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# A Class of Exponential Rayleigh Distribution and New Modified Weighted Exponential Rayleigh Distribution with Statistical Properties

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## Abstract

This paper deals with the mathematical method for extracting the Exponential Rayleighh (ER) distribution based on mixed between the cumulative distribution function of Exponential distribution and the cumulative distribution function of Rayleigh distribution using an application (maximum), as well as derived different statistical properties for $ER$ distribution ( Mode, The Median, $r^{th}$ moment, *The Variance*, Coefficient of Skewness , Coefficient of Kurtosis, Moment Generating Function, Factorial moment generating function, Quantile Function, Characteristic Function ). Then, we present a structure of a new distribution based on a modified weighted version of Azzalini's named Modified Weighted Exponential Rayleigh (MWER) distribution such that this new distribution is generalization of the ER distribution and provide some special models of the $MWER$ distribution, as well as derived different statistical properties for $MWER$ distribution.

**Keywords:** Exponential Rayleigh (ER) distribution, Modified Weighted Exponential Rayleigh (MWER) distribution.

## 1.Introduction

Statistical distributions are very important for parametric inferences and are usually applied to describe real world phenomena. Although they are useful in several scientific fields, it is observed that most common distributions, such as Exponential, Gamma, Weibull, Rayleigh and Lindley are not sufficiently flexible to accommodate various phenomena of nature for example, while exponential distribution is frequently defined as flexible, its hazard function is constant. For this cause, researchers have focused on the expansion of these common distributions in order to produce a more and more realistic and flexible models for data. In 1880, [1] introduced the Rayleigh distribution this distribution with one scale parameter is one of the most widely used distributions. Exponential and Rayleigh were an important distribution in statistics and operation research [2].

[3] mixed Exponential and Rayleigh distributions based on T-X families. [4]. presented the finite mixture of Exponential Rayleigh and Burr Type-XII distribution. [5] mixed multivariate Exponential distribution and the multivariate Rayleigh distribution to obtained Multivariate Rayleigh and Exponential Distributions. [6] introduced a new mixture distribution based on the tail of mixed between Exponential Rayleigh and Exponential Weibull distributions using an application (minimum).

There are various methods of inputting the shape parameter of a probability distribution model and they may result in a variety of weighted distributions. The weighted distributions are widely used in reliability, survival, bio-medicine, environment, and many other fields of immense practical interest in mathematics, probability, statistics. These distributions naturally arise as a result of observations created by a random process and recorded with some weight functions [7].

The aim of this paper, two distributions have been introduced Exponential Rayleigh distribution this distribution can be obtained based on mixed between cumulative distribution function of Exponential distribution and the cumulative distribution function of Rayleigh distribution using an application (maximum) and present a new distribution named Modified Weighted Exponential Rayleigh distribution built on a modified weighted version of Azzalini's (1985), this new distribution is a generalization of the Exponential Rayleigh distribution, as well as present the most important statistical properties of these two distributions, finally the conclusion of this paper is determined.

## 2. Exponential Rayleigh Distribution

A continuous non-negative random variable $Z$ is called to have an Exponential distribution with parameter $\alpha$, if its probability density function is given by [8]:

$$f(z;\alpha)_E = \begin{cases} \alpha\, e^{-\alpha z} & ; z \geq 0;\ \alpha > 0 \\ \text{zero otherwise.} & \end{cases} \quad \ldots(1)$$

Where $\alpha$ is a scale parameter.

The cumulative distribution function is:

$$F(z;\alpha)_E = 1 - e^{-\alpha z} \qquad ; z \geq 0;\ \alpha > 0 \quad \ldots(2)$$

A continuous non-negative random variable $Y$ is called to have a Rayleigh distribution with parameter $\lambda$, if its probability density function is given by [9]:

$$f(y;\lambda)_R = \begin{cases} \lambda y\, e^{-\frac{\lambda}{2}y^2} & ; y \geq 0;\ \lambda > 0 \\ \text{zero otherwise.} & \end{cases} \quad \ldots(3)$$

Where $\lambda$ is a scale parameter.

The cumulative distribution function is:

$$F(y;\lambda)_R = 1 - e^{-\frac{\lambda}{2}y^2} \qquad ; y \geq 0;\ \lambda > 0 \quad \ldots(4)$$

The Exponential Rayleigh distribution introduced by Mohammed and Hussein in (2019) depends on mixed of the tail (survival) function of Exponential distribution and the tail (survival) function of Rayleigh distribution using an application (minimum) [6]. This distribution can be also generated in another way depending on mixed between the cumulative distribution function of

Exponential distribution as in equation (2) and cumulative distribution function of Rayleigh distributions as in equation (4) using an application (maximum) as follows:

Let $X = max(Z, Y)$ where $Z \sim E(\alpha)$, $Y \sim R(\lambda)$ , $Z$ and $Y$ are two independent random variables then:

$$F(x;\alpha,\lambda)_{ER} = 1 - pr(max(Z,Y) > x)$$
$$F(x;\alpha,\lambda)_{ER} = 1 - [pr(Z > x).pr(Y > x)]$$
$$F(x;\alpha,\lambda)_{ER} = 1 - [(\int_x^\infty \alpha\, e^{-\alpha z}\, dz).(\int_x^\infty \lambda y\, e^{-\frac{\lambda}{2}y^2} dy)]$$
$$F(x;\alpha,\lambda)_{ER} = 1 - \ e^{-(\alpha x + \frac{\lambda}{2}x^2)} \qquad ; x \geq 0; \alpha, \lambda > 0 \qquad \ldots(5)$$

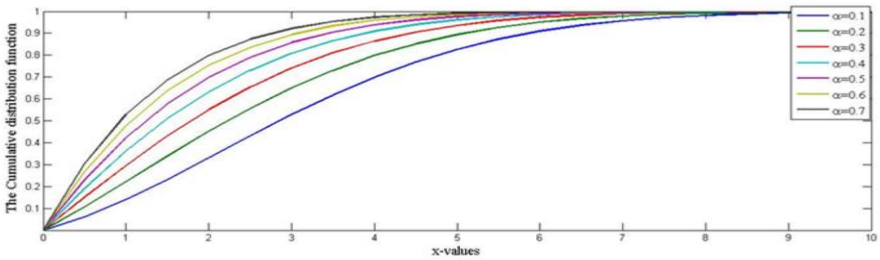


**Figure 1.** plot of the cumulative distribution function of $ER$ distribution for $\lambda = 0.1$ and different values of ($\alpha =$ 0.1,0.2,0.3,0.4,0.5, 0.6, 0.7 )[ MATLAB R2013a ].

The probability density function of Exponential Rayleigh $ER$ distribution is given by:

$$f(x;\alpha,\lambda)_{ER} = \begin{cases} (\alpha + \lambda x)\, e^{-\left(\alpha x + \frac{\lambda}{2}x^2\right)} & ; x \geq 0; \alpha, \lambda > 0 \\ \text{zero otherwise.} \end{cases} \qquad \ldots(6)$$

Where $\alpha$ $and$ $\lambda$ are scale parameters.

Such that,

- $f(x;\alpha,\lambda)_{ER} > 0$
- $\int_0^\infty f(x;\alpha,\lambda)_{ER}\ dx = \int_0^\infty (\alpha + \lambda x)\, e^{-\left(\alpha x + \frac{\lambda}{2}x^2\right)}\ dx = - \left[e^{-\left(\alpha x + \frac{\lambda}{2}x^2\right)}\right]_0^\infty = 1$

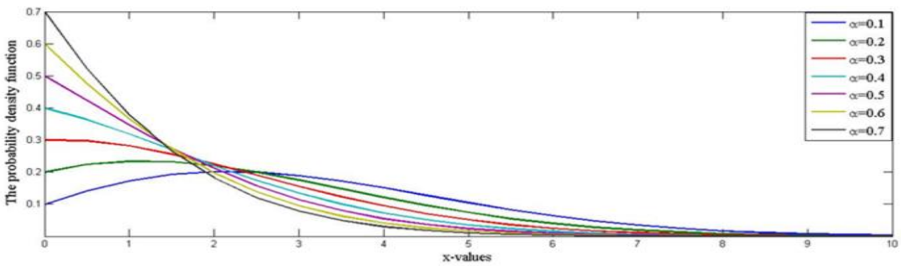


**Figure 2.** plot of the probability density function of $ER$ distribution for $\lambda = 0.1$ and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ )[MATLABR2013a].

- The Survival function is given by:

$$S(t;\alpha,\lambda)_{ER} = e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)} \qquad ; t \geq 0; \alpha,\lambda > 0 \qquad \ldots(7)$$

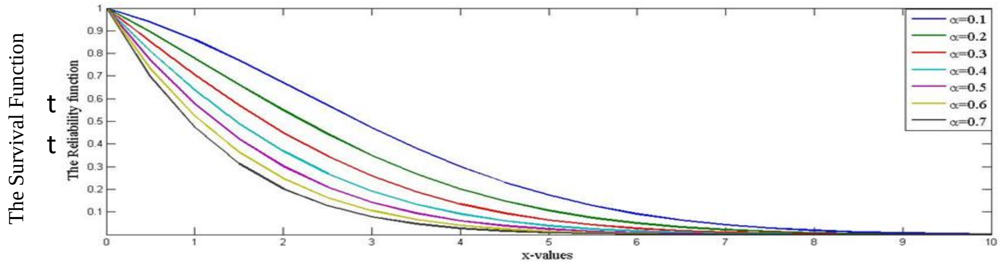


**Figure 3.** plot of the survival function of $ER$ distribution for $\lambda = 0.1$ and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

The Hazard rate function is obtained by:

$$h(t;\alpha,\lambda)_{ER} = \alpha + \lambda t \qquad ; t \geq 0; \alpha,\lambda > 0 \qquad \ldots(8)$$

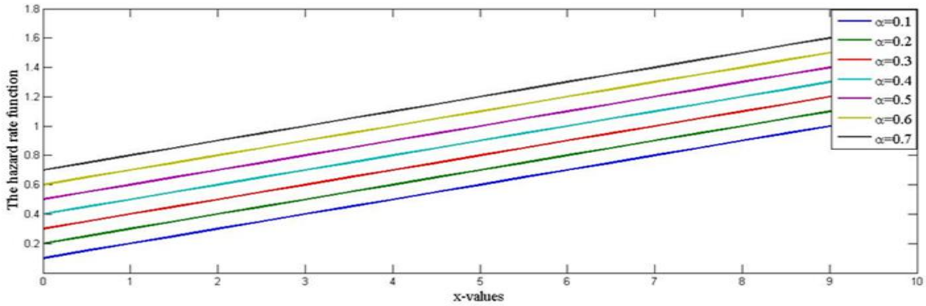


**Figure 4.** plot of hazard rate function of $ER$ distribution for $\lambda = 0.1$ and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

The Reverse hazard rate function is given by:

$$\emptyset(t;\alpha,\lambda)_{ER} = \frac{(\alpha+\lambda t)\, e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)}}{1-e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)}} \qquad ; t \geq 0; \alpha,\lambda > 0 \qquad \ldots(9)$$

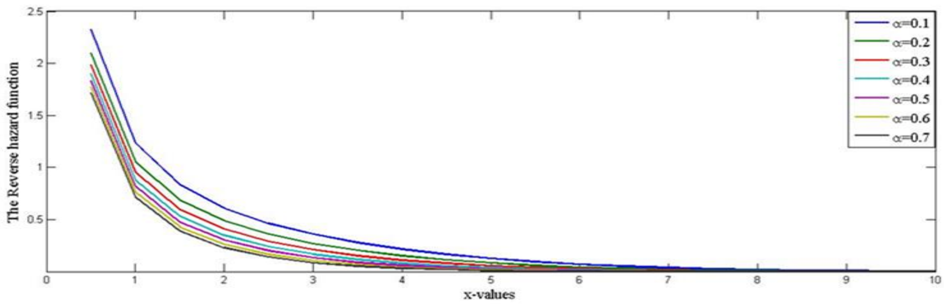


**Figure 5.** plot of the reverse hazard rate function of $ER$ distribution for $\lambda = 0.1$and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4,0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

## 2.1 Some Statistical Properties of $ER$ Distribution

### 2.1.1 The Mode

We can give the mode of $ER$ distribution as follows:

$$\frac{\partial f(x;\alpha,\lambda)_{ER}}{\partial x} = -(\alpha+\lambda x)^2\, e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)} + \lambda\, e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)}$$

$$[-(\alpha+\lambda x)^2 + \lambda]\, e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)} = 0 \qquad \ldots(10)$$

It is clear that $\frac{\partial f(x;\alpha,\lambda)_{ER}}{\partial x} = [-\{h(x;\alpha,\lambda)_{ER}\}^2 + h'(x;\alpha,\lambda)_{ER}]\, S(x;\alpha,\lambda)_{ER}$, Where $h(x;\alpha,\lambda)_{ER}$ is the hazard rate function given in equation (8), and $S(x;\alpha,\lambda)_{ER}$ is the Survival function was given in equation (7). Since $S(x;\alpha,\lambda)_{ER} \neq 0$. Thus dividing the equation (10) by $S(x;\alpha,\lambda)_{ER}$, we get:

$$\lambda^2 x^2 + 2\,\lambda\alpha x + (\alpha^2-\lambda) = 0$$

Based on the law of the constitution, we get:

$$x = \frac{-(2\,\lambda\alpha) \mp \sqrt{(2\,\lambda\alpha)^2 - 4\,\lambda^2(\alpha^2-\lambda)}}{2\lambda^2} \qquad \ldots(11)$$

Since $x > 0$, the negative value of $x$ is ignored. Suppose that $x = x_0$ that is a root of equation (11). This root based on the second derivative of the equation:

If $\frac{\partial^2 f(x;\alpha,\lambda)_{ER}}{\partial x^2}\Big|_{x=x_0} < 0$ the root is the local maximum. If $\frac{\partial^2 f(x;\alpha,\lambda)_{ER}}{\partial x^2}\Big|_{x=x_0} > 0$ the root is the local minimum. If $\frac{\partial^2 f(x;\alpha,\lambda)_{ER}}{\partial x^2}\Big|_{x=x_0} = 0$ the point is inflection.

### 2.1.2 The Median

The median of $ER$ distribution is given by:

$$1 - e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)} = \frac{1}{2}$$

$$\lambda\, x^2 + 2\,\alpha x - 2\ln 2 = 0$$

Based on the law of the constitution, we get:

$$x = \frac{-(2\alpha) \mp \sqrt{4\alpha^2 + 8\,\lambda\, ln\, 2}}{2\,\lambda} \qquad \ldots(12)$$

Since $x > 0$, the negative value of $x$ will be ignored.

### 2.1.3 The Moment about the Origin

The $r^{th}$ moment about the origin can be obtained by:

$$E(X^r)_{ER} = \int_0^\infty x^r(\alpha+\lambda x)\, e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)}\, dx \qquad \ldots(13)$$

Let

$$K(r,\alpha,\lambda) = x^r\, e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)} \qquad \ldots(14)$$

By Maclaurin series:

$$e^{-\alpha x} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} x^n \qquad \ldots(15)$$

Substituting equation (15) in equation (14), we get:

$$K(r,\alpha,\lambda) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\, x^{r+n}\, e^{-\frac{\lambda}{2}x^2} \qquad \ldots(16)$$

Substituting equation (16) in equation (13) we get:

$$E(X^r)_{ER} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \left[\int_0^\infty \alpha\, x^{r+n}\, e^{-\frac{\lambda}{2}x^2}\, dx + \int_0^\infty \lambda\, x^{r+n+1}\, e^{-\frac{\lambda}{2}x^2}\, dx\right] \qquad \ldots(17)$$

Now, solve the first integral as follows:

$$L_1 = \int_0^\infty \alpha\, x^{r+n}\, e^{-\frac{\lambda}{2}x^2}\, dx = \frac{\alpha\, 2^{\frac{r+n-1}{2}}}{\lambda^{\frac{r+n+1}{2}}}\, \Gamma\left(\frac{r+n+1}{2}\right) \qquad \ldots(18)$$

Now, solve the second integral as follows:

$$L_2 = \int_0^{\infty} \lambda\, x^{r+n+1}\ e^{-\frac{\lambda}{2}x^2}\ dx = \frac{2^{\frac{r+n}{2}}}{\lambda^{\frac{r+n}{2}}}\,\Gamma\left(\frac{r+n+2}{2}\right) \qquad \ldots(19)$$

Substituting equations (18) and (19) in equation (17) we get:

$$E(X^r)_{ER} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{r+n}{2}}}{\lambda^{\frac{r+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}}\, \Gamma\left(\frac{r+n+1}{2}\right) + \Gamma\left(\frac{r+n+2}{2}\right)\right] \qquad \ldots(20)$$

***The Mean:*** Let $r = 1$ in equation (20) we get the first moment which is called the mean, thus:

$$E(X)_{ER} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{1+n}{2}}}{\lambda^{\frac{1+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+2}{2}\right) + \Gamma\left(\frac{n+3}{2}\right)\right] \qquad \ldots(21)$$

***The Variance*:** The general form of $v(X)$ of $ER$ distribution is given by:

$$v(X)_{ER} = E(X^2)_{ER} - [E(X)_{ER}]^2$$

$$v(X)_{ER} = \left[\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{2+n}{2}}}{\lambda^{\frac{2+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+3}{2}\right) + \Gamma\left(\frac{n+4}{2}\right)\right]\right] - \left[\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{1+n}{2}}}{\lambda^{\frac{1+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+2}{2}\right) + \Gamma\left(\frac{n+3}{2}\right)\right]\right]^2 \qquad \ldots(22)$$

### 2.1.4 Coefficient of Skewness

The general form of the Coefficient of Skewness $(C.S)$ of $ER$ distribution is given by:

$$C.S_{ER} = \frac{E(X^3)_{ER}}{[E(X^2)_{ER}]^{\frac{3}{2}}} = \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{3+n}{2}}}{\lambda^{\frac{3+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}} \Gamma\left(\frac{n+4}{2}\right) + \Gamma\left(\frac{n+5}{2}\right)\right]}{\left[\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{2+n}{2}}}{\lambda^{\frac{2+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}} \Gamma\left(\frac{n+3}{2}\right) + \Gamma\left(\frac{n+4}{2}\right)\right]\right]^{\frac{3}{2}}} \qquad \ldots(23)$$

### 2.1.5 Coefficient of Kurtosis

The general form of the Coefficient of Kurtosis $(C.K)$ of $ER$ distribution is given by:

$$C.K_{ER} = \frac{E(X^4)_{ER}}{[E(X^2)_{ER}]^2} - 3 = \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{4+n}{2}}}{\lambda^{\frac{4+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}} \Gamma\left(\frac{n+5}{2}\right) + \Gamma\left(\frac{n+6}{2}\right)\right]}{\left[\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{2^{\frac{2+n}{2}}}{\lambda^{\frac{2+n}{2}}} \left[ \frac{\alpha}{\sqrt{2\lambda}} \Gamma\left(\frac{n+3}{2}\right) + \Gamma\left(\frac{n+4}{2}\right)\right]\right]^2} - 3 \qquad \ldots(24)$$

### 2.1.6 Moment Generating Function

The moment generating function of $ER$ distribution can be derived as follows:

$$M_X(t)_{ER} = E(e^{xt}) = \int_0^{\infty} e^{xt}\,(\alpha + \lambda x)\, e^{-\left(\alpha x + \frac{\lambda}{2}x^2\right)}\ dx$$

$$M_X(t)_{ER} = \int_0^{\infty} (\alpha + \lambda x)\, e^{-\left((\alpha - t)\,x + \frac{\lambda}{2}x^2\right)}\ dx \qquad \ldots(25)$$

Let

$$W((\alpha - t), \lambda) = e^{-\left((\alpha - t)\,x + \frac{\lambda}{2}x^2\right)} \qquad \ldots(26)$$

By Maclaurin series:

$$e^{-(\alpha - t)x} = \sum_{n=0}^{\infty} \frac{(-(\alpha - t))^n}{n!} x^n \qquad \ldots(27)$$

Substituting equation (27) in equation (26) we get:

$$W((\alpha-t),\lambda)=\sum_{n=0}^{\infty}\frac{(-(\alpha-t))^n}{n!}x^n\,e^{-\frac{\lambda}{2}x^2} \qquad \ldots(28)$$

Substituting equation (28) in equation (25) we get:

$$M_X(t)_{ER}=\sum_{n=0}^{\infty}\frac{(-(\alpha-t))^n}{n!}\left[\int_0^{\infty}\alpha\,x^n\;e^{-\frac{\lambda}{2}x^2}\;dx+\int_0^{\infty}\lambda\,x^{n+1}\;e^{-\frac{\lambda}{2}x^2}\;dx\right] \qquad \ldots(29)$$

Now, solve the first integral as follows:

$$L_1=\int_0^{\infty}\alpha\,x^n\;e^{-\frac{\lambda}{2}x^2}\;dx=\frac{\alpha\,2^{\frac{n-1}{2}}}{\lambda^{\frac{n+1}{2}}}\,\Gamma(\frac{n+1}{2}) \qquad \ldots(30)$$

Now, solve the second integral as follows:

$$L_2=\int_0^{\infty}\lambda\,x^{n+1}\;e^{-\frac{\lambda}{2}x^2}\;dx\;=\;\frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}}\,\Gamma(\frac{n+2}{2}) \qquad \ldots(31)$$

Substituting equations (30) and (31) in equation (29) yields:

$$M_X(t)_{ER}=\sum_{n=0}^{\infty}\frac{(-(\alpha-t))^n}{n!}\frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}}\left[\frac{\alpha}{\sqrt{2\lambda}}\,\Gamma\left(\frac{n+1}{2}\right)+\;\Gamma\left(\frac{n+2}{2}\right)\right] \qquad \ldots(32)$$

### 2.1.7 Factorial Moment Generating Function

The factorial moment generating function of $ER$ distribution can be obtained as follows:

$$M(t)_{ER}=E(t^x)=\int_0^{\infty}t^x\,(\alpha+\lambda x)\,e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)}\;dx$$

$$M(t)_{ER}=\int_0^{\infty}\;(\alpha+\lambda x)\,e^{-\left((\alpha-\ln(t))\,x+\frac{\lambda}{2}x^2\right)}\;dx \qquad \ldots(33)$$

Let

$$A((\alpha-\ln(t)),\lambda)=\;e^{-\left((\alpha-ln(t))\,x+\frac{\lambda}{2}x^2\right)} \qquad \ldots(34)$$

By Maclaurin series:

$$e^{-(\alpha-\ln(t))x}=\sum_{n=0}^{\infty}\frac{(-(\alpha-\ln(t)))^n}{n!}x^n \qquad \ldots(35)$$

Substituting equation (35) in equation (34) we get:

$$A((\alpha-\ln(t)),\lambda)=\sum_{n=0}^{\infty}\frac{(-(\alpha-\ln(t)))^n}{n!}x^n\,e^{-\frac{\lambda}{2}x^2} \qquad \ldots(36)$$

Substituting equation (36) in equation (33) we get:

$$M(t)_{ER}=\sum_{n=0}^{\infty}\frac{(-(\alpha-\ln(t)))^n}{n!}\left[\int_0^{\infty}\alpha\,x^n\;e^{-\frac{\lambda}{2}x^2}\;dx+\int_0^{\infty}\lambda\,x^{n+1}\;e^{-\frac{\lambda}{2}x^2}\;dx\right] \qquad \ldots(37)$$

Now, based on equations (30) and (31) we get:

$$M(t)_{ER}=\sum_{n=0}^{\infty}\frac{(-(\alpha-\ln(t)))^n}{n!}\frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}}\left[\frac{\alpha}{\sqrt{2\lambda}}\,\Gamma\left(\frac{n+1}{2}\right)+\;\Gamma\left(\frac{n+2}{2}\right)\right] \qquad \ldots(38)$$

### 2.1.8 Characteristic Function

The Characteristic function of $ER$ distribution can be derived as follows:

$$\emptyset_X(it)_{ER}=E(e^{itx})=\int_0^{\infty}e^{itx}\,(\alpha+\lambda x)\,e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)}\;dx$$

$$\emptyset_X(it)_{ER}=\int_0^{\infty}\;(\alpha+\lambda x)\,e^{-\left((\alpha-\mathrm{it})\,x+\frac{\lambda}{2}x^2\right)}\;dx \qquad \ldots(39)$$

Let

$$P((\alpha-\mathrm{it}),\lambda)=\;e^{-\left((\alpha-it)\,x+\frac{\lambda}{2}x^2\right)} \qquad \ldots(40)$$

By Maclaurin series:

$$e^{-(\alpha-\mathrm{it})x} = \sum_{n=0}^{\infty} \frac{(-(\alpha-\mathrm{it}))^n}{n!} x^n \qquad \ldots(41)$$

Substituting equation (41) in equation (40) gives:

$$P((\alpha-\mathrm{it}),\lambda) = \sum_{n=0}^{\infty} \frac{(-(\alpha-\mathrm{it}))^n}{n!} x^n \, e^{-\frac{\lambda}{2}x^2} \qquad \ldots(42)$$

Substituting equation (42) in equation (39) gives:

$$\emptyset_X(it)_{ER} = \sum_{n=0}^{\infty} \frac{(-(\alpha-\mathrm{it}))^n}{n!} \left[\int_0^{\infty} \alpha\, x^n \, e^{-\frac{\lambda}{2}x^2}\, dx + \int_0^{\infty} \lambda\, x^{n+1}\, e^{-\frac{\lambda}{2}x^2}\, dx\right] \qquad \ldots(43)$$

Now, based on equations (30) and (31) we get:

$$\emptyset_X(it)_{ER} = \sum_{n=0}^{\infty} \frac{(-(\alpha-\mathrm{it}))^n}{n!} \frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}} [\frac{\alpha}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+1}{2}\right) + \Gamma\left(\frac{n+2}{2}\right)] \qquad \ldots(44)$$

**2.1.9 Quantile Function**

The quantile function of $ER$ random variable is defined as a solution of $p(x \le x_{(q)}) = F(x_{(q)})_{ER}$ w.r.t. $x_{(q)}$, Therefore, via using the inverse transformation to equation (5), it can be found as:

$$x_{(q)} = F^{-1}(q) \qquad ; x_{(q)} > 0;\ 0 < q < 1$$

$$q = 1 - e^{-(\alpha x_{(q)} + \frac{\lambda}{2} x_{(q)}^2)}$$

$$ln\,(1-q) = -(\alpha x_{(q)} + \frac{\lambda}{2}\, x_{(q)}^2)$$

$$\lambda\, x_{(q)}^2 + 2\alpha x_{(q)} + 2\, ln\,(1-q) = 0$$

Based on law of the constitution, we get:

$$x_{(q)} = \frac{-(2\alpha) \mp \sqrt{4\alpha^2 - 8\,\lambda\, ln\,(1-q)}}{2\,\lambda} \qquad \ldots(45)$$

Since $x_{(q)} > 0$, the negative values of $x_{(q)}$ will be ignored.

## 3. Modified Weighted Exponential Rayleigh Distribution

This section discusses adding a shape parameter to Exponential Rayleigh $ER$ distribution and generating a Modified Weighted Exponential Rayleigh $MWER$ distribution as follows:

The general definition for extracting modified weighted non-negative models depending on a modified weighted version of Azzalini's (1985) can be summarized by [10]:

Let $g(x)$ be a probability density function and $\bar{G}(x)$ be corresponding reliability (survival) function such that the cumulative distribution function $G(x)$ exist. Then the modified weighted model of distribution is given by:

$$f(x)_{MW} = M\, g(x)\, \bar{G}(\theta x)$$

Where, $M$ is the normalizing constant and $\theta > 0$ is the shape parameter.

In our work this parameter $\theta$ does not depend on the degree of the random variable X. Now, consider a probability density function of $ER$ distribution as in equation (6) and the survival function as in equation (7), according to the previous definition for extracting modified weighted non-negative models, put $M = 1 + \theta$ extract the probability density function of Modified Weighted Exponential Rayleigh $MWER$ distribution as follows:

$$f(x;\alpha,\lambda,\theta)_{MWER} = M(\alpha + \lambda x)\, e^{-\left(\alpha x + \frac{\lambda}{2}x^2\right)}\, e^{-\left(\alpha\theta x + \frac{\lambda\theta}{2}x^2\right)}$$

$$f(x;\alpha,\lambda,\theta)_{MWER} = (1+\theta)(\alpha + \lambda x)\, e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)}$$

$$f(x;\alpha,\lambda,\theta)_{MWER} = \begin{cases} (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)} & ; x \ge 0 \\ \text{zero otherwise.} \end{cases} \qquad \ldots(46)$$

Where $\alpha, \lambda > 0$ are scale parameters and $\theta > 0$ is the shape parameter.
Such that,

- $f(x; \alpha, \lambda, \theta)_{MWER} > 0$
- $\int_0^\infty f(x; \alpha, \lambda, \theta)_{MWER}\, dx$

$$= \int_0^\infty (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx$$

$$= -\left[ e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)} \right]_0^\infty$$

$$= 1$$

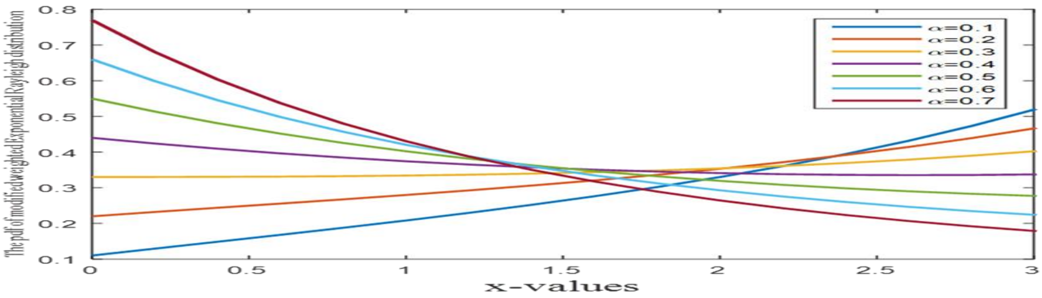


**Figure 6.** plot of the probability density function of $MWER$ distribution for $\lambda = \theta = 0.1$ and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

The cumulative distribution function of $MWER$ can be obtained by:

$$F(x; \alpha, \lambda, \theta)_{MWER} = 1 - e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)} \qquad ; x \geq 0;\ \alpha, \lambda, \theta > 0 \qquad \ldots(47)$$

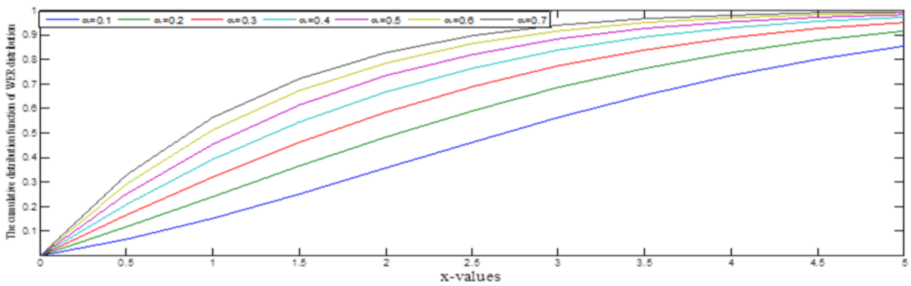


**Figure 7.** plot of the cumulative distribution function of $MWER$ distribution for $\lambda = \theta = 0.1$ and different values of ($\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

The Survival function of $MWER$ is given by:

$$S(t; \alpha, \lambda, \theta)_{MWER} = e^{-\left(\alpha(1+\theta)t + \frac{\lambda(1+\theta)}{2}t^2\right)} \qquad ; t \geq 0;\ \alpha, \lambda, \theta > 0 \qquad \ldots(48)$$

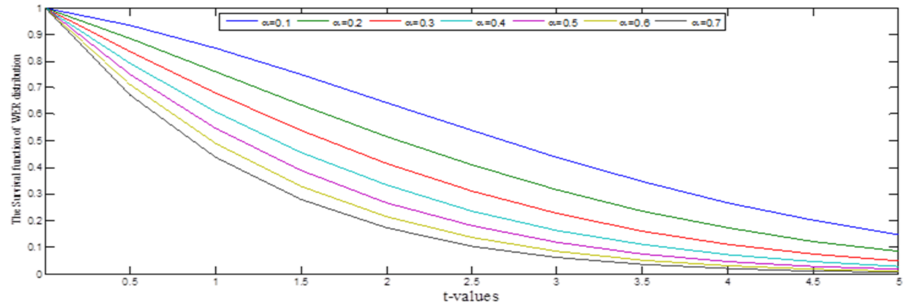


**Figure 8.** plot of the survival function of $MWER$ distribution for $\lambda = \theta = 0.1$ and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

The Hazard rate function of $MWER$ is given by:

$$h(t;\alpha,\lambda,\theta)_{MWER} = \alpha(1+\theta) + \lambda(1+\theta)t \qquad ; t > 0;\ \alpha,\lambda,\theta > 0 \qquad \dots(49)$$

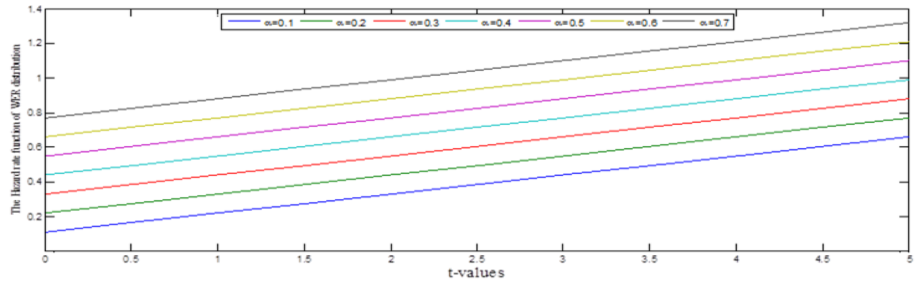


**Figure 9.** plot of hazard rate function of $MWER$ distribution for $\lambda = \theta = 0.1$ and different values of ($\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$) [ MATLAB R2013a ].

The Reverse hazard rate function of $MWER$ is given by:

$$\emptyset(t;\alpha,\lambda,\theta)_{MWER} = \frac{(\alpha(1+\theta)+\lambda(1+\theta)t)\ e^{-\left(\alpha(1+\theta)t+\frac{\lambda(1+\theta)}{2}t^2\right)}}{1-e^{-\left(\alpha(1+\theta)t+\frac{\lambda(1+\theta)}{2}t^2\right)}} \qquad ; t \geq 0;\ \alpha,\lambda,\theta > 0 \qquad \dots(50)$$

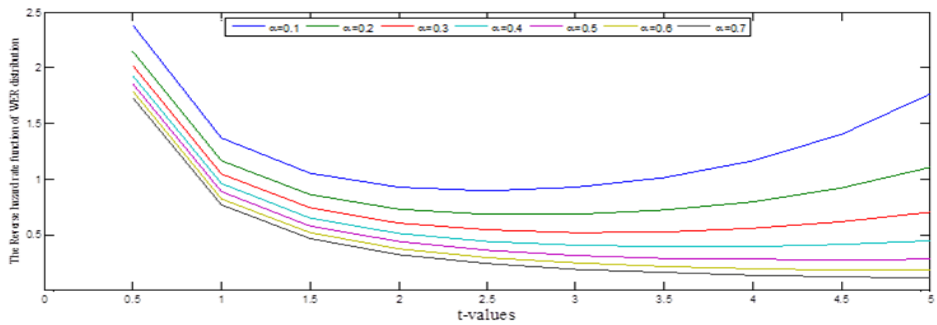


**Figure 10.** plot of the reverse hazard rate function of $MWER$ distribution for $\lambda = \theta = 0.1$ and different values of ( $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ ) [ MATLAB R2013a ].

### 3.1 Special Models

In this section, we provide special models of the $MWER$ distribution:

1. When $\alpha = \theta = 0$ the probability density function of Modified Weighted Exponential Rayleigh $MWER$ distribution reduces to give the probability density function of Rayleigh distribution [11]:

$$f(x;\lambda)_R = \begin{cases} \lambda x\, e^{-\frac{\lambda}{2}x^2} & ; x \geq 0;\ \lambda > 0 \\ \text{zero otherwise.} \end{cases}$$

Where $\lambda$ is scale parameter.

2. When $\lambda = \theta = 0$ the probability density function of Modified Weighted Exponential Rayleigh $MWER$ distribution reduces to give the probability density function of Exponential distribution [7]:

$$f(x;\alpha)_E = \begin{cases} \alpha\, e^{-\alpha x} & ; x \geq 0;\ \alpha > 0 \\ \text{zero otherwise.} \end{cases}$$

Where $\alpha$ is scale parameter.

3. When $\theta = 0$ the probability density function of Modified Weighted Exponential Rayleigh $MWER$ distribution reduces to give the probability density function of Exponential Rayleigh $ER$ distribution:

$$f(x;\alpha,\lambda)_{ER} = \begin{cases} (\alpha + \lambda x)\, e^{-\left(\alpha x + \frac{\lambda}{2}x^2\right)} & ; x \geq 0; \alpha, \lambda > 0 \\ \text{zero otherwise.} \end{cases}$$

Where $\alpha$ and $\lambda$ are scale parameters.

4. When $\lambda = 0$ the probability density function of Modified Weighted Exponential Rayleigh $MWER$ distribution reduces to give the probability density function of New Weighted Exponential distribution [12]:

$$f(x;\alpha,\theta)_{NWE} = \begin{cases} \alpha(1+\theta)\, e^{-\alpha(1+\theta)x} & ; x \geq 0;\ \alpha, \theta > 0 \\ \text{zero otherwise.} \end{cases}$$

Where $\alpha$ is scale parameter and $\theta$ is shape parameter.

5. When $\alpha = 0$ the probability density function of Modified Weighted Exponential Rayleigh $MWER$ distribution reduces to give the probability density function of new distribution named Modified Weighted Rayleigh distribution, this distribution is obtain depending on definition of modified weighted version of Azzalini's (1985) as follows:

Consider a probability density function of Rayleigh distribution as in equation (3) and the survival function:

$$S(x;\lambda)_R = e^{-\frac{\lambda}{2}x^2}$$

depending on definition of modified weighted version of Azzalini's (1985) and put $M = 1 + \theta$, define the probability density function of Modified Weighted Rayleigh $MWR$ distribution as follows:

$$f(x;\lambda,\theta)_{MWR} = M\lambda x\, e^{-\frac{\lambda}{2}x^2}\, e^{-\frac{\lambda\theta}{2}x^2}$$

$$f(x;\lambda,\theta)_{MWR} = \begin{cases} \lambda(1+\theta)x\, e^{-\frac{\lambda(1+\theta)}{2}x^2} & ; x \geq 0;\ \lambda,\theta > 0 \\ \text{zero otherwise.} \end{cases}$$

Where $\lambda$ is a scale parameter and $\theta$ is the shape parameter.
Such that;

- $f(x;\lambda,\theta)_{MWR} > 0$
- $\int_0^\infty f(x;\lambda,\theta)_{MWR}\, dx = \int_0^\infty \lambda(1+\theta)x\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx$

$$= -\left[ e^{-\frac{\lambda(1+\theta)}{2}x^2} \right]_0^\infty$$

$$= 1$$

**Table 1.** Special models of the Modified Weighted Exponential Rayleigh MWER distribution

| Distribution | $f(x)$ | $F(x)$ | $S(t)$ | $h(t)$ |
|---|---|---|---|---|
| $R$ | $\lambda x e^{-\frac{\lambda}{2}x^2}$ | $1-\ e^{-\frac{\lambda}{2}x^2}$ | $e^{-\frac{\lambda}{2}t^2}$ | $\lambda t$ |
| $E$ | $\alpha\, e^{-\alpha x}$ | $1-\ e^{-\alpha x}$ | $e^{-\alpha t}$ | $\alpha$ |
| $ER$ | $(\alpha+\lambda x)\, e^{-\left(\alpha x+\frac{\lambda}{2}x^2\right)}$ | $1-\ e^{-(\alpha x+\frac{\lambda}{2}x^2)}$ | $e^{-(\alpha t+\frac{\lambda}{2}t^2)}$ | $\alpha+\lambda t$ |
| $NWE$ | $\alpha(1+\theta)\, e^{-\alpha(1+\theta)x}$ | $1-\ e^{-\alpha(1+\theta)x}$ | $e^{-\alpha(1+\theta)t}$ | $\alpha(1+\theta)$ |
| $MWR$ | $\lambda(1+\theta)x\, e^{-\frac{\lambda(1+\theta)}{2}x^2}$ | $1-e^{-\frac{\lambda(1+\theta)}{2}x^2}$ | $e^{-\frac{\lambda(1+\theta)}{2}t^2}$ | $\lambda(1+\theta)t$ |

## 3.2 Some Statistical Properties of $MWER$ Distribution

### 3.2.1 The Mode

The mode of $MWER$ distribution can be derived as follows:

$$\frac{\partial f(x;\alpha,\lambda,\theta)_{MWER}}{\partial x}$$

$$= -\left(\alpha(1+\theta)+\lambda(1+\theta)x\right)^2 e^{-\left(\alpha(1+\theta)x+\frac{\lambda(1+\theta)}{2}x^2\right)} + \lambda(1+\theta)\, e^{-\left(\alpha(1+\theta)x+\frac{\lambda(1+\theta)}{2}x^2\right)}$$

$$\left[-\left(\alpha(1+\theta)+\lambda(1+\theta)x\right)^2 + \lambda(1+\theta)\right]\ e^{-\left(\alpha(1+\theta)x+\frac{\lambda(1+\theta)}{2}x^2\right)} = 0$$

…(51)

It is clear that $\frac{\partial f(x;\alpha,\lambda,\theta)_{MWER}}{\partial x} = \left[-\{h(x;\alpha,\lambda,\theta)_{MWER}\}^2 + h'(x;\alpha,\lambda,\theta)_{MWER}\right] S(x;\alpha,\lambda,\theta)_{MWER}$

Where $h(x;\alpha,\lambda,\theta)_{MWER}$ is the hazard rate function was given in equation (49), and $S(x;\alpha,\lambda,\theta)_{MWER}$ is the survival function was given in equation (48). Since $S(x;\alpha,\lambda,\theta)_{MWER} \neq 0$ Thus dividing the equation (51) by $S(x;\alpha,\lambda,\theta)_{MWER}$ yeilds:

$$\lambda^2(1+\theta)^2x^2 + 2\,\lambda\,\alpha(1+\theta)^2x + \alpha^2(1+\theta)^2 - \lambda(1+\theta) = 0$$

Based on the law of the Constitution, we get:

$$x = \frac{-2\,\lambda\,\alpha(1+\theta)^2 \mp \sqrt{4\alpha^2\lambda^2(1+\theta)^4 - 4\,\lambda^2(1+\theta)^2(\alpha^2(1+\theta)^2 - \lambda(1+\theta))}}{2\lambda^2\,(1+\theta)^2} \qquad \ldots(52)$$

The value of $x$ is ignor when $x < 0$, suppose that $x = x_0$ that is a root of equation (52), If $\frac{\partial^2 f(x;\alpha,\lambda,\theta)_{MWER}}{\partial x^2}\Big|_{x=x_0} < 0$ the root is the local maximum. If $\frac{\partial^2 f(x;\alpha,\lambda,\theta)_{MWER}}{\partial x^2}\Big|_{x=x_0} > 0$ the root is the local minimum. If $\frac{\partial^2 f(x;\alpha,\lambda,\theta)_{MWER}}{\partial x^2}\Big|_{x=x_0} = 0$ the point is inflection.

### 3.2.2 The Median

The median of $MWER$ distribution can be obtained as follows:

$$1 - e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)} = \frac{1}{2}$$

$$\lambda(1+\theta)\,x^2 + 2\,\alpha(1+\theta)x - 2\ln 2 = 0$$

Based on law of the Constitution, we get:

$$x = \frac{-2\alpha(1+\theta) \mp \sqrt{4\alpha^2(1+\theta)^2 + 8\,\lambda(1+\theta)\,ln\,2}}{2\,\lambda(1+\theta)} \qquad \ldots(53)$$

The value of $x$ is ignor when $x < 0$.

### 3.2.3 The Moment about the Origin

The $r^{th}$ moment about the origin can be defined as:

$$E(X^r)_{MWER} = \int_0^\infty x^r(\alpha(1+\theta) + \lambda(1+\theta)x)\,e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)}\,dx \qquad \ldots(54)$$

Let

$$D(r,\alpha,\lambda,\theta) = x^r\,e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)} \qquad \ldots(55)$$

By Maclaurin series:

$$e^{-\alpha(1+\theta)x} = \sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} x^n \qquad \ldots(56)$$

Substituting equation (56) in equation (55) we get:

$$D(r,\alpha,\lambda,\theta) = \sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} x^{r+n}\,e^{-\frac{\lambda(1+\theta)}{2}x^2} \qquad \ldots(57)$$

Substituting equation (57) in equation (54) we get:

$$E(X^r)_{MWER} = \sum_{n=0}^{\infty} \frac{\left(-\alpha(1+\theta)\right)^n}{n!}\left[\int_0^\infty \alpha(1+\theta)x^{r+n}\;e^{-\frac{\lambda(1+\theta)}{2}x^2}\,dx + \int_0^\infty \lambda(1+\theta)x^{r+n+1}\;e^{-\frac{\lambda(1+\theta)}{2}x^2}\,dx\right] \qquad \ldots(58)$$

Now, solve the first integral as follows:

$$L_1 = \int_0^\infty \alpha(1+\theta)x^{r+n}\;e^{-\frac{\lambda(1+\theta)}{2}x^2}\,dx = \frac{\alpha\,2^{\frac{r+n-1}{2}}}{\lambda^{\frac{r+n+1}{2}}\,(1+\theta)^{\frac{r+n-1}{2}}}\,\Gamma\!\left(\frac{r+n+1}{2}\right) \qquad \ldots(59)$$

Now, solve the second integral as follows:

$$L_2 = \int_0^\infty \lambda(1+\theta)\,x^{r+n+1}\;e^{-\frac{\lambda(1+\theta)}{2}x^2}dx = \frac{2^{\frac{r+n}{2}}}{\lambda^{\frac{r+n}{2}}\,(1+\theta)^{\frac{r+n}{2}}}\,\Gamma\!\left(\frac{r+n+2}{2}\right) \qquad \ldots(60)$$

Substituting equations (59) and (60) in equation (58) we get:

$$E(X^r)_{MWER} = \sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{r+n}{2}}}{\lambda^{\frac{r+n}{2}} (1+\theta)^{\frac{r+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{r+n+1}{2}\right) + \Gamma\left(\frac{r+n+2}{2}\right)\right] \quad \dots(61)$$

***The Mean:*** Let $r = 1$ in equation (61) we get the first moment which is called the mean, thus:

$$E(X)_{MWER} = \sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{1+n}{2}}}{\lambda^{\frac{1+n}{2}} (1+\theta)^{\frac{1+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+2}{2}\right) + \Gamma\left(\frac{n+3}{2}\right)\right] \quad \dots(62)$$

***The Variance:*** The general form of $v(X)$ of $MWER$ distribution is defined as:

$$v(X)_{MWER} = \left[\sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{2+n}{2}}}{\lambda^{\frac{2+n}{2}} (1+\theta)^{\frac{2+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+3}{2}\right) + \Gamma\left(\frac{n+4}{2}\right)\right]\right] - \left[\sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{1+n}{2}}}{\lambda^{\frac{1+n}{2}} (1+\theta)^{\frac{1+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+2}{2}\right) + \Gamma\left(\frac{n+3}{2}\right)\right]\right]^2 \quad \dots(63)$$

**3.2.4 Coefficient of Skewness**

The general form of the Coefficient of Skewness ($C.S$) of $MWER$ distribution can be obtained by:

$$C.S_{MWER} = \frac{\sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{3+n}{2}}}{\lambda^{\frac{3+n}{2}} (1+\theta)^{\frac{3+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}} \Gamma\left(\frac{n+4}{2}\right) + \Gamma\left(\frac{n+5}{2}\right)\right]}{\left[\sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{2+n}{2}}}{\lambda^{\frac{2+n}{2}} (1+\theta)^{\frac{2+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}} \Gamma\left(\frac{n+3}{2}\right) + \Gamma\left(\frac{n+4}{2}\right)\right]\right]^{\frac{3}{2}}} \quad \dots(64)$$

**3.2.5 Coefficient of Kurtosis**

The general form of the Coefficient of Kurtosis ($C.K$) of $MWER$ distribution is given by:

$$C.K_{MWER} = \frac{\sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{4+n}{2}}}{\lambda^{\frac{4+n}{2}} (1+\theta)^{\frac{4+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}} \Gamma\left(\frac{n+5}{2}\right) + \Gamma\left(\frac{n+6}{2}\right)\right]}{\left[\sum_{n=0}^{\infty} \frac{(-\alpha(1+\theta))^n}{n!} \frac{2^{\frac{2+n}{2}}}{\lambda^{\frac{2+n}{2}} (1+\theta)^{\frac{2+n}{2}}} \left[ \frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}} \Gamma\left(\frac{n+3}{2}\right) + \Gamma\left(\frac{n+4}{2}\right)\right]\right]^{2}} - 3 \quad \dots(65)$$

**3.2.6 Moment Generating Function**

The moment generating function of $MWER$ distribution can be found as follows:

$$M_X(t)_{MWER} = \int_0^{\infty} e^{xt}\, (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx$$

$$M_X(t)_{MWER} = \int_0^{\infty} (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left((\alpha(1+\theta)-t)\,x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx \quad \dots(66)$$

Let

$$T((\alpha(1+\theta)-t),\lambda) = e^{-\left((\alpha(1+\theta)-t)\,x + \frac{\lambda(1+\theta)}{2}x^2\right)} \quad \dots(67)$$

By Maclaurin series:

$$e^{-(\alpha(1+\theta)-t)x} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-t))^n}{n!} x^n \quad \dots(68)$$

Substituting equation (68) in equation (67) we get:

$$T((\alpha(1+\theta)-t),\lambda) = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-t))^n}{n!} x^n\, e^{-\frac{\lambda(1+\theta)}{2}x^2} \quad \dots(69)$$

Substituting equation (69) in equation (66) we get:

$$M_X(t)_{MWER} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-t))^n}{n!} \left[\int_0^{\infty} \alpha(1+\theta)\, x^n\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx + \int_0^{\infty} \lambda(1+\theta)\, x^{n+1}\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx\right] \quad \ldots(70)$$

Now, solve the first integral as follows:

$$L_1 = \int_0^{\infty} \alpha(1+\theta)\, x^n\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx = \frac{\alpha\, 2^{\frac{n-1}{2}}}{\lambda^{\frac{n+1}{2}} (1+\theta)^{\frac{n-1}{2}}}\, \Gamma\left(\frac{n+1}{2}\right) \quad \ldots(71)$$

Now, solve the second integral as follows:

$$L_2 = \int_0^{\infty} \lambda(1+\theta)\, x^{n+1}\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx = \frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}} (1+\theta)^{\frac{n}{2}}}\, \Gamma\left(\frac{n+2}{2}\right) \quad \ldots(72)$$

Substituting equations (71) and (72) in equation (70) yields:

$$M_X(t)_{MWER} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-t))^n}{n!} \frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}} (1+\theta)^{\frac{n}{2}}} \left[\frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+1}{2}\right) + \Gamma\left(\frac{n+2}{2}\right)\right] \quad \ldots(73)$$

### 3.2.7 Factorial Moment Generating Function

The factorial moment generating function of $MWER$ distribution can be obtained as follows:

$$M(t)_{MWER} = E(t^x) = \int_0^{\infty} t^x\, (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx$$

$$M(t)_{MWER} = \int_0^{\infty} (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left((\alpha(1+\theta)-\ln(t))\, x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx \quad \ldots(74)$$

Let

$$U((\alpha(1+\theta) - \ln(t)), \lambda) = e^{-\left((\alpha(1+\theta)-ln(t))\, x + \frac{\lambda(1+\theta)}{2}x^2\right)} \quad \ldots(75)$$

By Maclaurin series:

$$e^{-(\alpha(1+\theta)-\ln(t))x} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\ln(t)))^n}{n!} x^n \quad \ldots(76)$$

Substituting equation (76) in equation (75) we get:

$$U((\alpha(1+\theta) - \ln(t)), \lambda) = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\ln(t)))^n}{n!} x^n\, e^{-\frac{\lambda(1+\theta)}{2}x^2} \quad \ldots(77)$$

Substituting equation (77) in equation (74) we get:

$$M(t)_{MWER} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\ln(t)))^n}{n!} \left[\int_0^{\infty} \alpha(1+\theta)\, x^n\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx + \int_0^{\infty} \lambda(1+\theta)\, x^{n+1}\, e^{-\frac{\lambda(1+\theta)}{2}x^2}\, dx\right] \quad \ldots(78)$$

Now, based on equations (71) and (72) we get:

$$M(t)_{MWER} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\ln(t)))^n}{n!} \frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}} (1+\theta)^{\frac{n}{2}}} \left[\frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+1}{2}\right) + \Gamma\left(\frac{n+2}{2}\right)\right] \quad \ldots(79)$$

### 3.2.8 Characteristic Function

The characteristic function of $MWER$ distribution can be found as follows:

$$\emptyset_X(it)_{MWER} = E(e^{itx}) = \int_0^{\infty} e^{itx}\, (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left(\alpha(1+\theta)x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx$$

$$\emptyset_X(it)_{MWER} = \int_0^{\infty} (\alpha(1+\theta) + \lambda(1+\theta)x)\, e^{-\left((\alpha(1+\theta)-\mathrm{it})\, x + \frac{\lambda(1+\theta)}{2}x^2\right)}\, dx \quad \ldots(80)$$

Let

$$C((\alpha(1+\theta) - \mathrm{it}), \lambda) = e^{-\left((\alpha(1+\theta)-it)\, x + \frac{\lambda(1+\theta)}{2}x^2\right)} \quad \ldots(81)$$

By Maclaurin series:

$$e^{-(\alpha(1+\theta)-\text{it})x} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\text{it}))^n}{n!} x^n \qquad \ldots(82)$$

Substituting equation (82) in equation (8 1) gives:

$$C((\alpha(1+\theta)-\text{it}),\lambda) = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\text{it}))^n}{n!} x^n \, e^{-\frac{\lambda(1+\theta)}{2}x^2} \qquad \ldots(83)$$

Substituting equation (83) in equation (80) gives:

$$\emptyset_X(it)_{MWER} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\text{it}))^n}{n!} \left[\int_0^{\infty} \alpha(1+\theta)\, x^n \, e^{-\frac{\lambda(1+\theta)}{2}x^2} \, dx + \int_0^{\infty} \lambda(1+\theta)\, x^{n+1} \, e^{-\frac{\lambda(1+\theta)}{2}x^2} \, dx\right] \qquad \ldots(84)$$

Now, based on equations (71) and (72) we get:

$$\emptyset_X(it)_{MWER} = \sum_{n=0}^{\infty} \frac{(-(\alpha(1+\theta)-\text{it}))^n}{n!} \frac{2^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}(1+\theta)^{\frac{n}{2}}} \left[\frac{\alpha\sqrt{\theta}}{\sqrt{2\lambda}}\, \Gamma\left(\frac{n+1}{2}\right) + \Gamma\left(\frac{n+2}{2}\right)\right] \qquad \ldots(85)$$

### 3.2.9 Quantile Function

The quantile function of $MWER$ random variable is defined as a solution of $p(x \le x_{(q)}) = F(x_{(q)})_{MWER}$ w.r.t. $x_{(q)}$, Therefore, via using the inverse transformation to equation (47), it can be found as:

$$x_{(q)} = F^{-1}(q) \qquad ; x_{(q)} > 0;\ 0 < q < 1$$

$$q = 1 - e^{-\left(\alpha(1+\theta)x_{(q)} + \frac{\lambda(1+\theta)}{2}x_{(q)}^2\right)}$$

$$1 - q = e^{-\left(\alpha(1+\theta)x_{(q)} + \frac{\lambda(1+\theta)}{2}x_{(q)}^2\right)}$$

$$ln\,(1-q) = -(\alpha(1+\theta)x_{(q)} + \frac{\lambda(1+\theta)}{2}\, x_{(q)}^2)$$

$$\lambda\,(1+\theta)x_{(q)}^2 + 2\alpha(1+\theta)x_{(q)} + 2\,ln\,(1-q) = 0$$

Based on law of the constitution, we get:

$$x_{(q)} = \frac{-2\alpha(1+\theta) \mp \sqrt{4\alpha^2(1+\theta)^2 - 8\,\lambda\,(1+\theta)\,ln\,(1-q)}}{2\,\lambda\,(1+\theta)} \qquad \ldots(86)$$

The values of $x_{(q)}$ will be ignored when $x_{(q)} < 0$.

## 4.Conclusions

In this paper, introduce Exponential Rayleigh $ER$ distribution depending on mixed between cumulative distribution function of Exponential and Rayleigh distribution, as well as introduce a new class depending on a modified weighted version of Azzalini's (1985) named Modified Weighted Exponential Rayleigh $MWER$ distribution, such that the Exponential Rayleigh $ER$ distribution is special case of Modified Weighted Exponential Rayleigh $MWER$ distribution and provide some special models of the $MWER$ distribution. Different statistical properties such as the mode, the median, the $r^{th}$ moment about the origin, the moment generating function, factorial moment generating function, the characteristic function and quantile function are discuses and study for these two distributions.

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