IHJPAS. 36(2)2023

Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq

Strongly Pseudo Nearly Semei-2-Absorbing Submodule(I)

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Article history: Received 17 September 2022, Accepted 6 November 2022, Published in April 2023.

doi.org/10.30526/36.2.3021

Abstract

Let $\mathcal{H}$ be a module over a commutative ring R with identity. In this paper we intoduce the concept of Strongly Pseudo Nearly Semi-2-Absorbing submodule, where a proper submodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is said to be Strongly Pseudo Nearly Semi-2-Absorbing submodule of $\mathcal{H}$ if whenever $u^2\varkappa \in \mathcal{F}$, for $u \in R$, $\varkappa \in \mathcal{H}$ implies that either $u\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}):_R \mathcal{H}]$, this concept is a generalization of 2_Absorbing submodule, semi 2-Absorbing submodule, and strong form of (Nearly–2–Absorbing, Pseudo_2_Absorbing, and Nearly Semi–2–Absorbing) submodules. Several properties characterizations, and examples concerning this new notion are given. We study the relation between Strongly Pseudo Nearly Semei-2-Absorbing submodule and (2_Absorbing, Nearly_2_Absorbing, Pseudo_2_Absorbing, and Nearly Semi–2–Absorbing) submodules and the converse of this relation is true under certain condition. Also, we introduced many characterizations of Strongly Pseudo Nearly Semei-2-Absorbing submodules in some types of modules.

Keywords:2_Absorbing submodules, Pseudo_2_Absorbing submodules, Nearly_2_Absorbingsu bmodules, Nearly Semei-2-Absorbing submodules, STPNS-2-Absorbing submodules.

1. Introduction

Throughout this paper, all rings are assumed commutative with identity and all R-modules are left unitary. According Darani and Soheiline in 2011, we introduce a concept 2-Absorbing submodule. Many researchers have generalized the concept of 2_Absorbing submodules in different way. Recently, in 2018 Reem and Shwkea introduced the concept of Nearly–2–Absorbing submodules as new generalization of 2_Absorbing submodules. Also, in 2019 Haibat and Omer introduced the concept Pseudo_2_Absorbing submodules as new generalization of 2_Absorbing submodules. In 1967 Goodearl made a generalization of the concept of 2_Absorbing which is semi-2-Absorbing submodule. Also, in 2019 Akram and Haibat, Haibat and Omer made a generalization of the concept of 2_Absorbing an illusion Nearly semi-2-Absorbing and Pseudo semi-2-Absorbing. Our goal in this paper was studied the concept Strongly Pseudo Nearly Semei-2-Absorbing submodules as new generalization of 2_Absorbing submodule, strong form of both Nearly–2–Absorbing, Pseudo_2_Absorbing, Nearly Semi–2–Absorbing and Pseudo semi-2-Absorbing submodules are given Several properties and characterizations of this concepts in some types of modules.

2. Basic Properties

In this section we introduce the basic properties of the concept Strongly Pseudo Nearly Semei-2-Absorbing submodules.

Definition 2.1 [1].

Let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called 2-absorbing if whenever $uvh \in \mathcal{F}$, for u, $v \in R, h \in \mathcal{H}$, then either $uh \in \mathcal{F}$ or $vh \in \mathcal{F}$ or $uv \in [\mathcal{F}_R\!:\!\mathcal{H}]$.

Definition 2.2 [2].

$soc(\mathcal{H})$ is the intersection of all essential submodule of $\mathcal{H}$and a nonzero submodule $\mathbb{Q}$ of $\mathcal{H}$ is a essential in $\mathcal{H}$ if $\mathbb{Q} \cap \mathbb{C} \neq (0)$ for any nonzero submodule $\mathbb{C}$ of $\mathcal{H}$.

Definition 2.3 [3].

$J(\mathcal{H})$ is the intersection of all maximal submodule of $\mathcal{H}$ and a proper submodule $\mathcal{B}$ of an R._module $\mathcal{H}$ is called maximal submodule if whenever $\mathcal{B} \subseteq \mathcal{L} \subseteq \mathcal{H}$ for some submodule $\mathcal{L}$ of $\mathcal{H}$ then either $\mathcal{B} = \mathcal{L}$ or $\mathcal{L} = \mathcal{H}$.

Definition 2.4 [4].

Let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called Nearly-2-Absorbing submodule if whenever $uvm \in \mathcal{F}$, for $u, v \in R$, $h \in \mathcal{H}$, implies that either $uh \in \mathcal{F} + J(\mathcal{H})$ or $vh \in \mathcal{F} + J(\mathcal{H})$ or $uv \in [\mathcal{F} + J(\mathcal{H})_R\!:\!\mathcal{H}]$.

Definition 2.5 [5].

Let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called Pseudo-2-Absorbing submodule if whenever $uvm \in \mathcal{F}$, for $u, v \in R$, $h \in \mathcal{H}$, implies that either $uh \in \mathcal{F} + Soc(\mathcal{H})$ or $vh \in \mathcal{F} + Soc(\mathcal{H})$ or $uv \in [\mathcal{F} + Soc(\mathcal{H})_R\!:\!\mathcal{H}]$.

Definition 2.6 [2].

Let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called semi-2-absorbing if whenever $u^2y \in \mathcal{F}$, for$u \in R$, $y \in \mathcal{H}$ implies that $uy \in \mathcal{F}$ or $u^2 \in [\mathcal{F} :_R \mathcal{H}]$.

Definition 2.7 [6].

Let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called Nearly semi-2 absorbing submodule if whenever $\mathrm{a}^2\mathrm{y} \in \mathrm{N}$, for a ∈ R, y ∈ M implies that ay $\in \mathcal{F} + \mathrm{J}(\mathcal{H})$ or $\mathrm{a}^2 \in [\mathcal{F} + \mathrm{J}(\mathcal{H})_{\mathrm{R}}\!:\!\mathcal{H}]$.

Definition 2.8 [5].

A proper submodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is called Pseudo semi-2 absorbing submodule if whenever $\mathrm{a}^2\mathrm{y} \in \mathrm{N}$, for a ∈ R, y ∈ M implies that ay $\in \mathcal{F} + \mathrm{soc}(\mathcal{H})$ or $\mathrm{a}^2 \in [\mathcal{F} + \mathrm{soc}(\mathcal{H})_{\mathrm{R}}\!:\!\mathcal{H}]$.

Lemma 2.9 [7, Lemma. (2.3.15)].

Let $\mathcal{L}$, $\mathbb{Q}$ and $\mathcal{B}$ be submodule of an R_module $\mathcal{H}$ with $\mathbb{Q} \subseteq \mathcal{B}$. Then $(\mathcal{L} + \mathbb{Q}) \cap \mathcal{B} = (\mathcal{L} \cap \mathcal{B}) + \mathbb{Q} = (\mathcal{L} \cap \mathcal{B}) + (\mathbb{Q} \cap \mathcal{B})$.

Lemma 2.10 [7, Example .(12) (c)].

It well – known that an R _ module $\mathcal{H}$ is a semi simple if and only if for each submodule $\mathcal{F}$ of $\mathcal{H}$, $soc\left(\frac{\mathcal{H}}{\mathcal{F}}\right) = \frac{soc(\mathcal{H})+\mathcal{F}}{\mathcal{F}}$.

Lemma 2.11 [7, Example(12), P. 239].

Let $\mathcal{F}$ be a submodule of a semi simple R _ module $\mathcal{H}$ then $J\left(\frac{\mathcal{H}}{\mathcal{F}}\right) = \frac{J(\mathcal{H})+\mathcal{F}}{\mathcal{F}}$.

Definition 2.12 [8].
An element $a \in R$ is an idempotent element if $a^2 = a.a = a$. And if every element a of a ring R is an idempotent, then R is a Boolean ring.

Lemma 2.13 [9, Theorem(2.2)]
If R is a Boolean ring, then R is a regular ring.

Definition 2.14 [10].
An R-module H is called regular module if every submodule of H is a pure.

Lemma 2.15 [10].
If $\mathcal{H}$ is a regular R-module, then $J(\mathcal{H}) = 0$.

Definition 2.16 [2]
An R_module $\mathcal{H}$ is said to be a semi simple , if every submodule of $\mathcal{H}$ is a direct summand of $\mathcal{H}$, that is if $\mathcal{F}$ is a submodule of $\mathcal{H}$, then $\mathcal{H} = \mathcal{F} \oplus K$ for some submodule K of $\mathcal{H}$.

Lemma 2.17 [7, proposition . (9.14) (c)]
If $\mathcal{H}$ is a semi-simple R-module, then J($\mathcal{H}$) $= 0$.

Lemma 2.18[5, remark (1.2)].
It is clear that every 2-Absorbing submodule of an R-module $\mathcal{H}$ is Pseudo 2-Absorbing submodule.

Lemma 2.19 [4, proposition (2.8)].
Let A be a Nearly 2-Absorbing submodule of an R-module $\mathcal{H}$ with $J(\mathcal{H}) \subseteq A$. Then A is 2-Absorbing submodule.

Lemma 2.20 [6, proposition (2.11)].
Let $\mathcal{H}$ be an R-module and A is a proper submodule of $\mathcal{H}$ with $soc(\mathcal{H}) \subseteq A$. Then A is Semi 2-Absorbing of $\mathcal{H}$ if and only if A Nearly Semi 2-Absorbing of $\mathcal{H}$.

3. The Results

In this section we introduce the definition of Strongly Pseudo Nearly Semi-2-Absorbing submodule. Example, characterizations, some basic properties of this concept are given:

Definition 3.1
A proper submodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is said to be Strongly Pseudo Nearly Semi-2-Absorbing submodule of $\mathcal{H}$ (for short STPNS) if whenever $u^2\varkappa \in \mathcal{F}$, for $u \in R$, $\varkappa \in \mathcal{H}$ implies that either $u\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}):_R \mathcal{H}]$.

Remarks and Examples 3.2
1. Let $\mathcal{H} = Z_{36}$, $R = Z$ and the submodule $\mathcal{F} = \langle\bar{4}\rangle$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ since $soc(Z_{36}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{6}\rangle \cap Z_{36} = \langle\bar{6}\rangle$ and $J(Z_{36}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ that is for all $u \in Z$ and $\varkappa \in Z_{36}$ such that $u^2\varkappa \in \langle\bar{4}\rangle$, implies that either $u\varkappa \in \mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right) = \langle\bar{2}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ $u^2 \in \left[\mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right)_R : Z_{36}\right] = 2Z$. That is $2^2.\bar{4} \in \langle\bar{2}\rangle$, implies that $2.\bar{4} = \bar{8} \in \langle\bar{2}\rangle$ or $2^2 = 4 \in \left[\mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right)_R : Z_{36}\right] = 2Z$.

2. Every 2-Absorbing submodule is STPNS-2-Absorbing, but the converse is not true.

Proof

Assume that $\mathcal{F}$ is 2-Absorbing submodule of $\mathcal{H}$ and $u^2m \in \mathcal{F}$, for $u \in R$, $m \in \mathcal{H}$, that is $u.u.m \in \mathcal{F}$. Since $\mathcal{F}$ is 2-Absorbing submodule, then either $um \in \mathcal{F} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u.u\mathcal{H} =$

$u^2\mathcal{H} \subseteq \mathcal{F} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$implies that $u^2 \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))_R : \mathcal{H}]$. Hence $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$.
For the converse consider the following example:

Let $\mathcal{H} = Z_{36}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is STPNS_2_Absorbing submodule of $\mathcal{H}$ since $soc(Z_{36}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{6}\rangle \cap Z_{36} = \langle\bar{6}\rangle$ and $J(Z_{36}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ that is for all $u \in Z$ and $\varkappa \in Z_{36}$ such that $u^2\varkappa \in \langle\overline{12}\rangle$, implies that either $u\varkappa \in \mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$ $u^2 \in \left[\mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right)_R : Z_{36}\right] = 6Z$. That is $2^2.\bar{3} \in \langle\overline{12}\rangle$, implies that $2.\bar{3} = \bar{6} \in \mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right) = \bar{6} \in \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F} = \langle\overline{12}\rangle$ is not 2_Absorbing submodule of $\mathcal{H}$, since $2.3.\bar{2} \in \langle\overline{12}\rangle$, but $2.\bar{2} \notin \langle\overline{12}\rangle$ and $3.\bar{2} \notin \langle\overline{12}\rangle$ and $2.3 \notin [\langle\overline{12}\rangle_R : Z_{36}] = 12Z$.

3. Every STPNS-2-Absorbing submodule is Nearly-2-Absorbing, but the converse is not true.

Proof

Suppose that $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$and $u^2 m \in \mathcal{F}$, for $u \in R$, $m \in \mathcal{H}$. Since $\mathcal{F}$ is STPNS-2-Absorbing submodule, then $um \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + J(\mathcal{H})$. Hence $\mathcal{F}$ is Nearly-2-Absorbing submodule of $\mathcal{H}$.
For the converse consider the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{24}\rangle$ is Nearly-2-Absorbing submodule of M since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ that is for all $u, v \in Z$ and $m \in Z_{48}$ such that $uvm \in \langle\overline{24}\rangle$, implies that either $um \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or $vm \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$. That is $2.4.\bar{3} \in \langle 24\rangle$, implies that $2.\bar{3} = \bar{6} \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ and $4.\bar{3} = \overline{12} \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F} = \langle\overline{24}\rangle$ is not STPNS-2-Absorbing submodule of $\mathcal{H}$, since $2^2.\bar{6} \in \langle\overline{24}\rangle$, but $2.\bar{6} \notin \langle 24\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right) = \langle\overline{24}\rangle$ and $2^2 \notin \left[\langle\overline{24}\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right)_R : Z_{48}\right] = 24Z$.

4. Every STPNS-2-Absorbing submodule is Pseudo-2-Absorbing, but the converse is not true.

Proof

Suppose that $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$and $u^2 m \in \mathcal{F}$, for $u \in R$, $m \in \mathcal{H}$. Since $\mathcal{F}$ is STPNS-2-Absorbing submodule, then $um \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + soc(\mathcal{H})$. Hence $\mathcal{F}$ is pseudo-2-Absorbing submodule of $\mathcal{H}$.
For the converse consider the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is pseudo-2-Absorbing submodule of M since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ that is for all $u, v \in Z$ and $m \in Z_{48}$ such that $uvm \in \langle\overline{12}\rangle$, implies that either $um \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$ or $vm \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$or $uv \in \left[\mathcal{F} + \left(soc(Z_{48})\right)_R : Z_{48}\right] = 4Z$. That is $2.2.\bar{3} \in \langle\overline{12}\rangle$, implies that $2.2 \in \left[\mathcal{F} + \left(soc(Z_{48})\right)_R : Z_{48}\right] = 4Z$. But $\mathcal{F} = \langle\overline{12}\rangle$ is not STPNS-2-Absorbing submodule of $\mathcal{H}$, since $2^2.\bar{3} \in \langle\overline{12}\rangle$, but $2.\bar{3} \notin \langle\overline{12}\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right) = \langle\overline{12}\rangle$ and $2^2 \notin \left[\langle\overline{12}\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right)_R : Z_{48}\right] = 12Z$.

5. Every semi-2-Absorbing submodule is STPNS-2-Absorbing, but the converse is not true.

Proof

Let $\mathcal{F}$ be a semi-2-Absorbing submodule of $\mathcal{H}$, and $u^2\varkappa \in \mathcal{F}$ for $u \in R, \varkappa \in \mathcal{H}$. Since $\mathcal{F}$ is a semi-2-Absorbing, then either $u\varkappa \in \mathcal{F} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2\mathcal{H} \subseteq \mathcal{F} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$.

That is either $u\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$. Hence $\mathcal{F}$ is STPNS-2-absorbing submodule of $\mathcal{H}$.
For the converse consider the following example:

Let $\mathcal{H} = Z_{36}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is STPNS_2_Absorbing submodule of $\mathcal{H}$ since $soc(Z_{36}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{6}\rangle \cap Z_{36} = \langle\bar{6}\rangle$ and $J(Z_{36}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ that is for all $u \in Z$ and $\varkappa \in Z_{36}$ such that $u^2\varkappa \in \langle\overline{12}\rangle$, implies that either $u\varkappa \in \mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or $u^2 \in \left[\mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right)_R : Z_{36}\right] = 6Z$. That is $2^2.\bar{3} \in \langle\overline{12}\rangle$, implies that $2.\bar{3} = \bar{6} \in \mathcal{F} + \left(J(Z_{36}) \cap soc(Z_{36})\right) = \bar{6} \in \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F} = \langle\overline{12}\rangle$ is not semi-2_Absorbing submodule of $\mathcal{H}$, since $2^2.\bar{3} \in \langle\overline{12}\rangle$, but $2.\bar{3} \notin \langle\overline{12}\rangle$ and $2^2 \notin [\langle\overline{12}\rangle_R Z_{36}] = 12Z$.

6. Every STPNS-2-Absorbing submodule is Nearly semi-2-Absorbing but, the converse is not true.

Proof

Clear.

For the converse consider the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{24}\rangle$ is Nearly semi-2-Absorbing submodule of M since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ that is for all $u \in Z$ and $m \in Z_{48}$ such that $u^2m \in \langle\overline{24}\rangle$, implies that either $um \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or $u^2 \in [\langle\overline{24}\rangle + J(Z_{48})_R : Z_{48}] = 4Z$. That is $2^2.\bar{6} \in \langle\overline{24}\rangle$, implies that $2.\bar{6} = \overline{12} \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ and $2^2 \in [\langle\overline{24}\rangle + J(Z_{48})_R : Z_{48}] = 4Z$. But $\mathcal{F} = \langle\overline{24}\rangle$ is not STPNS-2-Absorbing submodule of $\mathcal{H}$, since $2^2.\bar{6} \in \langle\overline{24}\rangle$, but $2.\bar{6} \notin \langle 24\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right) = \langle\overline{24}\rangle$ and $2^2 \notin \left[\langle\overline{24}\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right)_R : Z_{48}\right] = 24Z$.

Proposition 3.3

Let $\mathcal{F}$ be a proper submodule of an R-module $\mathcal{H}$. Then $\mathcal{F}$ is a STPNS-2-Absorbing submodule of $\mathcal{H}$ if and only if $I^2\mathcal{L} \subseteq \mathcal{F}$ for I is an ideal of R and $\mathcal{L}$ is a submodule of $\mathcal{H}$ implies that either $I\mathcal{L} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}]$.

Proof

($\Longrightarrow$) Suppose that $\mathcal{F}$ is STPNS-2-Absorbing submodule of an R-module $\mathcal{H}$, and $I^2\mathcal{L} \subseteq \mathcal{F}$, for I is an ideal of R and $\mathcal{L}$ is a submodule of $\mathcal{H}$ and $I^2 \nsubseteq [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$.To prove that $I\mathcal{L} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$. Let $\varkappa \in I\mathcal{L}$, implies that $\varkappa = u_1\varkappa_1 + u_2\varkappa_2 + \cdots + u_n\varkappa_n$ for $u_i \in I$, $\varkappa_i \in L$ for all $i = 1,2,3,\dots,n$, thus ${u_i}^2\varkappa_i \in I^2\mathcal{L} \subseteq \mathcal{F}$, for ${u_i}^2 \notin [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$, but $\mathcal{F}$ is STPN-2-Absorbing of $\mathcal{H}$ and ${u_i}^2 \notin [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$, implies that $u_i\varkappa_i \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ for $i = 1,2,3,\dots,n$. Hence $\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ it follows that $I\mathcal{L} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$.

($\Longleftarrow$) Suppose that $u^2\varkappa \in \mathcal{F}$for $u \in R$, $\varkappa \in \mathcal{H}$, implies that $(u^2)(\varkappa) \subseteq \mathcal{F}$, by hypothesis we have either $(u)(\varkappa) \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $(u^2) \subseteq [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$. That is $u\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$. Hence $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$.

As a direct consequence of Proposition 3.3 we have the following corollaries.

Corollary 3.4

Let $\mathcal{F}$ be a proper submodule of an R-module $\mathcal{H}$. Then $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ if and only if $u^2\mathcal{L} \subseteq \mathcal{F}$ for $u \in R$ and $\mathcal{L}$ is a submodule of $\mathcal{H}$ implies that either $u\mathcal{L} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) :_R \mathcal{H}]$.

Corollary 3.5

Let $\mathcal{F}$ be a proper submodule of an R-module $\mathcal{H}$. Then $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$if and only if $\mathrm{I}^2\mathcal{H} \subseteq \mathcal{F}$ for I is an ideal of R, implies that either $\mathrm{I}\mathcal{H} \subseteq \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$ or $I^2 \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$.

Proposition 3.6

Let $\mathcal{H}$ be an R-module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa] \subseteq [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R r\varkappa]$ for each $\varkappa \in \mathcal{H}$ or $u^2 \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$.

Proof

($\Longrightarrow$) let $\omega \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa]$ and $u^2 \notin [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$, since $\omega \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa]$, then $u^2\omega\varkappa \in \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$. But $\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ and $u^2 \notin [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$, then $u\omega\varkappa \in \left(\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})\right) + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) = \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$.That is $\omega \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa]$ .Thus $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa] \subseteq [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa]$. It is clear that $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa] \subseteq [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa]$. Hence$[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa] = [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa]$.

($\Longleftarrow$) Let $u^2\varkappa \in \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$, for $u \in R, \varkappa \in \mathcal{H}$, then by hypothesis $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa] = [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa]$ or $u^2 \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$. If $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa] = [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa]$ and $u^2\varkappa \in \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$ then $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u^2\varkappa] = R$, it follows that $[\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R u\varkappa] = R$, hence $u\varkappa \in \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) \subseteq \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$, so $u\varkappa \in \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$ or $u^2\mathcal{H} \subseteq \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$. That is $\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$ is STPN-2-Absorbing submodule of $\mathcal{H}$.

Proposition 3.7

Let $\mathcal{H}$ be an R-module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$ with $\mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) \subseteq \mathcal{F}$. Then $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F}:_{\mathcal{H}} u^2] \subseteq [\mathcal{F}:_{\mathcal{H}} u]$ or $u^2 \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$.

Proof

($\Longrightarrow$) Suppose that $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$, let $\varkappa \in [\mathcal{F}:_R u^2]$ and $u^2 \notin [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$. Since $\varkappa \in [\mathcal{F}:_R u^2]$, implies that $u^2\varkappa \in \mathcal{F}$, but $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ and $u^2 \notin [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$, implies that $u\varkappa \in \mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H})$. Since $\mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) \subseteq \mathcal{F}$, then $\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}) = \mathcal{F}$, implies that $u\varkappa \in \mathcal{F}$, hence $\varkappa \in [\mathcal{F}:_{\mathcal{H}} u]$, thus $[\mathcal{F}:_{\mathcal{H}} u^2] \subseteq [\mathcal{F}:_{\mathcal{H}} u]$. Clear $[\mathcal{F}:_{\mathcal{H}} u] \subseteq [\mathcal{F}:_{\mathcal{H}} u^2]$, hence $[\mathcal{F}:_{\mathcal{H}} u^2] = [\mathcal{F}:_{\mathcal{H}} u]$.

($\Longleftarrow$) Let $u^2\varkappa \in \mathcal{F}$for $u \in R$, $\varkappa \in \mathcal{H}$, then $\varkappa \in [\mathcal{F}:_{\mathcal{H}} u^2] = [\mathcal{F}:_{\mathcal{H}} u]$, by hypothesis, implies that $\varkappa \in [\mathcal{F}:_{\mathcal{H}} u]$, then $u\varkappa \in \mathcal{F}$ or $u^2 \in [\mathcal{F} + \mathrm{J}(\mathcal{H}) \cap \mathrm{soc}(\mathcal{H}):_R \mathcal{H}]$. Hence $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$.

Remark 3.8

The intersection of two STPNS-2-Absorbing submodule of an R-module $\mathcal{H}$ need not to be STPNS-2-Absorbing submodule of $\mathcal{H}$. The following example explains that:

Consider the Z_module Z_{48} and the submodules $\mathcal{L}$=$\langle\bar{3}\rangle$ and $\mathbb{Q}$=$\langle\bar{4}\rangle$ are STPNS_2_Absorbing submodules of the Z_module Z_{48} (because $\langle\bar{3}\rangle$ and $\langle\bar{4}\rangle$ are 2-Absorbing of Z_{48}), but $\mathcal{L} \cap \mathbb{Q} = \langle\overline{12}\rangle$ is not STPNS_2_Absorbing, since $2^2.\bar{3} \in \langle\overline{12}\rangle$, but $2.\bar{3} \notin \langle\overline{12}\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right) = \langle\overline{12}\rangle$ and $2^2 \notin \left[\langle\overline{12}\rangle + \left(J(Z_{48}) \cap soc(Z_{48})\right)_R: Z_{48}\right] = 12Z$.

The inverse of above remark satisfy under certain condition.

Proposition 3.9

Let $\mathcal{L}$ and $\mathcal{F}$ be a proper submodules of an R-module $\mathcal{H}$, with $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{L}$ or $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{F}$. If $\mathcal{L}$ and $\mathcal{F}$ are STPNS-2-Absorbing submodules of $\mathcal{H}$, then $\mathcal{L} \cap \mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$.

Proof

Assume that $u^2\varkappa \in \mathcal{L} \cap \mathcal{F}$ for $u \in R, \varkappa \in \mathcal{H}$, implies that $u^2\varkappa \in \mathcal{L}$ and $u^2\varkappa \in \mathcal{F}$. Since both $\mathcal{L}$ and $\mathcal{F}$ are STPNS-2-Absorbing submodule of $\mathcal{H}$, then either $u\varkappa \in \mathcal{L} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2\mathcal{H} \subseteq \mathcal{L} + J(\mathcal{H}) \cap soc(\mathcal{H})$ and $\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2\mathcal{H} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$Thus either $u\varkappa \in (\mathcal{L} + J(\mathcal{H}) \cap soc(\mathcal{H})) \cap (\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq (\mathcal{L} + J(\mathcal{H}) \cap soc(\mathcal{H})) \cap (\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}))$. If $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{F}$, then $\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) = \mathcal{F}$, hence either $u\varkappa \in (\mathcal{L} + J(\mathcal{H}) \cap soc(\mathcal{H})) \cap \mathcal{F}$ or $u^2\mathcal{H} \subseteq (\mathcal{L} + J(\mathcal{H}) \cap soc(\mathcal{H})) \cap \mathcal{F}$. Thus by lemma 2.9we have either $u\varkappa \in (\mathcal{L} \cap \mathcal{F}) + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2\mathcal{H} \subseteq (\mathcal{L} \cap \mathcal{F}) + J(\mathcal{H}) \cap soc(\mathcal{H})$. Therefore $\mathcal{L} \cap \mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$. In similar way $\mathcal{L} \cap \mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$ if $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{L}$.

Proposition 3.10

Let $\mathcal{F}$ be STPNS_2_Absorbing submodule of an R_ module $\mathcal{H}$ and $\mathcal{L}$ is a submodule of $\mathcal{H}$ with $\mathcal{L} \subseteq \mathcal{F}$, then $\frac{\mathcal{F}}{\mathcal{L}}$ is STPNS_2_Absorbing submodule of an R_ module $\frac{\mathcal{H}}{\mathcal{L}}$.

Proof

Let $\mathcal{F}$ be STPNS_2_Absorbing submodule of $\mathcal{H}$, and $u^2(e + \mathcal{L}) = u^2e + \mathcal{L} \in \frac{\mathcal{F}}{\mathcal{L}}$ for $u \in R$ and $e + \mathcal{L} \in \frac{\mathcal{H}}{\mathcal{L}}$, $e \in \mathcal{H}$, implies that $u^2e \in \mathcal{F}$. Since $\mathcal{F}$ is STPNS_2_Absorbing submodule of $\mathcal{H}$, then either $ue \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2 \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))_R : \mathcal{H}]$. That is either $ue \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. It follows that either $u(e + \mathcal{L}) \in \frac{\mathcal{F}+(J(\mathcal{H})\cap soc(\mathcal{H}))}{\mathcal{L}}$ or $u^2(\frac{\mathcal{H}}{\mathcal{L}}) \subseteq \frac{\mathcal{F}+(J(\mathcal{H})\cap soc(\mathcal{H}))}{\mathcal{L}}$. That is either $u(e + \mathcal{L}) \in \frac{\mathcal{F}}{\mathcal{L}} + \frac{\mathcal{F}+(J(\mathcal{H})\cap soc(\mathcal{H}))}{\mathcal{L}} \subseteq \frac{\mathcal{F}}{\mathcal{L}} + (J\left(\frac{\mathcal{H}}{\mathcal{L}}\right) \cap soc\left(\frac{\mathcal{H}}{\mathcal{L}}\right)$ or $u^2\left(\frac{\mathcal{H}}{\mathcal{L}}\right) \subseteq \frac{\mathcal{F}+(J(\mathcal{H})\cap soc(\mathcal{H}))}{\mathcal{L}} \subseteq \frac{\mathcal{F}}{\mathcal{L}} + (J\left(\frac{\mathcal{H}}{\mathcal{L}}\right) \cap soc\left(\frac{\mathcal{H}}{\mathcal{L}}\right)$. That is either $u(e + \mathcal{L}) \subseteq \frac{\mathcal{F}}{\mathcal{L}} + (J\left(\frac{\mathcal{H}}{\mathcal{L}}\right) \cap soc\left(\frac{\mathcal{H}}{\mathcal{L}}\right)$ or $u^2 \in \left[\frac{\mathcal{F}}{\mathcal{L}} + (J\left(\frac{\mathcal{H}}{\mathcal{L}}\right) \cap soc\left(\frac{\mathcal{H}}{\mathcal{L}}\right))_R : \frac{\mathcal{H}}{\mathcal{L}}\right]$. Hence $\frac{\mathcal{F}}{\mathcal{L}}$ is STPNS_ 2_ Absorbing submodule of an R_ module $\frac{\mathcal{H}}{\mathcal{L}}$.

Proposition 3.11

Let $\mathcal{H}$ be an R_module , and $\mathcal{F}$ is a proper submodule of $\mathcal{H}$ with $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{F}$. Then $\mathcal{F}$ is STPNS_ 2_ Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F}_{\mathcal{H}} : \mathrm{I}]$ is STPNS_ 2_ Absorbing submodule of $\mathcal{H}$ for each ideal I of R .

Proof

($\Rightarrow$) Let $u^2\mathcal{L} \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}]$ for $\mathrm{u} \in \mathrm{R}$, $\mathcal{L}$ is a submodule of $\mathcal{H}$, then $u^2(\mathrm{I}\mathcal{L}) \subseteq \mathcal{F}$. Since $\mathcal{F}$ is STPNS_ 2_ Absorbing submodule of $\mathcal{H}$ then by proposition (3.3) either $u(\mathrm{I}\mathcal{L}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2 \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))_R : \mathcal{H}]$. But $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{F}$, implies that $\mathcal{F} + \left(J(\mathcal{H}) \cap soc(\mathcal{H})\right) = \mathcal{F}$ .Thus either $u(\mathrm{I}\mathcal{L}) \subseteq \mathcal{F}$ or $u^2 \in [\mathcal{F}_R : \mathcal{H}]$.That is either $u\mathcal{L} \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}]$ or $u^2\mathcal{H} \subseteq \mathcal{F} \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}]$. It follows that either $u\mathcal{L} \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}] \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}] + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2\mathcal{H} \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}] \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}] + J(\mathcal{H}) \cap soc(\mathcal{H})$. That is either $u\mathcal{L} \subseteq [\mathcal{F}_{\mathcal{H}} : \mathrm{I}] + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [[\mathcal{F}_{\mathcal{H}} : \mathrm{I}] + J(\mathcal{H}) \cap soc(\mathcal{H})_R : \mathcal{H}]$. Hence $[\mathcal{F}_{\mathcal{H}} : \mathrm{I}]$ is STPNS_ 2_ Absorbing submodule of $\mathcal{H}$.

($\Leftarrow$) Suppose that $[\mathcal{F}_{\mathcal{H}} : \mathrm{I}]$ is STPNS_ 2_ Absorbing submodule of $\mathcal{H}$ for every non_zero ideal I of R . Put $\mathrm{I} = \mathrm{R}$ we get $[\mathcal{F}_{\mathcal{H}} : \mathrm{R}] = \mathcal{F}$ is STPNS_ 2_ Absorbing submodule of $\mathcal{H}$.

4. The Relations of STPNS-2-Absorbing Submodules with 2-Absorbing Submodules And Other Form of Submodules

In this section we introduce the relations Of STPNS-2-Absorbing submodules with 2-Absorbing submodules and other form of submodules.

The converse of Remarks and Examples 3.2 (2) is true under certain conditions where given the following propositions.

Proposition 4.1
Let $\mathcal{H}$ be an R_module over a Boolean ring R with either $J(\mathcal{H}) = 0$ or $soc(\mathcal{H}) = 0$, and A is a proper submodule of $\mathcal{H}$. Then A is STPNS_2_Absorbing of $\mathcal{H}$ if and only if A is 2_Absorbing submodule of $\mathcal{H}$.

Proof

(⇒) Let $\alpha by \in A$ for $\alpha, b \in$ R, $y \in \mathcal{H}$. Since R is Boolean ring, then $(\alpha b)^2 y \in A$ with $(\alpha b)^2 = (ab) \notin [A + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$ and $by \notin A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\alpha y \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Since $J(\mathcal{H}) = 0$, then $J(\mathcal{H}) \cap soc(\mathcal{H}) = 0$ so $A + (J(\mathcal{H}) \cap soc(\mathcal{H})) = A$ if $soc(\mathcal{H}) = 0$, then $(\mathcal{H}) \cap soc(\mathcal{H}) = 0$ so $A + (J(\mathcal{H}) \cap soc(\mathcal{H})) = A$. In both case we get $(\alpha b)^2 = (ab) \notin [A:_R \mathcal{H}]$ and $by \notin A$, to prove that $\alpha y \in A$. Since A is a STPNS_ 2_Absorbing submodule of $\mathcal{H}$ and $(\alpha b)^2 \notin [A + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$, implies that $(\alpha b)y \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ that is $\alpha(by) \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. So R is Boolean ring then $\alpha(by) = \alpha^2(by) \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Now by Lemma 2.13, we have $a = \alpha^2 b$. Thus $\alpha^2(by) = ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, so $A + J(\mathcal{H}) \cap soc(\mathcal{H}) = A$ it follows that $ay \in A$. Hence A is 2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct.

Proposition 4.2
Let $\mathcal{H}$ be an R_module over a Boolean ring R, A is a proper submodule of $\mathcal{H}$. Then A is STPNS_2_Absorbing of $\mathcal{H}$ if and only if A is 2_Absorbing submodule of $\mathcal{H}$.

Proof

(⇒) Since R is a Boolean ring, then by Lemma 2.13 and 2.15 $J(\mathcal{H}) = 0$, so $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (0) \cap soc(\mathcal{H}) = (0)$. Thus the proof follows as in proposition 4.1.

(⇐) Direct.

The converse of Remarks and Examples 3.2 (3) is true under certain conditions where given the following propositions.

Proposition 4.3
Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $soc(\mathcal{H}) = \mathcal{H}$. Then A is Nearly_2_Absorbing if and only if A STPNS_2_Absorbing submodule of $\mathcal{H}$.

Proof

(⇒) Let $\alpha^2 y \in A$ for $\alpha \in$ R, $y \in \mathcal{H}$, that is $\alpha.a.y \in A$ with $a.a \notin [A + J(\mathcal{H}):_R \mathcal{H}]$. Since A is Nearly-2-Absorbing and $a.a \notin [A + J(\mathcal{H}):_R \mathcal{H}]$, then $\alpha y \in A + J(\mathcal{H})$. But $J(\mathcal{H}) \subseteq \mathcal{H}$, so $J(\mathcal{H}) \cap \mathcal{H} = J(\mathcal{H})$, that is $ay \in A + J(\mathcal{H}) \cap \mathcal{H}$. Since $soc(\mathcal{H}) = \mathcal{H}$ it follows that $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct.

The following corollary is a direct result of Proposition 4.3.

Corollary 4.4
Let $\mathcal{H}$ be a semi-simple, and $A \subset \mathcal{H}$. Then A is Nearly_2_Absorbing if and only if A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

Proposition 4.5
Let $\mathcal{H}$ be an R_module over a Boolean ring R, and $A \subset \mathcal{H}$. Then the following are Valente:

1. A is 2_Absorbing submodule of $\mathcal{H}$.
2. A is STPNS_2_Absorbing submodule of $\mathcal{H}$.
3. A is Nearly_2_Absorbing submodule of $\mathcal{H}$.

Proof

(1) $\Leftrightarrow$ (2) See Proposition 4.2.

(2) $\Leftrightarrow$ (3) See Corollary 4.4.

The converse of Remarks and Examples 3.2 (4) is true under certain conditions where given the following propositions.

Proposition 4.6
Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $J(\mathcal{H}) = \mathcal{H}$. Then A is Pseudo_2_Absorbing if and only if A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

Proof

($\Rightarrow$) Let $\text{a}^2 y \in A$ for $\text{a} \in$ R, $y \in \mathcal{H}$, that is $\text{a}.a.y \in A$ with $a.a \notin [A + J(\mathcal{H}):_R \mathcal{H}]$. Since A is Pseudo-2-Absorbing and $a.a \notin [A + soc(\mathcal{H}):_R \mathcal{H}]$, then $\text{a}y \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq \mathcal{H}$, so $\mathcal{H} \cap soc(\mathcal{H}) = soc(\mathcal{H})$, that is either $ay \in A + \mathcal{H} \cap soc(\mathcal{H})$. But $J(\mathcal{H}) = \mathcal{H}$, it follows that $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

($\Leftarrow$) Direct.

Proposition 4.7
Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $soc(\mathcal{H}) \subseteq A$ and $J(\mathcal{H}) \subseteq A$. Then the following are Valente:

1 A is 2_Absorbing submodule of $\mathcal{H}$.

2. A is Pseudo_2_Absorbing submodule of $\mathcal{H}$.

3. A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

4. A is Nearly_2_Absorbing submodule of $\mathcal{H}$.

Proof

(1) $\Rightarrow$ (2) Direct by Lemma 2.19.

(2) $\Rightarrow$ (3) Let $\text{a}^2 y \in A$ for $\text{a} \in$ R, $y \in \mathcal{H}$, that is $\text{a}.a.y \in A$ with $a.a \notin [A + soc(\mathcal{H}):_R \mathcal{H}]$. Since A is Pseudo-2-Absorbing and $a.a \notin [A + soc(\mathcal{H}):_R \mathcal{H}]$, then $\text{a}y \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq A$ then $A + soc(\mathcal{H}) = A$. Thus either $ay \in A \subseteq A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

(3) $\Rightarrow$ (4) Direct by Remarks and Examples 3.2 (3).

(4) $\Rightarrow$ (1) Direct by Lemma 2.19.

The converse of Remarks and Examples 3.2 (5) is true under certain conditions where given the following propositions.

Proposition 4.8

Let $\mathcal{H}$ be a semi-simple R_module, A is a proper submodule of $\mathcal{H}$. Then A is STPNS_2_Absorbing of $\mathcal{H}$ if and only if A is Semi 2_Absorbing submodule of $\mathcal{H}$.

Proof

(⇒) Let $\alpha^2 y \in A$ for $\alpha \in R$, $y \in \mathcal{H}$. Since A is STPNS_2_ Absorbing submodule of $\mathcal{H}$, then $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\alpha^2 \in [A + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Since $\mathcal{H}$ is semi-simple, then by Lemma 2.17 $J(\mathcal{H}) = 0$, so $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (0) \cap soc(\mathcal{H}) = (0)$. Thus either $ay \in A$ or $\alpha^2 \in [A :_R \mathcal{H}]$. Hence A is a Semi 2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct.

The converse of Remarks and Examples 3.2 (6) is true under certain conditions where given the following propositions.

Proposition 4.9

Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $soc(\mathcal{H}) = \mathcal{H}$. Then A is Nearly Semi _2_Absorbing if and only if A STPNS_2_Absorbing submodule of $\mathcal{H}$.

Proof

(⇒) Let $\alpha^2 y \in A$ for $\alpha \in R$, $y \in \mathcal{H}$. Since A is Nearly Semi _2_ Absorbing submodule of $\mathcal{H}$, then $ay \in A + J(\mathcal{H})$ or $\alpha^2 \in [A + J(\mathcal{H}):_R \mathcal{H}]$. But $J(\mathcal{H}) \subseteq \mathcal{H}$, so $J(\mathcal{H}) \cap \mathcal{H} = J(\mathcal{H})$, that is either $ay \in A + J(\mathcal{H}) \cap \mathcal{H}$ or $\alpha^2 \in [A + J(\mathcal{H}) \cap \mathcal{H} :_R \mathcal{H}]$. Since $soc(\mathcal{H}) = \mathcal{H}$ it follows that either $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\alpha^2 \in [A + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Thus A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct.

Proposition 4.10

Let $\mathcal{H}$ be an R_module over a Boolean ring R , and A is a proper submodule of $\mathcal{H}$ with $J(\mathcal{H}) \subseteq A$. Then the following are statement:

1 A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

2. A is Nearly Semi _2_Absorbing submodule of $\mathcal{H}$.

3. A is Semi 2_Absorbing submodule of $\mathcal{H}$.

4. A is 2_Absorbing submodule of $\mathcal{H}$.

Proof

(1) ⇒ (2) Direct by Remarks and Examples 3.2 (6).

(2) ⇒ (3) Direct by Lemma 2.20.

(3) ⇒ (4) Let $\alpha by \in A$ for $\alpha, b \in R$, $y \in \mathcal{H}$. Since R is Boolean ring, then $(\alpha b)^2 y \in A$ with $(\alpha b)^2 = (ab) \notin [A:_R \mathcal{H}]$ and $by \notin A$, to prove that $\alpha y \in A$. Since A is a Semi_ 2_Absorbing submodule of $\mathcal{H}$ and $(\alpha b)^2 \notin [A:_R \mathcal{H}]$, implies that $(\alpha b)y \in A$ that is $\alpha(by) \in A$.So R is Boolean ring then $\alpha(by) = \alpha^2(by) \in A$. Now by Lemma 2.13 we have $a = \alpha^2 b$. Thus $\alpha^2(by) = ay \in A$. Hence A is 2_Absorbing submodule of $\mathcal{H}$.

(4) ⇒ (1) Direct by Remarks and Examples 3.2 (2).

The following proposition we establish the relation between STPNS_2_Absorbing submodules and Pseudo semi 2_Absorbing submodules.

Proposition 4.11

Let $\mathcal{H}$ be an R-module and $A \subset \mathcal{H}$. If A is STPNS_2_Absorbing submodule of an R-module $\mathcal{H}$ then A is Pseudo semi 2_Absorbing submodule.

Proof

Suppose that $\mathcal{F}$ is STPNS-2-Absorbing submodule of $\mathcal{H}$, and let $u^2m \in \mathcal{F}$, for $u \in R$, $m \in \mathcal{H}$. Since $\mathcal{F}$ is STPNS-2-Absorbing submodule, then either $um \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + soc(\mathcal{H})$ or $u^2\mathcal{H} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + soc(\mathcal{H})$. Hence $\mathcal{F}$ is Pseudo semi-2-Absorbing submodule of $\mathcal{H}$.

The converse of Proposition 4.11 is true under certain conditions where given the following propositions.

Proposition 4.12

Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $J(\mathcal{H}) = \mathcal{H}$. Then A is Pseudo Semi _2_Absorbing if and only if A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

Proof

(⇒) Let $\alpha^2 y \in A$ for $\alpha \in R$, $y \in \mathcal{H}$. Since A is Pseudo Semi _2_ Absorbing submodule of $\mathcal{H}$, then $ay \in A + soc(\mathcal{H})$ or $\alpha^2 \in [A + soc(\mathcal{H}):_R \mathcal{H}]$. But $soc(\mathcal{H}) \subseteq \mathcal{H}$ then $soc(\mathcal{H}) \cap \mathcal{H} = soc(\mathcal{H})$ that is either $ay \in A + soc(\mathcal{H}) \cap \mathcal{H}$ or $\alpha^2 \in [A + soc(\mathcal{H}) \cap \mathcal{H}:_R \mathcal{H}]$. Since $J(\mathcal{H}) = \mathcal{H}$, it follows that either $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\alpha^2 \in [A + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Thus A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct by Proposition 4.11.

Proposition 4.13

Let $\mathcal{H}$ be an R_module over a Boolean ring R, and A a proper submodule of $\mathcal{H}$ with $soc(\mathcal{H}) \subseteq J(\mathcal{H})$ and $J(\mathcal{H}) \subseteq A$. Then the following are statement:

1. A is 2_Absorbing submodule of $\mathcal{H}$.
2. A is Pseudo_2_Absorbing submodule of $\mathcal{H}$.
3. A is Pseudo Semi _2_Absorbing submodule of $\mathcal{H}$.
4. A is STPNS_2_Absorbing submodule of $\mathcal{H}$.
5. A is Nearly_2_Absorbing submodule of $\mathcal{H}$.
6. A is Nearly Semi _2_Absorbing submodule of $\mathcal{H}$.
7. A is Semi 2_Absorbing submodule of $\mathcal{H}$.

Proof

(1) ⇒ (2) Direct by Lemma 2.18.

(2) ⇒ (3) Let $\alpha^2 y \in A$ for $\alpha \in R$, $y \in \mathcal{H}$, that is $\alpha.a.y \in A$ with $a.a \notin [A + soc(\mathcal{H}):_R \mathcal{H}]$. Since A is Pseudo-2-Absorbing and $a.a \notin [A + soc(\mathcal{H}):_R \mathcal{H}]$, then $\alpha y \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq J(\mathcal{H})$ then $(J(\mathcal{H}) \cap soc(\mathcal{H})) = soc(\mathcal{H})$. Thus we have either $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is A is STPNS-2-Absorbing submodule of $\mathcal{H}$.

(3) ⇒ (4) Let $\alpha^2 y \in A$ for $\alpha \in R$, $y \in \mathcal{H}$. Since A is Pseudo Semi_2_Absorbing submodule of $\mathcal{H}$, then $ay \in A + soc(\mathcal{H})$ or $\alpha^2 \in [A + soc(\mathcal{H}):_R \mathcal{H}]$. But $soc(\mathcal{H}) \subseteq J(\mathcal{H})$, then $soc(\mathcal{H}) \cap J(\mathcal{H}) = soc(\mathcal{H})$, so either $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\alpha^2 \in [A + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. That is A is STPNS_2_Absorbing submodule of $\mathcal{H}$.

(4) ⇒ (5) Direct by Remarks and Examples 3.2 (3).

(5) ⇒ (6) Let $\alpha^2 y \in A$ for $\alpha \in R$, $y \in \mathcal{H}$, that is $\alpha.a.y \in A$. Since A is Nearly-2-Absorbing, then either $a.a \in [A + J(\mathcal{H}):_R \mathcal{H}]$ or $\alpha y \in A + J(\mathcal{H})$. Hence A is Nearly Semi-2-Absorbing submodule of $\mathcal{H}$.

(6) ⇒ (7) Direct by Lemma 2.20

$(7) \Rightarrow (1)$ Let $\alpha by \in A$ for $\alpha, b \in R$, $y \in \mathcal{H}$. Since R is Boolean ring, then $(\alpha b)^2 y \in A$ with $(\alpha b)^2 = (ab) \notin [A:_R \mathcal{H}]$ and $by \notin A$, to prove that $\alpha y \in A$. Since A is a Semi-2-Absorbing submodule of $\mathcal{H}$ and $(\alpha b)^2 \notin [A:_R \mathcal{H}]$, implies that $(\alpha b)y \in A$ that is $\alpha(by) \in A$.So R is Boolean ring then $\alpha(by) = \alpha^2(by) \in A$. Now by Lemma 2.13 we have $a = \alpha^2 b$. Thus $\alpha^2(by) = ay \in A$. Hence A is 2-Absorbing submodule of $\mathcal{H}$.

5. Conclusion

We will present the most important propositions in this research:

. Let $\mathcal{H}$ be an R-module over a Boolean ring R, and $A \subset \mathcal{H}$. Then the following are statement:

1. A is 2-Absorbing submodule of $\mathcal{H}$.
2. A is STPNS-2-Absorbing submodule of $\mathcal{H}$.
3. A is Nearly-2-Absorbing submodule of $\mathcal{H}$.

. Let $\mathcal{H}$ be an R-module, and $A \subset \mathcal{H}$ with $soc(\mathcal{H}) \subseteq A$ and $J(\mathcal{H}) \subseteq A$. Then the following are statement:

1 A is 2-Absorbing submodule of $\mathcal{H}$.

2. A is Pseudo-2-Absorbing submodule of $\mathcal{H}$.
3. A is STPNS-2-Absorbing submodule of $\mathcal{H}$.
4. A is Nearly-2-Absorbing submodule of $\mathcal{H}$.

. Let $\mathcal{H}$ be an R-module over a Boolean ring R , and A is a proper submodule of $\mathcal{H}$ with $J(\mathcal{H}) \subseteq A$. Then the following are statement:

1 A is STPNS-2-Absorbing submodule of $\mathcal{H}$.

2. A is Nearly Semi -2-Absorbing submodule of $\mathcal{H}$.
3. A is Semi 2_Absorbing submodule of $\mathcal{H}$.
4. A is 2-Absorbing submodule of $\mathcal{H}$.

. Let $\mathcal{H}$ be an R-module over a Boolean ring R, and A a proper submodule of $\mathcal{H}$ with $soc(\mathcal{H}) \subseteq J(\mathcal{H})$ and $J(\mathcal{H}) \subseteq A$. Then the following are Valente:

1. A is 2-Absorbing submodule of $\mathcal{H}$.
2. A is Pseudo-2-Absorbing submodule of $\mathcal{H}$.
3. A is Pseudo Semi -2-Absorbing submodule of $\mathcal{H}$.
4. A is STPNS-2-Absorbing submodule of $\mathcal{H}$.
5. A is Nearly-2-Absorbing submodule of $\mathcal{H}$.
6. A is Nearly Semi -2-Absorbing submodule of $\mathcal{H}$.
7. A is Semi 2-Absorbing submodule of $\mathcal{H}$.

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