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Extension of Cap by Size and Degree in the Space $PG(3, 11)$

Jabbar Sharif Radhi
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.
e.b.abdulkrareem@uomustansiriyah.edu.iq

Emad Bakr Al-Zangana
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.
jabbarsharif13@gmail.com

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Abstract

A cap of size k and degree r in a projective space, (briefly; (k, r)-cap) is a set of k points with the property that each line in the space meet it in at most r points. The aim of this research is to extend the size and degree of complete caps and incomplete caps, (k, r)-caps of degree $r < 12$ in the finite projective space of dimension three over the finite field of order eleven, which already exist and founded by the action of subgroups of the general linear group over the finite field of order eleven and degree four, to $(k + i, r + 1)$-complete caps. These caps have been classified by giving the t_i-distribution and c_i-distribution. The Gap programming has been used to execute the designed algorithms and computations.

Keywords: Cap, Complete (Incomplete) cap, Companion matrix, Projective space, τ_i-distribution, c_i-distribution.

1. Introduction

In finite projective space, one of the important objects which has been many mathematicians did research on was (k, r)-cap. The operator k is the size of cap, and r is the degree of cap. These operators were the central of research, theoretically and by computer computations, where the researcher attempted to find the upper bound and lower bound of the size k for fixed degree r. Also, the classification of the caps as projectively distinct was their aim, but this goal is so difficult in the space of dimension greater that two.

Let $F_{11} = \{0,1,2,2^2, \dots, 2^9\}$ be the Galois field of order eleven, and C_f be the 4×4 matrix, $C_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2^8 & 2 & 1 & 2^2 \end{pmatrix}$ which is called the companion matrix over F_{11}. The points of the finite

projective space $\Sigma_{11} = PG(3,11)$ are constructing using C_f in the formula $P(i) \coloneqq [1,0,0,0]{C_f}^{i-1}$, $i = 1,2,\ldots,1464$, where $1464 = \theta(3,11) = 11^3 + 11^2 + 11 + 1$ is the order of space. This matrix is cyclic of order $\theta(3,11)$, so to refer to the space's points we can use numeral form $i \coloneqq P(i), i = 1,\ldots\theta(3,11)$. The lines in the space are found using the following formula:
Let $X = [x_0, x_1, x_2, x_3]$ and $Y = [y_0, y_1, y_2, y_3]$ be two point in Σ_{11} on a line l. A coordinate vector of l is $L = (l_{01}, l_{02}, l_{03}, l_{12}, l_{31}, l_{23})$, where $l_{ij} = x_i y_j - x_j y_i$. Then L is determined by l up to a factor of proportion. We will write $l = \mathbf{I}(L)$ to refer to l. To find points $Z = [z_0, z_1, z_2, z_3]$ on l, the following equation must be satisfied [1]:

$$Z \times \begin{pmatrix} 0 & -l_{23} & -l_{31} & -l_{12} \\ l_{23} & 0 & -l_{03} & l_{02} \\ l_{31} & l_{03} & 0 & -l_{01} \\ l_{12} & -l_{02} & l_{01} & 0 \end{pmatrix} = 0.$$

Definition 1.1:[2] A(k,r)-cap in $PG(n \geq 3, q)$ is the set of k points such that no $r+1$ points are collinear, but at most r points of which lie in any line. Here r is called degree of the (k,r)-cap. The (k,r)-cap is called complete cap if it is not contained in $(k+1,r)$-cap.

Definition 1.2:[2] Let K be a cap of degree r, an i-secant of a K in $PG(n,q)$ is a line such that $|k \cap \pi| = i$. The number of i-secants of K denoted by τ_i.

Definition 1.3:[2] Let Q be a point not on the (k,r)-cap, K. The number of i-secant of K passing through Q denoted by $\sigma_i(Q)$. The number $\sigma_r(Q)$ of r-secants is called the *index of Q with respect to K.*

Definition 1.4:[2] The set of all points of index i will be denoted by C_i and the cardinality of C_i denoted by c_i. The sequence $(t_0, \ldots, t_r)$ will represented the secant distribution and the sequences $(c_0, \ldots, c_d)$ refer to index distribution.

Definition 1.5:[2] The group of projectivities of $PG(n.q)$ is called projective general linear group, and denoted $PGL\,(n+1,q)$. The elements of $PGL(n+1,q)$ are non-singular matrices of dimension $n+1$. The matrix C_f is belong to Σ_{11}, so the $H = \langle C_f \rangle$ is cyclic subgroup of $PGL(4,11)$ of order $\theta(3,11)$, and for any integer t divided $\theta(3,11)$, $H_t = \langle {C_f}^t \rangle$ is cyclic subgroup of H of order k such that $t \cdot k = \theta(3,11)$.

[3] found caps using the action of fourteen subgroups H_t on the space Σ_{11}, where $t = 2,3,4,6,8,12,24,61,122,183,244,366,488,732$. These caps were just orbits come from action of the groups H_t on Σ_{11}, and these orbits denoted by $O_t[t,k]$. Also, they classified these caps by secant distribution and index distribution. Their results are summarized as in **Table 1**. Let *INCO*:=Incomplete cap, and *CO*:=Complete cap.

Table 1. Caps from subgroups $\langle T^t \rangle$ action on $PG(3,11)$

	$\langle T^t \rangle$	$(k, degree$ $p)$	$O_t[t,k]$	Type
1	$\langle T^2 \rangle$	$(732,12)$	$\{2n+1 \mid n=0,\ldots,731\}$	*INCO*
2	$\langle T^3 \rangle$	$(488,7)$	$\{3n+1 \mid n=0,\ldots,487\}$	*CO*
3	$\langle T^4 \rangle$	$(366,6)$	$\{4n+1 \mid n=0,\ldots,365\}$	*CO*
4	$\langle T^6 \rangle$	$(244,4)$	$\{6n+1 \mid n=0,\ldots,243\}$	*CO*
5	$\langle T^8 \rangle$	$(183,4)$	$\{8n+1 \mid n=0,\ldots,182\}$	*INCO*
6	$\langle T^{12} \rangle$	$(122,2)$	$\{12n+1 \mid n=0,\ldots,121\}$	*CO*
7	$\langle T^{24} \rangle$	$(61,2)$	$\{24n+1 \mid n=0,\ldots,60\}$	*INCO*
8	$\langle T^{61} \rangle$	$(24,12)$	$\{61n+1 \mid n=0,\ldots,23\}$	*INCO*
9	$\langle T^{122} \rangle$	$(12,12)$	$\{122n+1 \mid n=0,\ldots,11\}$	*INCO*
10	$\langle T^{183} \rangle$	$(8,4)$	$\{183n+1 \mid n=0,\ldots,7\}$	*INCO*
11	$\langle T^{244} \rangle$	$(6,6)$	$\{244n+1 \mid n=0,\ldots,5\}$	*INCO*
12	$\langle T^{366} \rangle$	$(4,4)$	$\{366n+1 \mid n=0,\ldots,3\}$	*INCO*
13	$\langle T^{488} \rangle$	$(3,3)$	$\{488n+1 \mid n=0,\ldots,2\}$	*INCO*
14	$\langle T^{732} \rangle$	$(2,2)$	$\{732n+1 \mid n=0,1\}$	*INCO*

The goal of this paper is to do extensions of the size of the orbits $O_i[i,t]$ and compute the τ_i-distribution and c_i-distribution to it.
Concerning the recent works on the caps resulting from the action of groups on the space $PG(3,q)$, there are some papers appear recently for example, [4,5], where they just find caps which exactly the orbits of group actions. The same idea has been used on the plane [6], and on the line $PG(1,27)$ [7].

To do extension of the size of (complete, incomplete) (k,r)-caps, $O_t[t,k]$ in the projective space $PG(3,11)$, we follow the following steps:

1. Determine the line $\mathbf{I}(L)$ that meets the orbit $O_i[i,t]$ in r points.
2. Find the extension points to $O_i[i,t]$ which are the set $L \backslash O_i[i,t] = Ex$.
3. Adding the first point say p_1 of Ex to $O_i[i,t]$. Let $Q_i^{p_1} = O_i[i,t] \cup \{p_1\}$.
4. Check the set $Q_i^{p_1}$ is (complete, incomplete) cap.
5. Find the τ_i-distribution and c_i-distribution of $Q_i^{p_1}$.
6. The incomplete caps $Q_i^{p_1}$ are extended to complete caps.

A system for computational discrete algebra, GAP [8] is used to implement the above steps.
Let $Y = \{2,3,4,6,8,12,24,61,122,183,244,366,488,732\}$ be the divisors of $\theta(3,11)$.

2. Extension of Caps

Theorem 2.1: Let $O_i[i,t]$ be the (i,r)-caps as in Table 1, i, t in Y and $r \neq 12$. Then there exist $(j, r+1)$-complete caps, $r = 2,3,4,6,7$ and $= 636,505,313,304,139,90,260,650,267,218,118$.

Proof: To do an extension of the size of each (i,r)-cap we first find a line that meets the cap in r-points, and then choose the first point belong to $L \backslash O_i[i,t]$. If the new cap is in complete, then we will add points of the index zero points to make it complete.

1. The line $\mathbf{I}(1,1,0,0,0,0)$ meet the cap $O_3[3,488]$ in 7 points, so we have five extension points can be adding to it in the following orders: $95,287,308,1167,1172$. Let $Q_3^{p_1} = O_3[3,488] \cup \{p_1\}, p_1 = 95$. The t_i-distribution is $(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)$ =(965,2909,997,6422,1022, 2917,989,5), so $O_3^{p_1}$ is (489,8)-cap and c_i values are $c_0 = 955$, $c_1 = 20$. Since $c_0 \neq 0$, then $O_3^{p_1}$ is incomplete. The (489,8)-cap will be complete when adding 147 points to it, and the points are 2,3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30, 32, 33, 35, 36, 38, 39, 41, 42, 44, 45, 47, 48, 50, 51, 53, 54, 56, 57, 59, 60, 62, 63, 65, 66, 68, 69, 71, 72, 74, 75, 77, 78, 80, 81, 83, 84, 86, 87, 89, 90, 92, 93, 95, 96, 98, 99, 101, 102, 104, 105, 107, 108, 110, 111, 113, 114, 116, 117, 119, 120, 122, 123, 125, 126,128, 129, 131, 132, 134, 135, 137, 138, 140, 141, 143, 144, 146, 147, 149, 150, 152, 153, 155, 156, 158, 159, 161,162, 164, 165, 167, 168, 170, 171, 173, 174, 176, 177, 179, 180, 182, 183, 185, 186, 188, 189, 191, 192, 194, 195,197, 198, 200, 201, 203, 204, 206, 207, 209, 210, 212, 213, 215, 216, 218, 219, 221, 222.

2. The line $\mathbf{I}(2^5, 1,0,0,0,0)$ meet the cap $O_4[4,366]$ in 6 points, so we have six extension points can be adding to it in the following orders: 239,346,542,766,955,1058. Let $O_4^{p_1}$= $O_4[4,366]\cup\{p_1\}$, p_1=239. The t_i-distribution is $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7)$= (785,1460,5082,1506,5106 1482,800,5), so $O_4^{p_1}$ is (367,7)-cap and c_i values are $c_0 = 1077$, $c_1 = 20$. Since $c_0 \neq 0$, then $O_4^{p_1}$ is incomplete. The (367,7)-cap will be complete when adding 138 points to it and the points are 2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31, 32, 34, 35, 36, 38, 39, 40, 42, 43, 44, 46, 47, 48, 50, 51, 52, 54, 55, 56, 58, 59, 60, 62, 63, 64, 66, 67, 68, 70, 71, 72, 74, 75, 76, 78, 79, 80, 82, 83, 84, 86, 87, 88, 90, 91, 92, 94, 95, 96, 98, 99, 100, 102, 103, 104, 106, 107, 108, 110, 111, 112, 114,115, 116, 118, 119, 120, 122, 123, 124, 126, 127, 128, 130, 131, 132, 134, 135, 136, 138, 139, 140, 142, 143, 144,146, 147, 148, 150, 151, 152, 154, 155, 156, 158, 159, 160, 162, 163, 164, 166, 167, 168, 170, 171,172, 174, 175,176, 178, 179, 180, 182, 183, 184.

3. The line $\mathbf{I}(2^8, 2^6, 1,0,0,0)$ meet the cap $O_6[6,244]$ in 4 points, so we have eight extension points can be adding to it in the following orders: 16,338,422,716,776,891,988,455. Let $O_6^{p_1}$= $O_6[6,244]\cup\{p_1\}$, $p_1 = 16$. The t_i-distribution is $(t_0, t_1, t_2, t_3, t_4, t_5)$= (2958,1485,7270,1510,2985,18), so $O_6^{p_1}$ is (245,5)-cap and c_i values are $c_0 = 1093$, $c_1 = 126$. Since $c_0 \neq 0$, then $O_6^{p_1}$ is incomplete. The (245,5)-cap will be complete when adding 68 points to it, and the points are 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 38,39, 40, 41, 42, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 62, 63, 64, 65, 66, 68, 69, 70, 71, 72, 74, 75, 76, 77, 78, 80, 81, 82, 83.

4. The line $\mathbf{I}(2^7, 2^6, 1,0,0,0)$ meet the cap $O_8[8,183]$ in 4 points, so we have eight extension points can be adding to it in the following orders: 318,860,1032,1158,1176,1252,1287,1444. Let $O_8^{p_1}$= $O_8[8,183]\cup\{p_1\}, p_1$=318. The t_i-distribution is $(t_0, t_1, t_2, t_3, t_4, t_5)$= (3683,4394,5487,1553,1100,9), so $O_8^{p_1}$ is (184,5)-cap and c_i values are $c_0 = 1217$, $c_1 = 63$. Since $c_0 \neq 0$, then $O_8^{p_1}$ is incomplete. The (184,5)-cap will be complete when adding 120 points to it, and the points are 2,3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60, 61, 62, 63, 64, 66, 67, 68, 69, 70, 71, 72, 74, 75, 76, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 122, 123, 124, 125, 126, 127,128, 130, 131, 132, 133, 134, 135, 136, 138.

5. The line $\mathbf{I}(1,0,1,0,0,0)$ meet the cap $O_{12}[12,122]$ in 2 points, so we have ten extension points can be adding to it in the following orders:2,58,120,144,315,1252,452,506,788, 1108,1426. Let $O_{12}^{p_1}$=$O_{12}[12,122]\cup\{p_1\}$, $p_1 = 2$. The t_i-distribution is (t_0, t_1, t_2, t_3) =(7315,1518,7338,55), so $O_{12}^{p_1}$ is (123,3)-cap and c_i values are $c_0 = 846$, $c_1 = 495$. Since $c_0 \neq 0$, then $O_{12}^{p_1}$ is incomplete. The (123,3)- cap will be complete when adding 16 points to it, and the points are 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19.

6. The line $\mathbf{I}(1,0,1,0,0,0)$ meet the cap $O_{24}[24,61]$ in 2 points, so we have ten extension points can be adding to it in the following orders: 2,58,120,144,315,1252,452,506,788,1108, 1426. Let $O_{24}^{p_1}$=$O_{24}[24,161]\cup\{p_1\}, p_1 = 2$. The t_i-distribution is $(t_0, t_1, t_2, t_3) = (9859,4500,1855,12)$, so $O_{24}^{p_1}$ is (25,3)-cap and c_i values are $c_0 = 1294$, $c_1 = 108$. Since $c_0 \neq 0$, then $O_{24}^{p_1}$ is incomplete. The (25,3)-cap will be complete when adding 65 points to it, and the points are 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63 64, 65, 66,67,68,69.

7. The line $\mathbf{I}(2^2, 2^3, 1,0,0,0)$ meet the cap $O_{183}[183,8]$ in 4 points, so we have eight extension points can be adding to it in the following orders: 123,245,611,855,315,977,1221,1343. Let $O_{183}^{p_1}$=$O_{183}[183,8]\cup\{p_1\}$, $p_1 = 123$. The t_i-distribution of this orbit is $(t_0, t_1, t_2, t_4, t_5)$ = $(15056, 1148, 20,1,1)$, so $O_{183}^{p_1}$ is (9,5)-cap and c_i values are $c_0 = 1448$, $c_1 = 7$. Since $c_0 \neq 0$, then $O_{183}^{p_1}$ is incomplete. The (9,5)-cap will be complete when adding 251 points to it, and the points are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32,33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61,62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254.

8. The line $\mathbf{I}(2^2, 2^3, 1,0,0,0)$ meet the cap $O_{244}[244,6]$ in 6 points, so we have six extension points can be adding to it in the following orders:123,367,611,855,1099,1343. Let $Q_{244}^{p_1}$= $O_{244}[244,6]\cup\{p_1\}, p_1$=123. The t_i -distribution of this orbit is $(t_0, t_1, t_7) = (15301, 924, 1)$, so $O_{183}^{p_1}$ is (7,7)-cap and c_i values are $c_0 = 1452$, $c_1 = 5$. Since $c_0 \neq 0$, then $O_{244}^{p_1}$ is incomplete The (7,7)-cap will be complete when adding 515 points to it And the points are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31,32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59,60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136,137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159,160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182,183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 2224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459,460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506,507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518.

9. The line $\mathbf{I}(2^2, 2^3, 1,0,0,0)$ meet the cap $O_{366}[366,4]$ in 4 points, so we have eight extension points can be adding to it in the following orders: 123,245,489,611,855,977,1221,1343. Let $O_{366}^{p_1}$= $O_{366}[366,4] \cup \{p_1\}, p_1$=123. The t_i-distribution is (t_0, t_1, t_5) =$(15565, 660, 1)$, so $O_{366}^{p_1}$ is (5,5)-cap and c_i values are $c_0 = 1452,\ c_1 = 7$. Since $c_0 \neq 0$, then $O_{366}^{p_1}$is incomplete. The (5,5) will be complete when adding 262 points to it, and the points are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32,33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61,62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90,91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115,116, 117, 118, 119, 120, 121, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162,163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185,186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254,255, 256, 257, 258, 259, 260, 261, 262, 263, 264.

10. The line $\mathbf{I}(2^2, 2^3, 1,0,0,0)$ meet the cap $O_{488}[488,3]$ in 3 points, so we have nine extension points can be adding to it in the following orders 123,245,376,611, 733,855, 1099,1221,1343. Let $O_{488}^{p_1}$= $O_{488}[488,3] \cup \{p_1\}, p_1$=123. The t_i-distribution of this orbit is (t_0, t_1, t_4)= $(15697, 528, 1)$, so $O_{488}^{p_1}$ is (4,4)-cap andc_i values are $c_0 = 1452,\ c_1 = 8$. Since $c_0 \neq 0$, then $O_{488}^{p_1}$ is incomplete. The(4,4)-cap will be complete when adding 214 points to it from the index points c_0, and the points are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32,33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185,186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216.

11. The line $\mathbf{I}(2^2, 2^3, 1,0,0,0)$ meet the cap $O_{732}[732,2]$ in 2 points, so we have ten extension points can be adding to it in the following orders: 123,245,367,489,611,855,977,1099, 1221,1343. Let $O_{732}^{p_1}$=$O_{732}[732,2]\cup\{p_1\}$, p_1=123. The t_i-distribution of this orbit is (t_0, t_1, t_3) =$(15829, 396, 1)$, so $O_{732}^{p_1}$ is (3,3)-cap and c_i values are $c_0 = 1452$, $c_1 = 9$. Since $c_0 \neq 0$, then $O_{732}^{p_1}$is incomplete. The(3,3)-cap will be complete when adding 115 points to it, and the points are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 3233, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115,116.

3. Conclusion

All the extension caps $O_3^{p_1}, O_4^{p_1}, O_6^{p_1}, O_8^{p_1}$ $O_{12}^{p_1}$, $O_{24}^{p_1}, O_{183}^{p_1}, O_{244}^{p_1}$,$O_{366}^{p_1}$, $O_{488}^{p_1}, O_{732}^{p_1}$ are incomplete caps, and can be completed by adding some points of index zero. The details are summarized in**Table 2.**

Let # denoted to the number of adding points to the incomplete cap to be complete.

Table 2. Details about the extension complete caps.

	$O_i^{p_1}$	τ_i-distribution	c_i- distribution	#
1	$O_3^{p_1}$	$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8)$	(c_0, c_1)	147
		(965,2909,997,6422,1022,2917,989,5)	(955,20)	
2	$O_4^{p_1}$	$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7)$	(c_0, c_1)	138
		(785,1460,5028,1506,5106, 1482,800,5)	(1077,20)	
3	$O_6^{p_1}$	$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$	(c_0, c_1)	68
		(2958,1485,7270,1510 ,2985,18)	(1093,126)	
4	$O_8^{p_1}$	$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$	(c_0, c_1)	120
		(3683,4394,5489, 1553,1100,9)	(1217,63)	
5	$O_{12}^{p_1}$	$(\tau_0, \tau_1, \tau_2, \tau_3)$	(c_0, c_1)	16

		(7315,1518,7338,55)	(846,495)	
6	$O_{24}^{p_1}$	$(\tau_0, \tau_1, \tau_2, \tau_3)$ (9859,4500,1855,12)	(c_0, c_1) (1294,108)	65
7	$O_{183}^{p_1}$	$(\tau_0, \tau_1, \tau_2, \tau_4, \tau_5)$ (15056,1148,20,1,1)	(c_0, c_1) (1448,7)	251
8	$O_{24}^{p_1}$	(τ_0, τ_1, τ_7) (15301,924,1)	(c_0, c_1) (1452,5)	515
9	$O_{366}^{p_1}$	(τ_0, τ_1, τ_5) (15565,660,1)	(c_0, c_1) (1452,7)	262
10	$O_{488}^{p_1}$	(τ_0, τ_1, τ_4) (15697,528,1)	(c_0, c_1) (1452,8)	214
11	$O_{732.}^{p_1}$	(τ_0, τ_1, τ_3) (15829,396,1)	(c_0, c_1) (1452,9)	115

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