite, think about the following illustration:

Let Ш $= Z_{48}$, Ʀ $= Z$ and the submodule $V = \langle\overline{8}\rangle$. Clear that V is not Pseudo-2-Absorbing, but V is EXNPQ2AB submodule of Ш.

6. Every Pseudo Quasi-2-Absorbing submodule of an Ʀ-$module$ Ш is EXNPQ2AB submodule however, the opposite is not true.

Proof.

Direct.

For the converse, see the example in (**5**).

7. The intersection of two EXNPQ2AB submodules of an Ʀ-*module* Ш need not to be EXNPQ2AB submodule of Ш, as the following example shows:

Consider the Z-*module* Z and the submodules $3Z$ and $4Z$ are EXNPQ2AB submodules of Z, but $3Z \cap 4Z = 12Z$ is not EXNPQ2AB submodule of the Z-*module* Z, since if $2.3.2.1 \in 12Z$, but $2.2.1 = 4 \notin 12Z + soc(Z) + J(Z)$ and $3.2.1 = 6 \notin 12Z + soc(Z) + J(Z)$ and $2.3.1 = 6 \notin= 12Z + soc(Z) + J(Z)$ (since $soc(Z) = (0)$ and $J(Z) = 0$).

Proposition 3.3

A *proper* submodule V of Ш is EXNPQ2AB submodule of Ш if and only if for any ɑ , ɓ ∈ Ʀ and $x \in$ Ш such that ɑɓ$x \notin V + soc$(Ш) $+ J$(Ш). Then $[V :_R$ ɑɓ$x] \subseteq [V + soc$(Ш) $+ J$(Ш)$:_R$ ɑ$x] \cup [V + soc$(Ш) $+ J$(Ш)$:_R$ ɓ$x]$.

Proof.

($\Longrightarrow$) Let V be EXNPQ2AB submodule of Ш and $t \in [V :_R$ ɑɓ$x]$, then ɑɓ$tx \in V$. Since V is EXNPQ2AB submodule of Ш and ɑɓ$x \notin V + soc$ (Ш) $+ J$(Ш), then either ɑ$tx \in V + soc$(Ш) $+ J$(Ш) or ɓ$tx \in V + soc$(Ш) $+ J$(Ш). Thus either $t \in [V + soc$(Ш) $+ J$(Ш)$:_R$ ɑ$x]$ or $t \in [V + soc$(Ш) $+ J$(Ш) $:_R$ ɓ$x]$. Hence, $t \in [V + soc$(Ш) $+ J$(Ш)$:_R$ ɑ$x] \cup [V + soc$ (Ш) $+ J$(Ш)$:_R$ ɓ$x]$. Then we get $[V :_R$ ɑɓ$x] \subseteq [V + soc$(Ш) $+ J$(Ш)$:_R$ ɑ$x] \cup [V + soc$(Ш) $+ J$(Ш)$:_R$ ɓ$x]$.

($\Longleftarrow$) Let ɑɓ$cx \in V$ for ɑ, ɓ, $c \in$ Ʀ, $x \in$ Ш and let ɑɓ$x \notin V + soc$(Ш) $+ J$(Ш). Since ɑɓ$cx \in V$, then $c \in [V :_R$ ɑɓ$x] \subseteq [V + soc$(Ш) $+ J$(Ш)$:_R$ ɑ$x] \cup [V + soc$(Ш) $+ J$(Ш)$:_R$ ɓ$x]$. It follows that either $c \in [V + soc$(Ш) $+ J$(Ш)$:_R$ ɑ$x]$ or $c \in [V + soc$(Ш) $+ J$(Ш)$:_R$ ɓ$x]$. That is either ɑ$cx \in V + soc$ (Ш) $+ J$(Ш) or ɓ$cx \in V + soc$ (Ш) $+ J$(Ш). Therefore, V EXNPQ2AB submodule of Ш.

Proposition 3.4

A *proper* submodule V of Ш is EXNPQ2AB submodule of Ш if and only if ɑɓ$c\mathbb{L} \subseteq V$, for ɑ, ɓ, $c \in$ Ʀ and $\mathbb{L}$ is a submodule of Ш, implies that either ɑ$c\mathbb{L} \subseteq V + soc$(Ш) $+ J$(Ш) or ɓ$c\mathbb{L} \subseteq V + soc$(Ш) $+ J$(Ш) or ɑɓ$\mathbb{L} \subseteq V + soc$(Ш) $+ J$(Ш).

Proof.

($\Longrightarrow$) Let V be EXNPQ2AB submodule of Ш and ɑɓ$c\mathbb{L} \subseteq V$, for ɑ, ɓ, $c \in$ Ʀ and $\mathbb{L}$ is a submodule of Ш. Suppose that ɑɓ$\mathbb{L} \nsubseteq V + soc$(Ш) $+ J$(Ш), ɑ$c\mathbb{L} \nsubseteq V + soc$(Ш) $+ J$(Ш) and ɓ$c\mathbb{L} \nsubseteq V + soc$(Ш) $+ J$(Ш). Then there is $e_1, e_2, e_3 \in \mathbb{L}$ such that ɑɓ$e_1 \notin V + soc$(Ш) $+ J$(Ш), ɑ$ce_2 \notin V + soc$(Ш) $+ J$(Ш) and ɓ$ce_3 \notin V + soc$(Ш) $+ J$(Ш). Now, ɑɓ$ce_1 \in V$ and since V is EXNPQ2AB submodule of Ш with ɑɓ$e_1 \notin V + soc$(Ш) $+ J$(Ш), then either ɓ$ce_1 \in V + soc$ (Ш) $+ J$(Ш) or ɑ$ce_1 \in V + soc$ (Ш) $+ J$(Ш). Also since ɑɓ$ce_2 \in V$ and ɑ$ce_2 \notin V + soc$(Ш) $+ J$(Ш), then either ɓ$ce_2 \in V + soc$ (Ш) $+ J$(Ш) or ɑɓ$e_2 \in V + soc$ (Ш) $+ J$(Ш). Again ɑɓ$ce_3 \in V$ and since V is EXNPQ2AB submodule of Ш with ɓ$ce_3 \notin V + soc$(Ш) $+ J$(Ш), then either ɑ$ce_3 \in V + soc$ (Ш) $+ J$(Ш) or ɑɓ$e_3 \in V + soc$ (Ш) $+ J$(Ш). Now, ɑɓ$c(e_1 + e_2 + e_3) \in V$ and V is EXNPQ2AB submodule of Ш, implies that either ɑɓ$(e_1 + e_2 + e_3) \in V + soc$ (Ш) $+ J$(Ш) or ɑ$c(e_1 + e_2 + e_3) \in V + soc$ (Ш) $+ J$(Ш) or ɓ$c(e_1 + e_2 + e_3) \in V + soc$ (Ш) $+ J$(Ш). If ɑɓ$(e_1 + e_2 + e_3) =$ ɑɓ$e_1 +$ ɑɓ$e_2 +$ ɑɓ$e_3 \in V + soc$ (Ш) $+ J$(Ш). But ɑɓ$e_2 \in V + soc$ (Ш) $+ J$(Ш) and ɑɓ$e_3 \in V + soc$ (Ш) $+ J$(Ш), then ɑɓ$e_1 \in V + soc$ (Ш) $+ J$(Ш), which is incongruent. If ɑ$c(e_1 + e_2 + +e_3) =$ ɑ$ce_1 +$ ɑ$ce_2 +$ ɑ$ce_3 \in V + soc$ (Ш) $+ J$(Ш). But ɑ$ce_1 \in V + soc$ (Ш) $+ J$(Ш) and ɑ$ce_3 \in V + soc$ (Ш) $+ J$(Ш), then

$\alpha ce_2 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ which is contradiction. If $\beta c(e_1 + e_2 + +e_3) = \beta ce_1 + \beta ce_2 + \beta ce_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$. But $\beta ce_1 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ and $\beta ce_2 \in V + soc(\mathbb{W}) + J(\mathbb{W})$, then $\beta ce_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ which is contradiction. Hence $\alpha c\mathbb{L} \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\beta c\mathbb{L} \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\alpha\beta\mathbb{L} \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$.

($\Longleftarrow$) Let $\alpha\beta cn \in V$ for $\alpha, \beta, c \in \mathbb{R}$, $n \in \mathbb{W}$, then $\alpha\beta c(n) \subseteq V$, hence by hypothesis either $\alpha c(n) \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\beta c(n) \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\alpha\beta(n) \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$. That is either $\alpha cn \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\beta cn \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\alpha\beta n \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Therefore V is EXNPQ2AB submodule of $\mathbb{W}$.

Proposition 3.5

Let $\mathbb{W}$ be *module* and V be a *proper* submodule of $\mathbb{W}$. Then V is EXNPQ2AB submodule of $\mathbb{W}$ if and only if for every submodule A of $\mathbb{W}$ and for every ideals I_1, I_2, I_3 of $\mathbb{R}$ such that $I_1I_2I_3A \subseteq V$, implies that either $I_1I_2A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $I_1I_3A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $I_2I_3A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$.

Proof.

($\Longrightarrow$) Let $I_1I_2I_3A \subseteq V$, where I_1, I_2, I_3 are ideals of $\mathbb{R}$ and A is a submodule of $\mathbb{W}$, with $I_1I_2A \nsubseteq [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} \mathbb{W}]$. To demonstrate that $I_1I_3A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $I_2I_3A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$. Suppose that $I_1I_3A \nsubseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ and $I_2I_3A \nsubseteq V + soc(\mathbb{W}) + J(\mathbb{W})$, that is there exist $a_1, a_2, a_3 \in A$ and a nonzero $r \in I_1$, $s \in I_2$ and $t \in I_3$ such that $rsa_1 \notin V + soc(\mathbb{W}) + J(\mathbb{W})$ and $rta_2 \notin V + soc(\mathbb{W}) + J(\mathbb{W})$ and $sta_3 \notin V + soc(\mathbb{W}) + J(\mathbb{W})$. Now, $rsta_1 \in V$ and $rsa_1 \notin V + soc(\mathbb{W}) + J(\mathbb{W})$, implies that either $rta_1 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $sta_1 \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Also $rsta_2 \in V$ and $rta_2 \notin V + soc(\mathbb{W}) + J(\mathbb{W})$, implies that either $rsa_2 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $sta_2 \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Again, $rsta_3 \in V$ and $sta_3 \notin V + soc(\mathbb{W}) + J(\mathbb{W})$, implies that either $rta_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $rsa_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Now, $rst(a_1 + a_2 + a_3) \in V$ and V is EXNPQ2AB, then either $rs(a_1 + a_2 + a_3) \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $rt(a_1 + a_2 + a_3) \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $st(a_1 + a_2 + a_3) \in V + soc(\mathbb{W}) + J(\mathbb{W})$. If $rs(a_1 + a_2 + a_3) = rsa_1 + rsa_2 + rsa_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ and rsa_2, $rsa_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$, hence $rsa_1 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ which is a contradiction. If $rt(a_1 + a_2 + a_3) = rta_1 + rta_2 + rta_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ and rta_1, $rta_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$, hence $rta_2 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ which is a contradiction. If $st(a_1 + a_2 + a_3) = sta_1 + sta_2 + sta_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ and $sta_1, sta_2 \in V + soc(\mathbb{W}) + J(\mathbb{W})$, hence $sta_3 \in V + soc(\mathbb{W}) + J(\mathbb{W})$ which is a contradiction. Thus either $I_1I_2A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $I_1I_3A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $I_2I_3A \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$.

($\Longleftarrow$) Clear.

Proposition 3.6

Let $\mathbb{W}$ be *module* and V be a *proper* submodule of $\mathbb{W}$. Then V is EXNPQ2AB submodule of $\mathbb{W}$ if and only if for any $r, s \in \mathbb{R}$ and $I$of $\mathbb{R}$ and $x \in \mathbb{W}$ with $rsIx \subseteq V$ implies that either $rsx \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $rIx \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $sIx \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$.

Proof.

($\Longrightarrow$) Let $rsIx \subseteq V$ for $r, s \in \mathbb{R}$ and I is an ideal of $\mathbb{R}$ and $x \in \mathbb{W}$, it follows that $I \subseteq [V :_{\mathbb{R}} rsx]$. If $rsx \in V \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$, hence $rsx \in V + soc(\mathbb{W}) + J(\mathbb{W})$, then we are done. Suppose that $rsx \notin V + soc(\mathbb{W}) + J(\mathbb{W})$, then by **Proposition 3.3** $[V :_{\mathbb{R}} rsx] \subseteq [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} rx] \cup [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} sx]$. But $rsIx \subseteq V$, then $I \subseteq [V :_{\mathbb{R}} rsx] \subseteq [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} rx] \cup [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} sx]$, hence $I \subseteq [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} rx] \cup [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} sx]$, it follows that either $I \subseteq [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} rx]$ or $I \subseteq [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} sx]$, thus either $rIx \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$ or $sIx \subseteq V + soc(\mathbb{W}) + J(\mathbb{W})$.

($\Longleftarrow$) Let $rstx \in V$ for $r,s,t \in Ꞧ$ and $x \in Ɯ$, that is $rs\langle t\rangle x \subseteq V$. It follows by hypothesis either $rsx \in V + soc(Ɯ) + J(Ɯ)$ or $r\langle t\rangle x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $s\langle t\rangle x \subseteq V + soc(Ɯ) + J(Ɯ)$. Hence either $rsx \in V + soc(Ɯ) + J(Ɯ)$ or $rtx \in V + soc(Ɯ) + J(Ɯ)$ or $stx \in V + soc(Ɯ) + J(Ɯ)$. Therefore V is EXNPQ2AB submodule of Ɯ.

From the **Proposition 3.5** and **Proposition 3.6** we get the following corollaries.

Corollary 3.7
Let Ɯ be an Ꞧ-*module* and V be a *proper* submodule of Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for each $r \in Ꞧ$, $x \in Ɯ$ and every ideals I, J of Ꞧ with $rIJx \subseteq V$, implies that either $rIx \subseteq V + soc(Ɯ) + J(Ɯ)$ or $rJx \subseteq V + soc(Ɯ) + J(Ɯ)$ or $IJx \subseteq V + soc(Ɯ) + J(Ɯ)$.

Corollary 3.8
Let Ɯ be an Ꞧ-*module* and V be a *proper* submodule of Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for every ideals I_1, I_2, I_3 of Ꞧ and $x \in Ɯ$ such that $I_1I_2I_3x \subseteq V$ implies that either $I_1I_2x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $I_1I_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $I_2I_3x \subseteq V + soc(Ɯ) + J(Ɯ)$.

Corollary 3.9
Let Ɯ be an Ꞧ-*module* and V be a *proper* submodule of Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for any $r, s \in Ꞧ$ and any ideal I of Ꞧ and every submodule A of Ɯ with $rsIA \subseteq V$ implies that either $rsA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $rIA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $sIA \subseteq V + soc(Ɯ) + J(Ɯ)$.

Corollary 3.10
Let Ɯ be an Ꞧ-*module* and V be a *proper* submodule of Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for each $r \in Ꞧ$ and any ideals I, J of Ꞧ and every submodule A of Ɯ with $rIJA \subseteq V$ implies that either $rIA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $rJA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $IJA \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proposition 3.11
Let V be EXNPQ2AB submodule of an Ꞧ- *module* Ɯ and L is a submodule of Ɯ with L $\subseteq V$, then $\frac{V}{L}$ is EXNPQ2AB submodule of an Ꞧ-module $\frac{Ɯ}{L}$.

Proof.

Let V be EXNPQ2AB submodule of Ɯ and $\alpha IJ(e + L) = \alpha IJe + L \subseteq \frac{V}{L}$ for $\alpha \in Ꞧ, I, J$ are ideals of Ꞧ and $e + L \in \frac{Ɯ}{L}$, $e \in Ɯ$, implies that $\alpha IJe \subseteq V$. Since V is EXNPQ2AB submodule of Ɯ, then by **Corollary 3.7** either $\alpha Ie \subseteq V + soc(Ɯ) + J(Ɯ)$ or $\alpha Je \subseteq V + soc(Ɯ) + J(Ɯ)$ or $IJe \subseteq V + soc(Ɯ) + J(Ɯ)$. It follows, either $\alpha I(e + L) \subseteq \frac{V + soc(Ɯ) + J(Ɯ)}{L} \subseteq \frac{V}{L} + \frac{V + soc(Ɯ) + J(Ɯ)}{L} \subseteq \frac{V}{L} + soc\left(\frac{Ɯ}{L}\right) + J(\frac{Ɯ}{L})$ or $\alpha J(e + L) \subseteq \frac{V}{L} + soc\left(\frac{Ɯ}{L}\right) + J(\frac{Ɯ}{L})$ or $IJ(e + L) \subseteq \frac{V + soc(Ɯ) + J(Ɯ)}{L} \subseteq \frac{V}{L} + \frac{V + soc(Ɯ) + J(Ɯ)}{L} \subseteq \frac{V}{L} + soc\left(\frac{Ɯ}{L}\right) + J(\frac{Ɯ}{L})$, that is either $\alpha I(e + L) \subseteq \frac{V}{L} + soc\left(\frac{Ɯ}{L}\right) + J(\frac{Ɯ}{L})$ or $\alpha J(e + L) \subseteq \frac{V}{L} + soc\left(\frac{Ɯ}{L}\right) + J(\frac{Ɯ}{L})$ or $IJ \subseteq \frac{V}{L} + soc\left(\frac{Ɯ}{L}\right) + J(\frac{Ɯ}{L})$. Hence by **Corollary 3.7** $\frac{V}{L}$ is EXNPQ2AB submodule of $\frac{Ɯ}{L}$.

Proposition 3.12
Let Ш is a $semi\ simple$ Ʀ-$module$ V and Ḥ are submdules for Ш such that Ḥ $\subseteq V$, and V is a $proper$ submodule of Ш. If Ḥ and $\frac{V}{Ḥ}$ are EXNPQ2AB submodules of Ш and $\frac{Ш}{Ḥ}$ respectively, then V is EXNPQ2AB submodules of Ш**.**

Proof.

Suppose Ḥ and $\frac{V}{Ḥ}$ are EXNPQ2AB submodules for Ш and $\frac{Ш}{Ḥ}$ respectively, and let $I_1I_2I_3m \subseteq V$, for I_1, I_2, I_3 are ideals of R, $m \in$ Ш. So $I_1I_2I_3(m + Ḥ) = I_1I_2I_3m + Ḥ \subseteq \frac{V}{Ḥ}$. If $I_1I_2I_3m \subseteq Ḥ$ and Ḥ is EXNPQ2AB submodules of Ш, implies that by **Corollary 3.8** either $I_1I_2m \subseteq Ḥ + (soc(Ш) + J(Ш)) \subseteq V + (soc(Ш) + J(Ш))$ or $I_1I_3m \subseteq Ḥ + (soc(Ш) + J(Ш)) \subseteq V + (soc(Ш) + J(Ш))$ or $I_2I_3m \subseteq Ḥ + (soc(Ш) + J(Ш)) \subseteq V + (soc(Ш) + J(Ш))$, hence V is EXNPQ2AB submodules for Ш. So, we may assume that $I_1I_2I_3m \nsubseteq Ḥ$. It follows that $I_1I_2I_3(m + Ḥ) \subseteq \frac{N}{Ḥ}$**,** but $\frac{V}{Ḥ}$ is EXNPQ2AB submodules of $\frac{Ш}{Ḥ}$ **,** again by **Corollary 3.8** either $I_1I_2(m + Ḥ) \subseteq \frac{V}{Ḥ} + \left(soc\left(\frac{Ш}{Ḥ}\right) + J\left(\frac{Ш}{Ḥ}\right)\right)$ or $I_1I_3(m + Ḥ) \subseteq \frac{V}{Ḥ} + \left(soc\left(\frac{Ш}{Ḥ}\right) + J\left(\frac{Ш}{Ḥ}\right)\right)$ or $I_2I_3(m + Ḥ) \subseteq \frac{V}{Ḥ} + \left(soc\left(\frac{Ш}{Ḥ}\right) + J\left(\frac{Ш}{Ḥ}\right)\right)$. Since Ш is a $semi\ simple$ then by **Lemma 2.10** either $I_1I_2(m + Ḥ) \subseteq \frac{V}{Ḥ} + \frac{Ḥ+soc(Ш)}{Ḥ} + \frac{Ḥ+J(Ш)}{Ḥ}$ or $I_1I_3(m + Ḥ) \subseteq \frac{V}{Ḥ} + \frac{Ḥ+soc(Ш)}{Ḥ} + \frac{Ḥ+J(Ш)}{Ḥ}$ or $I_2I_3(m + Ḥ) \subseteq \frac{V}{Ḥ} + \frac{Ḥ+soc(Ш)}{Ḥ} + \frac{Ḥ+J(Ш)}{Ḥ}$. But $Ḥ \subseteq V$, it follows that $Ḥ + soc(Ш) \subseteq V + soc(Ш)$ and $Ḥ + J(Ш) \subseteq V + J(Ш)$, hence $\frac{V}{Ḥ} + \frac{Ḥ+soc(Ш)}{Ḥ} + \frac{Ḥ+J(Ш)}{Ḥ} \subseteq \frac{V}{Ḥ} + \frac{V+soc(Ш)}{Ḥ} + \frac{V+J(Ш)}{Ḥ} = \frac{V+soc(Ш)+J(Ш)}{Ḥ}$. Thus either $I_1I_2(m + Ḥ) \subseteq \frac{V+(soc(Ш)+J(Ш))}{Ḥ}$ or $I_1I_3(m + Ḥ) \subseteq \frac{V+(soc(Ш)+J(Ш))}{Ḥ}$ or $I_1I_3(m + Ḥ) \subseteq \frac{V+(soc(Ш)+J(Ш))}{Ḥ}$, it follows that either $I_1I_2m \subseteq V + (soc(Ш) + J(Ш))$ or $I_1I_3m \subseteq V + (soc(Ш) + J(Ш))$or $I_2I_3m \subseteq V + (soc(Ш) + J(Ш))$. Hence by **Corollary 3.8** V is EXNPQ2AB submodules of Ш**.**

According to the following proposition, under specific circumstances, the intersection of two EXNPQ2AB submodules is an EXNPQ2AB submodule.

Proposition 3.13
Let Ш be an Ʀ-$module$ either E or V is $maximal\ essential$ submodule of Ш and E not contained in V. If E and V are EXNPQ2AB submodules of Ш, then $V \cap E$ is EXNPQ2AB submodule of Ш.

Proof.

Since E not contained in V, then $V \cap E$ is a $proper$ submodule of V and since V is a $proper$ submodule of Ш, hence $V \cap E$ is a $proper$ submodule of Ш. Now, let $rsIx \subseteq E \cap V$, for $r, s \in$ Ʀ, $x \in$ Ш and I is an ideal of Ʀ, it follows that $rsIx \subseteq E$ and $rsIx \subseteq V$. But both E and V are EXNPQ2AB submodules of Ш, then by **Proposition 3.6** we have either $rsx \in E + soc(Ш) + J(Ш)$ or $rIx \subseteq E + soc(Ш) + J(Ш)$ or $sIx \subseteq E + soc(Ш) + J(Ш)$ and $rsx \in V + soc(Ш) + J(Ш)$ or $rIx \subseteq V + soc(Ш) + J(Ш)$ or $sIx \subseteq V + soc(Ш) + J(Ш)$. Thus either $rsx \in (E + soc(Ш) + J(Ш)) \cap (V + soc(Ш) + J(Ш))$ or $rIx \subseteq (E + soc(Ш) + J(Ш)) \cap (V + soc(Ш) + J(Ш))$ or $sIx \subseteq (E + soc(Ш) + J(Ш)) \cap (V + soc(Ш) + J(Ш))$. Since either E or V is $maximal\ essential$ submodule of Ш, then either $soc(Ш) \subseteq E$ or $soc(Ш) \subseteq V$. Suppose that E is $maximal\ essential$ submodule of Ш, so that by **Lemma 2.11** $soc(Ш) \subseteq E$ and since E is $maximal$ submodule of Ш, then $J(Ш) \subseteq E$. It follows' that $E + soc(Ш) + J(Ш) = E$. Hence either $rsx \in E \cap (V + soc(Ш) + J(Ш))$ or $rIx \subseteq E \cap (V + soc(Ш) + J(Ш))$ or $sIx \subseteq E \cap (V + soc(Ш) + J(Ш))$. Therefore by **Lemma 2.12** we get either $rsx \in (E \cap V) + (soc(Ш) + J(Ш))$ or

$rIx \subseteq (E \cap V) + (soc(\text{W}) + J(\text{W}))$ or $sIx \subseteq (E \cap V) + (soc(\text{W}) + J(\text{W}))$. Hence by **Proposition 3.6** $V \cap E$ is EXNPQ2AB submodule of W.

Proposition 3.14
Let W be an R-*module* with $soc(\text{W})$ is Quasi-2-Absorbing submodule of W . If $V \subset \text{W}$ such that $V \subseteq soc(\text{W})$, then V is EXNPQ2AB submodule of W.

Proof.

Let $I_1 I_2 I_3 A \subseteq V$, for I_1, I_2, I_3 are ideals of R and A is a submodule of W . Since $V \subseteq soc\,(\text{W})$, it follows that $I_1 I_2 I_3 A \subseteq soc\,(\text{W})$. But $soc(\text{W})$ is Quasi-2-Absorbing subomdule of W, then either $I_1 I_3 A \subseteq soc\,(\text{W}) \subseteq V + soc\,(\text{W}) + J(\text{W})$ or $I_2 I_3 A \subseteq soc\,(\text{W}) \subseteq V + soc\,(\text{W}) + J(\text{W})$ or $I_1 I_2 A \subseteq soc\,(\text{W}) \subseteq V + soc\,(\text{W}) + J(\text{W})$. That is either $I_1 I_3 A \subseteq V + soc(\text{W}) + J(\text{W})$ or $I_2 I_3 A \subseteq V + soc(\text{W}) + J(\text{W})$ or $I_1 I_2 A \subseteq V + soc(\text{W}) + J(\text{W})$.Therefore by **Proposition 3.5** V is EXNPQ2AB submodule of W.

Proposition 3.14
Let W be an R-*module* with $J(\text{W})$ is Quasi-2-Absorbing submodule of W . If $V \subset \text{W}$ such that $V \subseteq J(\text{W})$, then V is EXNPQ2AB submodule of W.

Proof.

Let $\alpha\beta c A \subseteq V$, for α , $\beta, c \in$ R and A is a submodule of W . Since $V \subseteq J(\text{W})$, it follows that $\alpha\beta A \subseteq J(\text{W})$. But $J(\text{W})$ is Quasi-2-Absorbing subomdule of W , then either $\alpha c A \subseteq J(\text{W}) \subseteq V + soc(\text{W}) + J(\text{W})$ or $\beta c A \subseteq J(\text{W}) \subseteq V + soc(\text{W}) + J(\text{W})$ or $\alpha\beta A \subseteq J(\text{W}) \subseteq V + soc(\text{W}) + J(\text{W})$. That is either $\alpha c A \subseteq V + soc(\text{W}) + J(\text{W})$ or $\beta c A \subseteq V + soc(\text{W}) + J(\text{W})$ or $\alpha\beta A \subseteq V + soc(\text{W}) + J(\text{W})$. Therefore by **Proposition 3.4** V is EXNPQ2AB submodule of W.

4. The Relationship between the Concept of EXNPQ2AB Submodules and Other Concepts.

The relationships between EXNPQ2AB submodules, 2-Absorbing submodules and other types of submodules are discussed in this section.

Proposition 4.1
Let W be a cyclic R-*module* and V is an *essential* submodule of W with $J(\text{W}) \subseteq V$. Then V is 2-Absorbing if and only if V is EXNPQ2AB submodule of W.

Proof.

(⇒) By **Remarks and Examples 3.2(1).**

(⇐) Let $\text{W} = \text{R}w$ for some $w \in \text{W}$ and assume that V is EXNPQ2AB submodule of W. Let $\alpha b h \in V$ for $\alpha, b \in$ R, $h \in$ W, then *there exist* an element $c \in$ R such that $\alpha b h = \alpha b c w \in V$. Since V is EXNPQ2AB submodule of W, then either $\alpha c w \in V + soc(\text{W}) + J(\text{W})$ or $b c w \in V + soc(\text{W}) + J(\text{W})$ or $\alpha b w \in V + soc(\text{W}) + J(\text{W})$. That is either $ab \in [V + soc(\text{W}) + J(\text{W}) :_{\text{R}} w] = [V + soc(\text{W}) + J(\text{W}) :_{\text{R}} \text{W}]$ or $\alpha h \in V + soc(\text{W}) + J(\text{W})$ or $bh \in V + soc(\text{W}) + J(\text{W})$. Since V is *essential* submodule of W, then $soc(\text{W}) \subseteq V$ and by hypotheses $J(\text{W}) \subseteq V$, we get $V + soc(\text{W}) = V$ and $V + J(\text{W}) = V$, thus $V + soc(\text{W}) + J(\text{W}) = V$. Hence either $ab \in [V :_{\text{R}} \text{W}]$ or $bh \in V$ or $\alpha h \in V$. Therefore V is 2-Absorbing submodule of W.

The Proof of the **Proposition 4.2 and 4.3** is staightforward.

Proposition 4.2
Let W be a cyclic R-module and V is maximal submodule of W with $soc(\text{W}) \subseteq V$. Then V is 2-Absorbing if and only if V is EXNPQ2AB submodule of W.

Proposition 4.3
Let $\mathbb{W}$ be a cyclic $\mathbb{R}$-module and V is a proper submodule of $\mathbb{W}$ with $soc(\mathbb{W}) + J(\mathbb{W}) \subseteq V$. Then V is 2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proposition 4.4
Let $\mathbb{W}$ be $\mathbb{R}$-$module$ and V is an essential submodule of $\mathbb{W}$ with $J(\mathbb{W}) \subseteq V$. Then V is Quasi-2-Absorbing if and only if V is EXNPQ-2-Absorbing.

Proof.

($\Rightarrow$) By **Remarks and Examples 3.2(2).**

($\Leftarrow$) Let $abch \in V$ for $a, b, c \in \mathbb{R}$ and $h \in \mathbb{W}$, hence either $abh \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bch \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $ach \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Since V is $essential$ submodule of $\mathbb{W}$, then $soc(\mathbb{W}) \subseteq V$ and by hypotheses $J(\mathbb{W}) \subseteq V$, we get $V + soc(\mathbb{W}) = V$ and $V + J(\mathbb{W}) = V$, thus $V + soc(\mathbb{W}) + J(\mathbb{W}) = V$. Hence either $abh \in V$ or $bch \in V$ or $ach \in V$. Therefor V is Quasi-2-Absorbing of $\mathbb{W}$.

Proposition 4.5
Let $\mathbb{W}$ be $\mathbb{R}$-module and V is a $proper$ submodule of $\mathbb{W}$ with $soc(\mathbb{W}) + J(\mathbb{W}) \subseteq V$. Then V is Quasi-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

Direct.

Proposition 4.6
Let $\mathbb{W}$ be a cyclic $\mathbb{R}$-$module$ with V is a $proper$ submodule of $\mathbb{W}$ and $soc(\mathbb{W}) \subseteq J(\mathbb{W})$. Then V is Nearly-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

($\Rightarrow$) By **Remarks and Examples 3.2(3).**

($\Leftarrow$) Let $\mathbb{W} = \mathbb{R}w$ for some $w \in \mathbb{W}$ and assume that V is EXNPQ2AB submodule of $\mathbb{W}$. Let $abh \in V$ for $a, b \in \mathbb{R}$, $h \in \mathbb{W}$, then $there\ exist$ an element $c \in \mathbb{R}$ such that $abh = abcw \in V$. Since V is EXNPQ2AB submodule of $\mathbb{W}$, then either $acw \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bcw \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $abw \in V + soc(\mathbb{W}) + J(\mathbb{W})$. That is either $ab \in [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} w] = [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} \mathbb{W}]$ or $ah \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bh \in V + soc(\mathbb{W}) + J(\mathbb{W})$. But $soc(\mathbb{W}) \subseteq J(\mathbb{W})$, thus $soc(\mathbb{W}) + J(\mathbb{W}) = J(\mathbb{W})$. Hence either $ab \in [V + J(\mathbb{W}) :_{\mathbb{R}} \mathbb{W}]$ or $bh \in V + J(\mathbb{W})$ or $ah \in V + J(\mathbb{W})$. Therefore V is Nearly-2-Absorbing submodule of $\mathbb{W}$.

Proposition 4.7
Let $\mathbb{W}$ be a cyclic $\mathbb{R}$-$module$ with $V \subset \mathbb{W}$ and $soc(\mathbb{W}) \subseteq V$. Then V is Nearly-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.**Proof.**

($\Rightarrow$) Clear.

($\Leftarrow$) Let $\mathbb{W} = \mathbb{R}w$ for some $w \in \mathbb{W}$ and assume that V is EXNPQ2AB submodule of $\mathbb{W}$. Let $abh \in V$ for $a, b \in \mathbb{R}$, $h \in \mathbb{W}$, then $there\ exist$ an element $c \in \mathbb{R}$ such that $abh = abcw \in V$. Since V is EXNPQ2AB submodule of $\mathbb{W}$, then either $acw \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bcw \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $abw \in V + soc(\mathbb{W}) + J(\mathbb{W})$. That is either $ab \in [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} w] = [V + soc(\mathbb{W}) + J(\mathbb{W}) :_{\mathbb{R}} \mathbb{W}]$ or $ah \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bh \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Since $soc(\mathbb{W}) \subseteq V$, then $V + soc(\mathbb{W}) = V$, so $V + soc(\mathbb{W}) + J(\mathbb{W}) = V + J(\mathbb{W})$. Hence either $ab \in [V + J(\mathbb{W}) :_{\mathbb{R}} \mathbb{W}]$ or $bh \in V + J(\mathbb{W})$ or $ah \in V + J(\mathbb{W})$. Therefore V is Nearly-2-Absorbing submodule of $\mathbb{W}$.

The Proof of the **Proposition 4.8 and 4.9** is staightforward.

Proposition 4.8

Let $\mathbb{W}$ be a cyclic $Ɽ$-*module* with V is *proper* of $\mathbb{W}$ and $soc(\mathbb{W}) = (0)$. Then V is Nearly-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proposition 4.9

Let $\mathbb{W}$ be a cyclic $Ɽ$-*module* and V is an *essential* submodule of $\mathbb{W}$. Then V is Nearly-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proposition 4.10

Let $\mathbb{W}$ be $Ɽ$-*module*, $soc(\mathbb{W}) \subseteq J(\mathbb{W})$ and $V \subset \mathbb{W}$. Then V is Nearly Quasi-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

($\Rightarrow$) By **Remarks and Examples 3.2(4).**

($\Leftarrow$) Let $\mathrm{a}bch \in V$ for $\mathrm{a}, b, c \in Ɽ$, $h \in \mathbb{W}$. Since V is EXNPQ2AB, then either $\mathrm{a}bh \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bch \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\mathrm{a}ch \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Since $soc(\mathbb{W}) \subseteq J(\mathbb{W})$, then $soc(\mathbb{W}) + J(\mathbb{W}) = J(\mathbb{W})$, thus either $\mathrm{a}bh \in V + J(\mathbb{W})$ or $bch \in V + J(\mathbb{W})$ or $\mathrm{a}ch \in V + J(\mathbb{W})$. Hence V is Nearly Quasi-2-Absorbing of $\mathbb{W}$.

Proposition 4.11

Let $\mathbb{W}$ be $Ɽ$-*module*, $soc(\mathbb{W}) \subseteq V$ and $V \subset \mathbb{W}$. Then V is Nearly Quasi-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

Direct.

Proposition 4.12

Let $\mathbb{W}$ be a cyclic $Ɽ$-*module* and V is an *maximal* submodule of $\mathbb{W}$. Then V is Pseudo-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

($\Rightarrow$) By **Remarks and Examples 3.2(5).**

($\Leftarrow$) Let $\mathbb{W} = Ɽw$ for some $w \in \mathbb{W}$ and assume that V is EXNPQ2AB submodule of $\mathbb{W}$. Let $\mathrm{a}bh \in V$ for $\mathrm{a}, b \in Ɽ$, $h \in \mathbb{W}$, then *there exist* an element $c \in Ɽ$ such that $\mathrm{a}bh = \mathrm{a}bcw \in V$. Since V is EXNPQ2AB submodule of $\mathbb{W}$, then either $\mathrm{a}cw \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bcw \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $\mathrm{a}bw \in V + soc(\mathbb{W}) + J(\mathbb{W})$. That is either $ab \in [V + soc(\mathbb{W}) + J(\mathbb{W}):_{\mathrm{R}} w] = [V + soc(\mathbb{W}) + J(\mathbb{W}):_{\mathrm{R}} \mathbb{W}]$ or $\mathrm{a}h \in V + soc(\mathbb{W}) + J(\mathbb{W})$ or $bh \in V + soc(\mathbb{W}) + J(\mathbb{W})$. Since V is an *maximal* submodule of $\mathbb{W}$, then $J(\mathbb{W}) \subseteq V$, hence $V + J(\mathbb{W}) = V$, so $V + soc(\mathbb{W}) + J(\mathbb{W}) = V + soc(\mathbb{W})$. Hence either $ab \in [V + soc(\mathbb{W}):_{\mathrm{R}} \mathbb{W}]$ or $bh \in V + soc(\mathbb{W})$ or $\mathrm{a}h \in V + soc(\mathbb{W})$. Therefore, V is Pseudo-2-Absorbing submodule of $\mathbb{W}$.

Proposition 4.13

Let $\mathbb{W}$ be a cyclic $Ɽ$-*module* with V is *proper* of $\mathbb{W}$ and $J(\mathbb{W}) = (0)$. Then V is Pseudo-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

Direct.

Proposition 4.14

Let $\mathbb{W}$ be $Ɽ$-module and V is a maximal submodule of $\mathbb{W}$. Then, V is Pseudo Quasi-2-Absorbing if and only if V is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

(⇒) By **Remarks and Examples 3.2(6).**

(⇐) Clear.

Finally, we will present a Proposition that equals all the previous concepts.

Proposition 4.15
Let Ɯ be a multiplicatiion Ʀ-*module* and V is *a proper* submodule of Ɯ with $J(Ɯ) = V$ and $soc(Ɯ) \subseteq V$. Consequently, the following claims are equal:

1. V is a 2-Absorbing submodule of Ɯ.

2. Quasi-2-Absorbing submodule of Ɯ.

3. Nearly Quasi-2-Absorbing submodule of Ɯ.

4. Nearly-2-Absorbing submodule of Ɯ.

5. EXNPQ-2-Absorbing submodule of Ɯ.

6. Pseudo-2-Absorbing submodule of Ɯ.

7. Pseudo Quasi-2-Absorbing submodule of Ɯ.

Proof.

(**1** ⇔ **2**) By **Lemma 2.14.**

(**2** ⇒ **3**) Clear.

(**3** ⇒ **4**) Let $ɑbx \in V$ for $ɑ, b \in Ʀ$, $x \in Ɯ$ and assume that V is Nearly Quasi-2-Absorbing submodule of Ɯ. That is $ɑb(x) \subseteq V$, since Ɯ is a multiplication Ʀ-*module*, then *there exist* an ideal I of Ʀ such that $(x) = IƜ$, hence $ɑbIƜ \subseteq V$. Since V is Nearly Quasi-2-Absorbing submodule of Ɯ, then either $ɑIƜ \subseteq V + J(Ɯ)$ or $bIƜ \subseteq V + J(Ɯ)$ or $ɑbƜ \subseteq V + J(Ɯ)$. That is either $ab \in [V + J(Ɯ):_{Ʀ} Ɯ]$ or $ɑx \in V + J(Ɯ)$ or $bx \in V + J(Ɯ)$. Therefore V is Nearly-2-Absorbing submodule of Ɯ.

(**4** ⇒ **5**) By **Remarks and Examples 3.2(3).**

(**5** ⇒ **6**) Let $ɑbx \in V$ for $ɑ, b \in Ʀ$, $x \in Ɯ$ and assume that V is EXNPQ2AB submodule of Ɯ. That is $ɑb(x) \subseteq V$, since Ɯ is a multiplicatiIon Ʀ-*module*, then *there exist* an ideal I of Ʀ such that $(x) = IƜ$, hence $ɑbIƜ \subseteq V$. Since V is EXNPQ2AB submodule of Ɯ, then either $ɑIƜ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $bIƜ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɑbƜ \subseteq V + soc(Ɯ) + J(Ɯ)$. That is either $ab \in [V + soc(Ɯ) + J(Ɯ):_{Ʀ} Ɯ]$ or $ɑx \in V + soc(Ɯ) + J(Ɯ)$ or $bx \in V + soc(Ɯ) + J(Ɯ)$. But $J(Ɯ) \subseteq V$, then $V + J(Ɯ) = V$ and $V + soc(Ɯ) + J(Ɯ) = V + soc(Ɯ)$. Thus either $ab \in [V + soc(Ɯ) :_{Ʀ} Ɯ]$ or $bx \in V + soc(Ɯ)$ or $ɑx \in V + soc(Ɯ)$. Therefore V is Pseudo-2-Absorbing submodule of Ɯ.

(**6** ⇒ **7**) Let $ɑɓcẍ \in V$, for $ɑ, ɓ, c \in Ʀ$, $ẍ \in Ɯ$. That is $ɑɓ(cẍ) \in V$. But V is Pseudo-2-Absorbing submodule of Ɯ, then either $ɑ(cẍ) \in V + soc(Ɯ)$ or $ɓ(cẍ) \in V + soc(Ɯ)$ or $ɑɓƜ \subseteq V + soc(Ɯ)$. Thus either $ɑcẍ \in V + soc(Ɯ)$ or $ɓcẍ \in V + soc(Ɯ)$ or $ɑɓẍ \in V + soc(Ɯ)$ for all $ẍ \in Ɯ$. Hence V is Pseudo Quasi-2-Absorbing submodule of Ɯ.

(**7** ⇒ **1**) Let $ɑbx \in V$ for $ɑ, b \in Ʀ$, $x \in Ɯ$ and assume that V is Pseudo Quasi-2-Absorbing submodule of Ɯ. That is $ɑb(x) \subseteq V$, since Ɯ is a multiplication Ʀ-*module*, then t*here exist* an ideal I of Ʀ such that $(x) = IƜ$, hence $ɑbIƜ \subseteq V$. Since V is Pseudo Quasi-2-Absorbing submodule of Ɯ, then either $ɑIƜ \subseteq V + soc(Ɯ)$or $bIƜ \subseteq V + soc(Ɯ)$or $ɑbƜ \subseteq V + soc(Ɯ)$. That is either $ab \in [V + soc(Ɯ):_{Ʀ} Ɯ]$ or $ɑx \in V + soc(Ɯ)$ or $bx \in V + soc(Ɯ)$. But $soc(Ɯ) \subseteq V$, hence $V + soc(Ɯ) = V$. Thus either $ab \in [V:_{Ʀ} Ɯ]$ or $bx \in V$ or $ɑx \in V$. Therefore, V is 2-Absorbing submodule of Ɯ.

5. Conclusion

The term EXNPQ2AB submodules is a novel generalization of (2-Absorbing, Quasi-2-Absorbing, Nearly-2-Absorbing, Nearly Quasi-2-Absorbing, Pseudo-2-Absorbing and Pseudo Quasi-2-Absorbing) submodules that we introduce in this article. Using examples, we also discuss the opposite generalization introduced in several different characterizations. There are given connections between this generalization and other types of modules.

References

1. Darani, A.Y; Soheilniai. F. 2-Absorbing and Weakly 2-AbsorbingSubmodules, *Tahi Journal. Math*, **2011**,*(9)*, 577-584.
2. Lu, C. P. Prime Submodules of Modules, *Comm. Math, University Spatula*, **1981**, *(33)*, 61-69.
3. Goodearl, K. R., (1976), Ring Theory, Marcel Dekker, Inc. New York and Basel., **1976**, *206*.
4. Kash, F. Modules and Rings. London Math. Soc. Monographs New York , Academic Press , **1982**, *370*.
5. Reem, T; Shwkea, M. Nearly 2-Absorbing Submodules and Related Concept, *Tikrit Journal for Pure Sci*, **2018**, *2(3),* 215-221.
6. Osama A. Naji; Mohammed M. Al-Ashker and Arwa E. Ashour. On n-2-Absorbing Submodules, M. Sc. Thesis,The Islamic University -Gaza. **, 2016**, *42*.
7. Haibat K. Mohammedali and Omar A.Abdalla. Pseudo-2-Absorbing and Pseudo Semi-2-Absorbing Submodules, *AIP Conference Proceedings*, **2019**, *2096,020006*, 1-9.
8. Haibat, K. M and Khalaf, H .A. Nearly Quasi2-Absorbing Submodules, *Tikrit Journal. for Pure. Sci*, **2018**, *23(9)*, 99-102.
9. Anderson, E. W and Fuller, K. R. Rings and Categories of Modules , *springer Verlag, New York*. **1992**.
10. Payman, M. H. Hollow Modules and Semi Hollow Modules. M. Sc. Thesis, University of Baghbad. **2005**.
11. El-Bast, Z. A ; Smith, P. F. Multiplication Modules, Comm. In Algebra, **1988**, *16(4)*, 755-779.
12. Faranak. F. Quasi 2-absorbing Submodules, *KYUNGPOOK Mathematical Junral*, **2019**, *59(2)*, 209-214.