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# Convergence To Approximate Solutions of Multivalued Operators

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**Abstract**
The goal of this article is to study a new explicit iterative processes method approach to zeroes for solving maximal monotone(M.M ) multivalued operators in Hilbert spaces, utilizing a finite family of different types of mappings as nonexpansive map,contraction map, resolvent operator and nearest point metric projection map. On the other hand, The results in this paper are develop and extend the main and important findings of previous studies. Then, utilizing various structural conditions in Hilbert space and variational inequality problems, also,we examine the strong convergence to nearest projection map for these explicit iterative processes methods under the presence of two important conditions for convergence, namely closure and convexity. The findings reported in this research strengthen, improve and extend key previous findings from the literature.

**Keywords**: Projection Mapping, Iterative Method, Nonexpamsive Mapping, Monotone Operators ,s-convergence, Fixed Point.

## 1.Introduction

Assume$H$ is a true Hilbert space and M is the biggest monotone mapping. Many scholars have explored the challenge of finding zeroes, including [1] and [2]. The Proximal method Algorithm is a popular approach for solving $0 \in Mx$. Rockafellar demonstrated w-convergence of the peri-point technique in 1976, but it did not s-converge [3]. The human-like iterative approach is properly arranged in the f- point process to solve numerous nonlinear issues [4]. However, Mann-likealgorithm processes in Hilbert space are only w- smoothed. [5] presented a viscosity,for resolving f- point of a nonlinear maps . The contour of Hilbert space identifies common solution s- convergence theorems. The f- point theory solutions have been demonstrated to be a successful and influential way for addressing a wide range of real-world problems that can be broken down into identical f- point problems.In order to get a rough resolution of f-point difficulties, many algorithm procedures must be devised [6- 18]. The purpose of this paper is to expand and improve the proximal methods of multivalued operators. One of the most important examples of f- point theory is the challenge of solving zero of (V.I) problem as

$$\begin{cases} a_{2n+1} = a_n b + \delta_n a_{2n} + \gamma_n Res^N_{\beta_n}(a_{2n}) + e_n & n \geq 0 \\ a_{2n} = \lambda_n b + \rho_n a_{2n-1} + \delta_n Res^M_{\mu_n}(a_{2n-1}) + e'_n & n \geq 0 \end{cases}$$

For given u , $a_0 \in H$, where ($e_n$) and ($e'_n$) are sequences of calculation mistakes , N and M are M.M operators and $a_n, \delta_n, \gamma_n$ are seq lies in (0,1) and $\beta_n, \mu_n \in (0,\infty)$ . Here $Res^N_\beta = (I + \beta N)^{-1}, \beta > 0$ ( the Resovent operator of N) .

Now, we will review some concepts and lemmas known . Let $\emptyset \neq$C be a closed,convex in H and $Pro_C$ denote the nearest mapping from H onto C,A mapping $K$is called nonexpansive if $\|K_u - K_v\| \leq \|u - v\|, \forall u, v \in H$ and $\alpha$ −inverse strongly monotone if there exist $\alpha > 0$ such that

$\langle S_u - S_v, u - v\rangle \geq \alpha\|K_u - K_v\|^2, \forall u, v \in$H .

**Lemma 1 . [16]** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $R^+$ satisfying the following relation :

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n , n \geq n_o$$

Where$\{\alpha_n\}_{n=1}^{\infty} \sqsubset (0,1)$ and$\{\delta_n\}_{n=1}^{\infty} \subset R$ satisfy the following condition : $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty} sup\delta_n \leq 0$ or $\sum_{n=1}^{\infty}\alpha_n\,\delta_n = \infty$ . Then $\lim_{n\to\infty}\alpha_n = 0$ .

**lemma 2 [18]** Let $a \in H$ be given . The nearest mapping is characterized by:

(i) $Pro_C a \in C, \langle a - Pro_C a, Pro_C a - c\rangle \geq 0$ , for all $c \in C$ .

(ii) $Pro_C$ is firmly nonexpansive.

**Lemma 3 [19]** Let$\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be bounded sequence in H and let$\{\beta_n\}_{n=1}^{\infty}$ be a sequence in (0,1) with

$0 \leq \lim_{n\to\infty} inf\beta_n \leq \lim_{n\to\infty} sup\beta_n \leq 1.$

Suppose $a_{n+1} = (1 - \beta_n)b_n + \beta_n a_n$ for all integers $n \geq 0$ and $\lim_{n\to\infty}$ sup $(\|b_{n+1} - b_n\| - \|a_{n+1} - a_n\|) \leq 0$ .Then $\lim_{n\to\infty}\|b_n - a_n\| = 0$ .

## 2 .Main Results

In this part, we introduced a new methods and we study the s- convergence,

**Theorem : 2.1** Let $\mathcal{M}_i: H \longrightarrow 2^H$ be a M.M. operators such that $F(Pro_s) \bigcap_{i=1}^{l} \mathcal{M}_i^{-1}0 \neq \emptyset$.If the sequences $\{\xi_n^i\}_{n=2}^{\infty}$, $\{R_{n_i}\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$ $and\{Q_n\}_{n=1}^{\infty}$are seqs in (0,1)satisfy the following conditions :

(i) $\sum_{n=1}^{\infty}|\xi_{n+1}^i - \xi_n^i| < \infty\, for\ all\ i = 1,2, \dots l$ .

(ii) $\sum_{n=1}^{\infty}|R_{n+1}^i - R_n^i| < \infty$ for all $i = 1,2, \dots l$ .

(iii) $\sum_{n=1}^{\infty}|Q_{n+1} - Q_n| < \infty$ .

$\|S_{(n+1)i}f(x_{n+1}) - S_{ni}f(x_n)\| \leq S_{ni}\|f(x_{n+1}) - f(x_n)\|$and

$\|(1 - S_{(n+1)i})\, g(x_{n+1}) - (1 - S_{ni})\, g(x_n)\| \leq (1 - S_{ni})\, \|\, g(x_{n+1}) - g(x_n)\|$.Then the sequences $\{x_n\}$ defined by $x \in C$ and

$y_n^{i-1} = S_{ni}f(x_n) + (1 - S_{ni})\, g(x_n)$

$y_n^m = R_{n_1}y_n^0 + \sum_{i=2}^{l} R_{n_i}\, Res_{i,n}y_n^{i-1}$ where $f, g: H \to S$ is nonexpansive

$x_{n+1} = Q_n Pro_\Psi x + (1 - Q_n)y_n^m,\ n, m \geq 0$

Is s-converges to $Pro_\Psi x$ .

Proof : Let $e \in S$. Then we have

$$\|x_{n+1} - e\| = \|Q_n Pro_\Psi x + (1 - Q_n)y_n^m - e\|$$

$= \|Q_n Pro_\Psi x + (1 - Q_n)y_n^m - [(1 - Q_n)e + Q_n e]\|$

$\leq Q_n\|Pro_\Psi x - e\| + (1 - Q_n)\|y_n^m - e\| \leq Q_n\|x - e\| + (1 - Q_n)\|y_n^m - e\|$ (2.1)

$$\|y_n^m - e\| = \left\| R_{n_1}y_n^0 + \sum_{i=2}^{l} R_{n_i}\, Res_{i,n}y_n^{i-1} - e \right\|$$

$= \left\| R_{n_1} y_n^0 + \sum_{i=2}^{l} R_{n_i} Res_{i,n} y_n^{i-1} - [R_{n_1} e + \sum_{i=1}^{m} R_{n_i} e] \right\|$

$$= \left\| (R_{n_1}) y_n^0 + \sum_{i=2}^{l} R_{n_i} \, Res_{i,n} y_n^{i-1} - [R_{n_1} e + \sum_{i=2}^{m} R_{n_i} \, e] \right\|$$

$\leq \ (R_{n_1}) \| y_n^o - e \| + \sum_{i=2}^{l} R_{n_i} \| \, y_n^{i-1} - e \|$

$$\leq \ \sum_{i=1}^{l} R_{n_i} \| \, y_n^{i-1} - e \| \leq \ \| \, y_n^{i-1} - e \| = \| \ S_{ni} f(x_n) + (1 - S_{ni}) \, g(x_n) \ - e \|$$

$\leq \ \| S_{ni} f(x_n) - e \| + k(1 - S_{ni}) \| \ g(x_n) - e \|$

$\leq S_{ni} \, \| x_n - e \| + (1 - S_{ni}) \| \, x_n - e \| \ \leq \| \ x_n - p \| \qquad (2.2)$

From (2.1) and (2.2) . we get

$$\| x_{n+1} - e \| \leq Q_n \| Pro_{\Psi} x - e \| + (1 - Q_n) \ \| \, x_n - e \|$$
$$\leq \max\{ \| x - e \|, \| \, x_n - e \| \}$$
$$.$$
$$.$$
$$\leq \max\{ \| x - e \|, \| x_0 - e \| \}$$

Hence ,$\{x_n\}_{n=1}^{\infty}$ is bounded . So $\{y_n^m\}_{n=1}^{\infty}$ and $\{Res_{i,n} y_n^i\}_{n=1}^{\infty}$ is bounded .

Now ,

$\left\| y_{n+1}^i - y_n^i \right\| \ \forall \, i = 1,2, \dots l$ .

$\| y_{n+1}^m - y_n^m \| \ = \left\| R_{n_{1+1}} y_{n+1}^0 + \sum_{i=2}^{l} R_{n_i+1} \, Res_{i,n+1} y_{n+1}^{i-1} - R_{n_1} y_n^o - \sum_{i=2}^{l} R_{n_i} Res_{i,n} y_n^{i-1} \ \right\|$

$$= \left\| \begin{matrix} R_{n_1+1} y_{n+1}^0 - R_{n_1+1} y_n^0 + R_{n_1+1} y_n^0 + \sum_{i=2}^{l} R_{n_i+1} \, Res_{i,n+1} y_{n+1}^{i-1} - \\ \sum_{i=2}^{l} R_{n_i+1} \, Res_{i,n} y_n^{i-1} + \sum_{i=2}^{l} R_{n_i} \, Res_{i,n} y_n^{i-1} \ - R_{n_1} y_n^0 - \sum_{i=2}^{l} R_{n_i+1} \, Res_{i,n} y_n^{i-1} \end{matrix} \right\|$$

$$\leq R_{n_1+1} \| y_{n+1}^0 - y_n^0 \| + \left| R_{n_1+1} - R_{n_1} \right| (\| y_n^0 \|) + \sum_{i=2}^{l} R_{n_i+1} \ \left\| Res_{i,n+1} y_{n+1}^{i-1} - Res_{i,n} y_n^{i-1} \right\|$$
$$+ \sum_{i=2}^{l} \ |( R_{n_i} - \ R_{\ n_i+1}) | \left\| Res_{i,n} y_n^{i-1} \right\|$$

$$\leq R_{n_1+1} \| y_{n+1}^0 - y_n^0 \| + \left( \sum_{i=2}^{l} \ |( R_{n_i} - \ R_{n_i+1}) | \left\| Res_{i,n} y_n^{i-1} \right\| + \left| R_{n_1+1} - R_{n_1} \right| (\| y_n^0 \|) \right)$$
$$+ \sum_{i=2}^{l} R_{n_i+1} \ \left\| Res_{i,n+1} y_{n+1}^{i-1} - Res_{i,n} y_n^{i-1} \right\|$$

$$\leq R_{n_1+1} \| y_{n+1}^0 - y_n^0 \| + \left( \left( \sum_{i=2}^{l} \ |( R_{n_i} - \ R_{n_i+1}) | + \left| R_{n_0+1} - R_{n_0} \right| \right) K \right)$$
$$+ \sum_{i=2}^{l} R_{n_i+1} \ \left\| Res_{i,n+1} y_{n+1}^{i-1} - Res_{i,n} y_n^{i-1} \right\|$$

$$\le R_{n_1+1}\mathrm{K} + \left(\left(\sum_{i=1}^{l} \left|\left( R_{n_i} - R_{n_i+1}\right)\right| + \left|R_{n_1+1} - R_{n_1}\right|\right) K\right)$$

$$+ \sum_{i=2}^{l} R_{n_i+1} \left\|Res_{i,n+1}y_{n+1}^{i-1} - Res_{i,n}y_n^{i-1}\right\|$$

$K = max\{\left\|Res_{i,n}y_n^{i-1}\right\|, \|y_n^0\|, \|y_{n+1}^0 - y_n^0\|\}$

Next , to find, $\left\|Res_{i,n+1}y_{n+1}^{i-1} - Res_{i,n}y_n^{i-1}\right\|$

For $\xi_{n+1}^i \ge \xi_n^i$ , we have

$$\left\|Res_{i,n+1}y_{n+1}^{i-1} - Res_{i,n}y_n^{i-1}\right\|$$

$$= \left\|Res_{i,n}\left(\frac{\xi_n^i}{\xi_{n+1}^i} y_{n+1}^{i-1} + \left(1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right) Res_{i,n+1}y_{n+1}^{i-1}\right) - Res_{i,n}y_n^{i-1}\right\|$$

$$\le \left\|\frac{\xi_n^i}{\xi_{n+1}^i} y_{n+1}^{i-1} + \left(1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right) Res_{i,n+1}y_{n+1}^{i-1} - y_n^{i-1}\right\|$$

$$= \left\|\frac{\xi_n^i}{\xi_{n+1}^i} y_{n+1}^{i-1} + \left(1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right) Res_{i,n+1}y_{n+1}^{i-1} - \frac{\xi_n^i}{\xi_{n+1}^i} y_n^{i-1} + \left(1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right) y_n^{i-1} \right\|$$

$$= \left\|\frac{\xi_n^i}{\xi_{n+1}^i} y_{n+1}^{i-1} - \frac{\xi_n^i}{\xi_{n+1}^i} y_n^{i-1} + \left(1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right) Res_{i,n+1}y_{n+1}^{i-1} + \left(1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right) y_n^{i-1}\right\|$$

$\le \frac{\xi_n^i}{\xi_{n+1}^i} \left\|y_{n+1}^{i-1} - y_n^{i-1}\right\| + \left|1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right| \left(\left\|Res_{i,n+1}y_{n+1}^{i-1} + y_{n+1}^{i-1})\right\|\right)$

$\le \left\|y_{n+1}^{i-1} - y_n^{i-1}\right\| + \left|1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right| \left(\left\|Res_{i,n+1}y_{n+1}^{i-1} + y_{n+1}^{i-1}\right\|\right)$

$\le \left\|y_{n+1}^{i-1} - y_n^{i-1}\right\| + \left|1 - \frac{\xi_n^i}{\xi_{n+1}^i}\right| S$ where $S = \sup\{\left\|Res_{i,n+1}y_{n+1}^{i-1} + y_{n+1}^{i-1}\right\|, n \ge 0\}$

$\le \left\|y_{n+1}^{i-1} - y_n^{i-1}\right\| + \left|\frac{\xi_{n+1}^i - \xi_n^i}{\xi_{n+1}^i}\right| S$

$\le \left\|y_{n+1}^{i-1} - y_n^{i-1}\right\| + \frac{S}{\xi_{n+1}^i}\left|\xi_{n+1}^i - \xi_n^i\right|$

$\le \left\|y_{n+1}^{i-1} - y_n^{i-1}\right\| + \frac{S}{\xi}\left|\xi_{n+1}^i - r_n^i\right|$

$\le \left\|S_{(n+1)i}f(x_{n+1}) + (1 - S_{(n+1)i})\, g(x_{n+1}) - S_{ni}f(x_n) + (1 - S_{ni})\, g(x_n)\right\| + \frac{S}{\xi}\left|\xi_{n+1}^i - \xi_n^i\right|$

$\le \left\|S_{(n+1)i}f(x_{n+1}) - S_{ni}f(x_n)\right\| + \left\|(1 - S_{(n+1)i})\, g(x_{n+1}) - (1 - S_{ni})\, g(x_n)\right\| + \frac{S}{\xi}\left|\xi_{n+1}^i - \xi_n^i\right|$

$\le S_{ni}\|(x_{n+1}) - (x_n)\| + (1 - S_{ni})\|\,(x_{n+1}) - (x_n)\| + \frac{S}{\xi}\left|\xi_{n+1}^i - \xi_n^i\right|$

$\le \|x_{n+1} - x_n\| + \frac{S}{\xi}\left|\xi_{n+1}^i - \xi_n^i\right|$

$\|y_{n+1}^m - y_n^m\|$

$\le R_{n_0+1}\mathrm{K} + \left(\left(\sum_{i=1}^{l}\left( \left|\left( R_{n_i} - R_{n_i+1}\right)\right| + \left|R_{n_0+1} - R_{n_0}\right|\right) K\right) + \sum_{i=1}^{l} R_{n_i+1}\left(\|x_{n+1} - x_n\| + \frac{S}{\xi}\left|\xi_{n+1}^i - \xi_n^i\right|\right)\right.$

$\|y_{n+1}^m - y_n^m\| \le \|x_{n+1} - x_n\| + R_{n_0+1}\mathrm{K} + \left(\left(\sum_{i=1}^{l}\left( \left|\left( R_{n_i} - R_{n_i+1}\right)\right| + \left|R_{n_0+1} - R_{n_0}\right|\right) K\right) + \frac{S}{\xi}\sum_{i=1}^{l} R_{n_i+1} \left|\xi_{n+1}^i - \xi_n^i\right|\right.$ (2.3)

Now,we have

$\left\|y_{n+1}^{i}-y_{n}^{i}\right\| \leq \|x_{n+1}-x_{n}\|+\frac{S}{\xi}\sum_{i=1}^{l}\left(\left|\xi_{n+1}^{i}-\xi_{n}^{i}\right|\right)+R_{n_0+1}\mathrm{K}+\left(\left(\sum_{i=0}^{l}\left(\ \left|\left(R_{n_i}-\ R_{n_i+1}\right)\right|\right)K\right)\right.$

Now, we estimate $\|x_{n+2}-x_{n+1}\|$ . since $x_{n+1}=\alpha_n x+(1-\alpha_n)y_n^m$

$=\|x_{n+2}-x_{n+1}\|=\|Q_{n+1}Pro_{\Psi}x+(1-Q_{n+1})y_{n+1}^{m}-Q_n x+(1-Q_n)y_n^m\|$

$\leq 1-Q_n)\|y_{n+1}^{m}-y_n^m\|+|Q_{n+1}-Q_n|(\|Pro_{\Psi}x\|+\ \|y_{n+1}^{m}\|)$

$\leq (1-Q_n)\|y_{n+1}^{m}-y_n^m\|+\mathrm{L}|Q_{n+1}-Q_{n+1}|$

Where L $=\|Pro_{\Psi}x\|+\ \|y_{n+1}^{m}\|$ .So, we get

$\|x_{n+2}-x_{n+1}\|\leq(1-Q_n)\|x_{n+1}-x_n\|+R_{n_0+1}\mathrm{K}+\left(\left(\sum_{i=0}^{l}\left(\ \left|\left(R_{n_i}-\ R_{n_i+1}\right)\right|\right)K\right)+\right.$
$\frac{S}{\xi}\sum_{i=1}^{l}R_{n_i+1}\quad\left|\xi_{n+1}^{i}-\xi_{n}^{i}\right|+\mathrm{L}|Q_{n+1}-Q_{n+1}|$

So, $\|x_{n+1}-x_n\|\to 0\ as\ n\to\infty$

$$\begin{aligned}
\|x_n-y_n^m\| &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-y_n^m\| \\
&\leq \|x_{n+1}-x_n\|+\|Q_n x+(1-Q_n)y_n^m-y_n^m\| \\
&= \|x_{n+1}-x_n\|+\|Q_n x+y_n^m-Q_n y_n^m-y_n^m\| \\
&= \|x_{n+1}-x_n\|+\|Q_n x-Q_n y_n^m\| \\
&\leq \|x_{n+1}-x_n\|+Q_n\|x-y_n^m\|
\end{aligned}$$

We have $\|x_n-y_n^m\|\to 0\quad as\ \ n\to\infty$ .

Now , we show that $\left\|y_n^{i-1}-Res_{i,n}y_n^{i-1}\right\|\to 0$ for all $i=1,2,..M$ . In fact ,

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{n\to\infty} sup\left\|y_n^{i-1}-Res_{i,n}y_n^{i-1}\right\|=\lim_{k\to\infty}\left\|y_{n_k}^{i-1}-Res_{i,n_k}y_{n_k}^{i-1}\right\|,$$

And let $\left\{x_{n_{k_l}}\right\}$ be a subsequence of $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} sup\left\|x_{n_k}-e\right\|=\lim_{l\to\infty}\left\|x_{n_{k_l}}-e\right\|.$$

Then we have

$$\begin{aligned}
&\left\|x_{n_{k_l}}-e\right\|=\left\|x_{n_{k_l}}-y_{n_{k_l}}^{m}+y_{n_{k_l}}^{m}-e\right\| \\
&\leq \left\|x_{n_{k_l}}-y_{n_{k_l}}^{m}\right\|+\left\|y_{n_{k_l}}^{m}-e\right\| \\
&\leq \left\|x_{n_{k_l}}-y_{n_{k_l}}^{m}\right\|+\left\|R_{n_0}y_n^{o}+\sum_{i=2}^{m}R_{n_i}\ Res_{i,n}y_n^{i}-e\right\| \\
&\leq \left\|x_{n_{k_l}}-y_{n_{k_l}}^{m}\right\|+\left\|R_{n_0}y_{nkl}^{o}-e\right\|+\ \left\|\sum_{i=2}^{l}R_{n_i}\ Res_{i,n}y_{nkl}^{i}-e\right\| \\
&\leq \left\|x_{n_{k_l}}-y_{n_{k_l}}^{m}\right\|+R_{n_0}\left\|y_{nkl}^{o}-e\right\|+\ \sum_{i=2}^{l}R_{n_i}\ \left\|Res_{i,n}y_{nkl}^{i}-e\right\| \\
&\leq \left\|x_{n_{k_l}}-y_{n_{k_l}}^{m}\right\|+R_{n_0}\left\|y_{nkl}^{o}-e\right\|+\ \sum_{i=2}^{l}R_{n_i}\ \left\|y_{nkl}^{i}-e\right\|
\end{aligned}$$

$$\leq \left\|x_{n_{k_l}} - y^m_{n_{k_l}}\right\| + R_{n_0}\|y^1_{nkl} - e\| + \sum_{i=1}^{l} R_{n_i}\ \|y^i_{nkl} - e\|$$

$$\leq \left\|x_{n_{k_l}} - y^m_{n_{k_l}}\right\| + R_{n_0}\|y^1_{nkl} - e\| + \sum_{i=2}^{l} R_{n_i}\ \|y^i_{nkl} - e\|$$

$$\leq \left\|x_{n_{k_l}} - y^m_{n_{k_l}}\right\| + \sum_{i=0}^{l} R_{n_i}\ \|y_{nkl} - e\|$$

$$\left\|x_{n_{k_l}} - e\right\| = \left\|x_{n_{k_l}} - y^M_{n_{k_l}}\right\| + \|y_{nkl} - e\|$$

Therefore

$$\lim_{l\to\infty} \left\|x_{n_{k_l}} - e\right\| \leq \lim_{l\to\infty} \left\|y^m_{n_{k_l}} - e\right\|$$

$$\left\|y^m_{n_{k_l}} - e\right\|^2 = \left\|R_{n_0} y^1_{nkl} + \sum_{i=2}^{l} R_{n_{ikl}}\ Res_{i,n} y^i_{nkl} - e\right\|^2 =$$

$$\left(1 - \sum_{i=1}^{ml} \beta_{n_{i_{kl}}}\right)\left\|y^1_{n_{k_l}} - e\right\|^2 + \sum_{i=1}^{l} R_{n_{i_{kl}}} \left\|Res_{i,n_{k_l}} y^i_{n_{k_l}} - e\right\|^2$$

$$-\left(1 - \sum_{i=1}^{l} R_{n_{i_{kl}}}\right)\sum_{i=1}^{l} R_{n_{i_{kl}}} \left\|y^i_{n_{k_l}} - Res_{i,n_{k_l}} y^i_{n_{k_l}}\right\|^2$$

$$\leq \left(1 - \sum_{i=1}^{l} R_{n_{i_{kl}}} + \sum_{i=1}^{l} R_{n_{i_{kl}}}\right)\left\|y^i_{n_{k_l}} - e\right\|^2 - \left(1 - \sum_{i=1}^{l} R_{n_{i_{kl}}}\right)\sum_{i=1}^{l} R_{n_{i_{kl}}} \left\|y^i_{n_{k_l}} - Res_{i,n_{k_l}} y^1_{n_{k_l}}\right\|^2$$

$$\leq \left\|y^{i-1}_{n_{k_l}} - e\right\|^2 - \left(1 - \sum_{i=1}^{l} R_{n_{i_{kl}}}\right)\sum_{i=1}^{l} R_{n_{i_{kl}}} \left\|y^i_{n_{k_l}} - Res_{i,n_{k_l}} y^i_{n_{k_l}}\right\|^2$$

$$\leq \left\|x_{n_{k_l}} - e\right\|^2 - \left(1 - \sum_{i=1}^{l} R_{n_{i_{kl}}}\right)\sum_{i=1}^{l} R_{n_{i_{kl}}} \left\|y^1_{n_{k_l}} - Res_{i,n_{k_l}} y^i_{n_{k_l}}\right\|^2$$

Hence, $\left(1 - \sum_{i=1}^{l} R_{n_{i_{kl}}}\right)\sum_{i=1}^{l} R_{n_{i_{kl}}} \left\|y^i_{n_{k_l}} - Res_{i,n_{k_l}} y^i_{n_{k_l}}\right\|^2$

$\leq \left\|x_{n_{k_l}} - e\right\|^2 - \left\|y^i_{n_{k_l}} - e\right\|^2 \to 0$ , as $l \to \infty$

Therefore, we have $\left\|y^i_{n_{k_l}} - Res_{i,n_{k_l}} y^i_{n_{k_l}}\right\| \to 0$ . which implies that

$\lim_{n\to\infty} sup\|y^i_n - Res_{i,n} y^i_n\| = 0$. Hence $\|y^i_n - Res_{i,n} y^i_n\| \to 0$ , for all i .

As the same way, we obtain $\|x_n - Res_{i,n} x_n\| \to 0$.Next we show that $\lim_{n\to\infty} sup < x - Pro_\Psi(x)\ , x_n - Pro_\Psi(x) > \leq 0$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\} \rightharpoonup x^*$ and $x^* \in S$

$$\lim_{n\to\infty} sup < v - Pro_\Psi(v)\ , x_n - Pro_\Psi(v) > = \lim_{k\to\infty} < v - Pro_\Psi(v)\ , x_{nk} - Pro_\Psi(v) > .$$

$= < v - Pro_\Psi(v), x^* - Pro_\Psi(v) \leq 0.$

Finally , we show that $\|x_n - Pro_{\Psi}(v)\| \to 0$ . From lemma, We have

$$\|x_{n+1} - Pro_s(v)\|^2 = \|Q_n Pro_{\Psi} x + (1-Q_n) y_n^m - Pro_{\Psi}(v)\|^2$$
$$= \|Q_n Pro_{\Psi} x + (1-Q_n) y_n^m - [(1-Q_n) Pro_{\Psi}(v) - Q_n\, Pro_{\Psi}(v)]\|^2$$
$$\leq \|Q_n (Pro_s x - Pro_s(v) + (1-Q_n)(y_n^m - Pro_s(v))\|^2$$
$$\leq (1-Q_n)\|y_n^m - Pro_{\Psi}(v)\|^2 + Q_n\| x - Pro_{\Psi}(v)\|^2$$
$$\leq (1-Q_n)\|x_n - Pro_{\Psi}(v)\|^2 + Q_n\| x - Pro_{\Psi}(v)\|^2$$

That $\|x_{n+1} - Pro_{\Psi}(v)\| \to 0 \; as\; n \to \infty$. This completes the proof .

**Corollary : 2.2** .If $\mathcal{M}_i$ are M.M. operators such that

$$F(Pro_s) \cap_{i=1}^{l} \mathcal{M}_i^{-1} 0 \neq \emptyset$$

Under (i-ii) in theorem (3.1). Then $\{x_n\}$ defined by $x \in C$ and

$$\begin{cases} y_n^{i-1} = S_{ni}(x_n) + (1 - S_{ni})\,(x_n) \\ y_n^m = R_{n_1} y_n^0 + \sum_{i=2}^{l} R_{n_i}\, Res_{i,n} y_n^{i-1} \\ x_{n+1} = Q_n Pro_{\Psi} x + (1-Q_n) y_n^m,\; n,m \geq 0 \end{cases}$$

Converges- s to $Pro_{\Psi} x$ .

**Corollary : 2.3** .If $\mathcal{M}_i$ are M.M. operators such that $\mathcal{M}_i^{-1} 0 \neq \emptyset$.Under (i-ii) in theorem (3.1). Then the sequences $\{x_n\}$ defined by $x \in C$ and

$$\begin{cases} y_n^{i-1} = S_{ni}(x_n) + (1 - S_{ni})\,(x_n) \\ y_n^m = R_{n_1} y_n^0 + \sum_{i=2}^{l} R\, Res_{i,n} y_n^{i-1} \\ x_{n+1} = Q_n x + (1-Q_n) y_n^m,\; n,m \geq 0 \end{cases}$$

s-converges to $x$ .

## 3. Conclusion

In this article, a new iterative methods of approximation fixed point are presented. On the other hand, the s-convergence by using multivalued operators also proven.

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