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Extend Nearly Pseudo Quasi-2-Absorbing submodules(II)

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Abstract

The concept of the Extend Nearly Pseudo Quasi-2-Absorbing submodules was recently introduced by Omar A. Abdullah and Haibat K. Mohammadali in 2022, where he studies this concept and it is relationship to previous generalizationsm especially 2-Absorbing submodule and Quasi-2-Absorbing submodule, in addition to studying the most important Propositions, charactarizations and Examples. Now in this research, which is considered a continuation of the definition that was presented earlier, which is the Extend Nearly Pseudo Quasi-2-Absorbing submodules, we have completed the study of this concept in multiplication modules. And the relationship between the Extend Nearly Pseudo Quasi-2-Absorbing submodule and Extend Nearly Pseudo Quasi-2-Absorbing ideal. We also studied more result of Extend Nearly Pseudo Quasi-2-Absorbing submodule in multiplication module. In the end, we obtained new Propositions and distinguished results in studying this concept.

Keywords: EXNPQ-2-Absorbing submodule, multiplication modules, non-singular modules, faithful module, projective module, good rings and local rings.

1. Introduction

In recent years, many generalizations have appeared about the concept of the 2-Absorbing submodule such as (Pseudo Quasi-2-Absorbing, Nearly Quasi-2-Absorbing and Soc-QP2-Absorbing) submodules see [1, 2 and 3]. The concept of the Extend Nearly Pseudo Quasi-2-Absorbing submodules is one of the recent generalizations that were recently introduced by us, researchers, Omar and Haibat see [4]. Where we dealt with in the previous research basic properties with relationships. The present work is divided into three parts. Part one is preliminaries part, we present in this part of the work the necessary background needed later consisting of definitions, propositions and remarks (without proof) and in the second part we introduced and studied the concept of the Extend Nearly Pseudo Quasi-2-Absorbing submodule in multiplication module. Also we got a lot of important results like Propositions 3.2, 3.6 and 3.7. In the end we presented more result of Extend Nearly Pseudo Quasi-2-Absorbing submodule in multiplication modules. See Propositions 4.1, 4.2 and 4.10.

2. Preliminaries

The following list some fundamental definitions and notations that will be utilized in this paper.

Definition 2.1[4].

A proper submodule V of an Ʀ-module Ɯ is said to be Extend Nearly Pseudo Quasi-2-Absorbing (for short EXNPQ2AB) submodule of Ɯ if whenever $ɑƃc\mathfrak{x} \in V$, where $ɑ, ƃ, c \in$ Ʀ, $\mathfrak{x} \in$ Ɯ, implies that either $ɑc\mathfrak{x} \in V + soc(Ɯ) + J(Ɯ)$ or $ƃc\mathfrak{x} \in V + soc(Ɯ) + J(Ɯ)$ or $ɑƃ\mathfrak{x} \in V + soc(Ɯ) + J(Ɯ)$. And an ideal Ƥ of a ring Ʀ is called EXNPQ2AB ideal of Ʀ, if Ƥ is an EXNPQ2AB Ʀ-submodule of an Ʀ-module Ʀ.

Definition 2.2[5].

An Ʀ-module Ɯ is multiplicatiion, if every submodule V of Ɯ is of the form $V =$ ƤƜ for some ideal Ƥ of Ʀ. Equivalently Ɯ is a multiplicatiion Ʀ-module if every submodule V of Ɯ of the form $V = [V :_{Ʀ} Ɯ]Ɯ$.

Definition 2.3[6].

An Ʀ-module Ɯ is faithful if $ann_{Ʀ}(Ɯ) = (0)$, where $ann_{Ʀ}(Ɯ) = \{r \in Ʀ : rw = (0)\}$.

Definition 2.4[6].

An Ʀ-module Ɯ is finitely generated if $Ɯ = Ʀx_1 + Ʀx_2 + \cdots + Ʀx_n$ for $x_1, x_2, \ldots, x_n \in Ɯ$.

Definition 2.5[7].

An Ʀ-module Ɯ is called concellation module if ƤƜ = ƁƜ for any ideals Ƥ and Ɓ of Ʀ implies that Ƥ = Ɓ.

Lemma 2.6[5, Coro. (2.14) (i)].

Let Ɯ be faithful multiplication Ʀ-module, then $soc(Ʀ)Ɯ = soc(Ɯ)$.

Lemma 2.7 [8, Coro. (2.14) (i)].

Let Ɯ be faithful multiplication Ʀ-module, then $J(Ʀ)Ɯ = J(Ɯ)$.

Definition 2.8[6].

An Ʀ-module Ɯ is a projective if *for any* Ʀ-epimorphism f from an Ʀ-module Ɯ on to an Ʀ-module $\overline{Ɯ}$ and for any homomorphism g from an Ʀ-module $\overline{\overline{Ɯ}}$ to $\overline{Ɯ}$, there exists a homomorphism h from $\overline{\overline{Ɯ}}$ to Ɯ such that $f \circ h = g$.

Lemma 2.9[6, Theo. (9.2.1) (g)].

For any projective Ʀ-module Ɯ, we have $J(Ʀ)Ɯ = J(Ɯ)$.

Lemma 2.10[8, Prop. (3.24)].

For any projective Ʀ-module Ɯ, we have $soc(Ʀ)Ɯ = soc(Ɯ)$.

Remark 2.11[6].

Ʀ is a good ring if $J(Ʀ)Ɯ = J(Ɯ)$.

Definition 2.12[9].

Aring Ʀ is Artinian if Ʀ satisfies (DCC) is an ideals of Ʀ, that is if $\{Ƥ_{\propto}\}_{\propto \in \Lambda}$ is a family of ideals of Ʀ such that $Ƥ_1 \supseteq Ƥ_2 \supseteq \cdots$, then $\exists ɱ \in Ζ^{+}$ such that $Ƥn = Ƥɱ$ for any $n \geq ɱ$.

Definition 2.13[10].

Aring Ʀ is said to be local ring Ʀ if Ʀ has a unique maximal ideal.

Lemma 2.14[6, Coro. (9.7.3) (b)].

If Ʀ is an Artinian ring, then Ʀ is a good ring.

Lemma 2.15[11, Prop. (1.12)].

If Ɯ is an Ʀ-module over local ring Ʀ, then $J(Ʀ)Ɯ = J(Ɯ)$.

Definition 2.16[12].

An Ʀ-module Ɯ is *non-singular* if $Ȥ(Ɯ) = Ɯ$, where $Ȥ(Ɯ) = \{x \in Ɯ : xƤ = (0)$, for some essential ideal Ƥ of R$\}$.

Lemma 2.17[12, Coro. (1.26)].

Let Ɯ be is a non-singular Ʀ-modules, then $soc(Ʀ)Ɯ = soc(Ɯ)$.

Lemma 2.18[13, Coro of Theo. (9)].

Let Ɯ be a finitely generated multiplication Ʀ-module Ƥ and Ɓ are ideals of Ʀ. Then $ƤƜ \subseteq ƁƜ$ if and only if $Ƥ \subseteq Ɓ + ann_{Ʀ}(Ɯ)$.

Definition 2.19[14].

An Ʀ-module Ɯ is called a Z-regular if for each $e \in Ɯ$ *there exists* $f \in Ɯ' = Hom_{Ʀ}(Ɯ, Ʀ)$ such that $e = f(e)e$.

Definition 2.20[15].

An Ʀ-module Ɯ is called weak cancellation if $ƁƜ = ƤƜ$, implies that $Ɓ + ann_{Ʀ}(Ɯ) = Ƥ + ann_{Ʀ}(Ɯ)$ for Ɓ, Ƥ are ideals in Ʀ.

Lemma 2.21[8, Prop. (3.25)].

Let Ɯ be a Z-regular Ʀ-module, then $soc(Ɯ) = soc(Ʀ)Ɯ$.

Lemma 2.22[7, Prop. (3.9)].

If Ɯ is a multiplication Ʀ-module, then Ɯ is finitely generated if and only if Ɯ is weak cancellation.

Lemma 2.23[7, Prop. (3.1)].

If Ɯ is a multiplication Ʀ-module, then Ɯ is concellation if and only if Ɯ is faithful finitely generated.

Proposition 2.24[4, Prop. (3.4)].

A proper submodule V of Ɯ is EXNPQ2AB submodule of Ɯ if and only if $ɑɓc\mathcal{L} \subseteq V$, for ɑ, ɓ, $c \in$ Ʀ and $\mathcal{L}$ is a submodule of Ɯ, implies that either $ɑc\mathcal{L} \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɓc\mathcal{L} \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɑɓ\mathcal{L} \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proposition 2.25[4, Prop. (3.5)].

Let Ɯ be module and $V \subset Ɯ$. Then V is EXNPQ2AB submodule of Ɯ if and only if for every submodule A of Ɯ and for every ideals $Ƥ_1, Ƥ_2, Ƥ_3$ of Ʀ such that $Ƥ_1Ƥ_2Ƥ_3A \subseteq V$, implies that either $Ƥ_1Ƥ_2A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proposition 2.26[4, Coro. (3.7)].

Let Ɯ be an Ʀ-module and $V \subset Ɯ$. Then V is EXNPQ2AB submodule of Ɯ if and only if for each $r \in Ʀ$, $x \in Ɯ$ and every ideals $Ƥ, J$ of Ʀ with $rƤJx \subseteq V$, implies that either $rƤx \subseteq V + soc(Ɯ) + J(Ɯ)$ or $rJx \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ƤJx \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proposition 2.27[4, Coro. (3.8)].

Let Ɯ be an Ɽ-module and $V \subset$ Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for every ideals $Ƥ_1, Ƥ_2, Ƥ_3$ of Ɽ and $x \in$ Ɯ such that $Ƥ_1Ƥ_2Ƥ_3x \subseteq V$ implies that either $Ƥ_1Ƥ_2x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proposition 2.28[4, Coro. (3.9)].

Let Ɯ be an Ɽ-module and $V \subset$ Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for any $r, s \in$ Ɽ and any ideal Ƥ of Ɽ and every submodule A of Ɯ with $rsƤA \subseteq V$ implies that either $rsA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $rƤA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $sƤA \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proposition 2.29[4, Coro. (3.10)].

Let Ɯ be an Ɽ-module and $V \subset$ Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if for each $r \in$ Ɽ and any ideals $Ƥ, J$ of Ɽ and every submodule A of Ɯ with $rƤJA \subseteq V$ implies that either $rƤA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $rJA \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ƤJA \subseteq V + soc(Ɯ) + J(Ɯ)$.

3. Main Results

In this part we introduced some characterizations of Extend Nearly Pseudo Quasi-2-Absorbing submodules in multiplication modules.

Proposition 3.1

Let Ɯ be a multiplication Ɽ-module and $V \neq$ Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if whenever $Ꜧ_1Ꜧ_2Ꜧ_3A \subseteq V$ for some submodules $Ꜧ_1$,$Ꜧ_2$,$Ꜧ_3, A$ of Ɯ, implies that either $Ꜧ_1Ꜧ_2A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ꜧ_1Ꜧ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ꜧ_2Ꜧ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proof.

(⟹) Let $Ꜧ_1Ꜧ_2Ꜧ_3A \subseteq V$ for some submodules $Ꜧ_1$, $Ꜧ_2$, $Ꜧ_3$, A of Ɯ. Since Ɯ is a multiplication, then $Ꜧ_1 = Ƥ_1Ɯ$, $Ꜧ_2 = Ƥ_2Ɯ$, $Ꜧ_3 = Ƥ_3Ɯ$ and $A = Ƥ_4Ɯ$ for some ideals $Ƥ_1$, $Ƥ_2$, $Ƥ_3$ and $Ƥ_4$of Ɽ. That is $Ꜧ_1Ꜧ_2Ꜧ_3A = Ƥ_1Ƥ_2Ƥ_3(Ƥ_4Ɯ) \subseteq V$. But V is EXNPQ2AB submodule of Ɯ, hence from **Proposition 2.25** we get either $Ƥ_1Ƥ_2(Ƥ_4Ɯ) \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_3(Ƥ_4Ɯ) \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3(Ƥ_4Ɯ) \subseteq V + soc(Ɯ) + J(Ɯ)$. Next, following either $Ꜧ_1Ꜧ_2A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ꜧ_1Ꜧ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ꜧ_2Ꜧ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$. Ƥ

(⟸) Let $Ƥ_1Ƥ_2Ƥ_3A \subseteq V$ for $Ƥ_1, Ƥ_2, Ƥ_3$ are ideals of Ɽ and A is a submodule of Ɯ. Put $Ꜧ_1 = Ƥ_1Ɯ$, $Ꜧ_2 = Ƥ_2Ɯ$ and $Ꜧ_3 = Ƥ_3Ɯ$. That is $Ꜧ_1Ꜧ_2Ꜧ_3A \subseteq V$. Now, by hypotheses either $Ꜧ_1Ꜧ_2A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ꜧ_1Ꜧ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ꜧ_2Ꜧ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$, thus $Ƥ_1Ƥ_2A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3A \subseteq V + soc(Ɯ) + J(Ɯ)$. Therefore by **Proposition 2.25** V is EXNPQ2AB submodule of Ɯ.

Proposition 3.2

Let Ɯ be a multiplication Ɽ-module and $V \neq$ Ɯ. Then V is EXNPQ2AB submodule of Ɯ if and only if whenever $Ƒ_1Ƒ_2Ƒ_3x \subseteq V$ for some submodules $Ƒ_1$, $Ƒ_2$, $Ƒ_3$ of Ɯ,$x \in$ Ɯ, then either $Ƒ_1Ƒ_2x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƒ_1Ƒ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƒ_2Ƒ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proof.

(⟹) Let V is EXNPQ2AB submodule of Ɯ and $Ƒ_1Ƒ_2Ƒ_3x \subseteq V$ for some submodules $Ƒ_1$,$Ƒ_2$, $Ƒ_3$ of Ɯ and $x \in$ Ɯ. Since Ɯ is a multiplication, then $Ƒ_1 = Ƥ_1Ɯ$, $Ƒ_2 = Ƥ_2Ɯ$ and $Ƒ_3 = Ƥ_3Ɯ$ for some ideals $Ƥ_1$, $Ƥ_2$and $Ƥ_3$ of $Ʀ$. That is $Ƒ_1Ƒ_2Ƒ_3x = Ƥ_1Ƥ_2Ƥ_3x \subseteq V$. But V is EXNPQ2AB submodule of $Ɯ$, hence from **Proposition 2.27** we get either $Ƥ_1Ƥ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_2x \subseteq V + soc(Ɯ) + J(Ɯ)$. Next, following either $Ƒ_1Ƒ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƒ_2Ƒ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƒ_1Ƒ_2x \subseteq V + soc(Ɯ) + J(Ɯ)$.

($\Leftarrow$) Let $Ƥ_1Ƥ_2Ƥ_3x \subseteq V$ for $Ƥ_1, Ƥ_2, Ƥ_3$ are ideals of $Ʀ$ and $x \in Ɯ$. Put $Ƒ_1 = Ƥ_1Ɯ$, $Ƒ_2 = Ƥ_2Ɯ$ and $Ƒ_3 = Ƥ_3Ɯ$. That is $Ƒ_1Ƒ_2Ƒ_3x \subseteq V$. Now, by hypotheses either $Ƒ_1Ƒ_2x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƒ_1Ƒ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƒ_2Ƒ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$, thus $Ƥ_1Ƥ_2x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3x \subseteq V + soc(Ɯ) + J(Ɯ)$. Therefore by **Proposition 2.27** V is EXNPQ2AB submodule of $Ɯ$.

Proposition 3.3

Let $Ɯ$ be a multiplication $Ʀ$-module and $V \neq Ɯ$. Then V is EXNPQ2AB submodule of $Ɯ$ if and only if whenever $ɱ_1ɱ_2ɱ_3Ꜧ \subseteq V$ for some $ɱ_1, ɱ_2, ɱ_3 \in Ɯ$, $Ꜧ$ is a submodule of $Ɯ$, implies that either $ɱ_1ɱ_2Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɱ_1ɱ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɱ_2ɱ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$.

Proof.

($\Rightarrow$) Let $ɱ_1ɱ_2ɱ_3Ꜧ \subseteq V$ for some $ɱ_1, ɱ_2, ɱ_3 \in Ɯ$ and $Ꜧ$ is a submodule of $Ɯ$. That is $(ɱ_1)(ɱ_2)(ɱ_3)Ꜧ \subseteq V$ Since $Ɯ$ is a multiplication, then $(ɱ_1) = Ƥ_1Ɯ$, $(ɱ_2) = Ƥ_2Ɯ$, $(ɱ_3) = Ƥ_3Ɯ$ and $Ꜧ = Ƥ_4Ɯ$ for some ideals $Ƥ_1$, $Ƥ_2$, $Ƥ_3$ and $Ƥ_4$ of $Ʀ$. That is $(ɱ_1)(ɱ_2)(ɱ_3)Ꜧ = Ƥ_1Ƥ_2Ƥ_3(Ƥ_4Ɯ) \subseteq V$. But V is EXNPQ2AB submodule of $Ɯ$, hence from **Proposition 2.25** we get either $Ƥ_1Ƥ_3(Ƥ_4Ɯ) \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3(Ƥ_4Ɯ) \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_2(Ƥ_4Ɯ) \subseteq V + soc(Ɯ) + J(Ɯ)$. Next, following either $ɱ_1ɱ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɱ_2ɱ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɱ_1ɱ_2Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$.

($\Leftarrow$) Let $Ƥ_1Ƥ_2Ƥ_3Ꜧ \subseteq V$ for $Ƥ_1, Ƥ_2, Ƥ_3$ are ideals of $Ʀ$ and $Ꜧ$ is a submodule of $Ɯ$. Put $(ɱ_1) = Ƥ_1Ɯ$, $(ɱ_2) = Ƥ_2Ɯ$ and $(ɱ_3) = Ƥ_3Ɯ$. That is $(ɱ_1)(ɱ_2)(ɱ_3)Ꜧ \subseteq V$. That is $ɱ_1ɱ_2ɱ_3Ꜧ \subseteq V$. Now, by hypotheses either $ɱ_1ɱ_2Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɱ_1ɱ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $ɱ_2ɱ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$, thus $(ɱ_1)(ɱ_2)Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $(ɱ_1)(ɱ_3)Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $(ɱ_2)(ɱ_3)Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$. Then $Ƥ_1Ƥ_2Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_1Ƥ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$ or $Ƥ_2Ƥ_3Ꜧ \subseteq V + soc(Ɯ) + J(Ɯ)$. Therefore by **Proposition 2.25** V is EXNPQ2AB submodule of $Ɯ$.

Remark 3.4

If V is an EXNPQ2AB submodule of an $Ʀ$-module $Ɯ$, then $[V:_Ʀ Ɯ]$ need not to be EXNPQ2AB ideal of $Ʀ$.

The following example shows that:

Let $Ɯ = Z_{48}$, $Ʀ = Z$ and the submodule $V = \langle\overline{16}\rangle$ is EXNPQ2AB submodule of $Ɯ$, since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{8}\rangle \cap Z_{48} = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. Then $\langle\overline{16}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\overline{16}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$, hence for all $ɑ, б, e \in Z$ and $ɱ \in Z_{48}$ $such\ that\ ɑбem \in \langle\overline{16}\rangle$, implies that either $ɑбɱ \in \langle\bar{2}\rangle$ or $ɑeɱ \in \langle\bar{2}\rangle$ or $бeɱ \in \langle\bar{2}\rangle$. But $[\langle\overline{16}\rangle:_Ʀ Z_{48}] = 16Z$ is not an

EXNPQ2AB *ideal* of Z, since $2.4.2.1 \in 16\mathbb{Z}$, for $1,2,4 \in \mathbb{Z}$, implies that $2.4.1 \notin 16\mathbb{Z}$ and $2.2.1 \notin 16\mathbb{Z}$ and $4.2.1 \notin 16\mathbb{Z}$.

Under certain conditions, the above observation is fulfilled.

Proposition 3.5

Let $F \neq W$ and W is faithful multiplication R-module. Then F is EXNPQ2AB submodule of W if and only if $[F:_R W]$ is EXNPQ2AB ideal of R.

Proof.

$(\Rightarrow)$ Let F is EXNPQ2AB submodule of W, and $P_1P_2P_3P_4 \subseteq [F:_R W]$ for some ideals P_1, P_2, P_3 and P_4 of R, then $P_1P_2P_3P_4W \subseteq F$. But W is a multiplication, then $P_1P_2P_3P_4W = F_1F_2F_3F_4 \subseteq F$, by taking $P_1W = F_1$, $P_2W = F_2$, $P_3W = F_3$ and $P_4W = F_4$. But F is EXNPQ2AB submodule of W, then by **Proposition 3.1** either $F_1F_3F_4 \subseteq F + soc(W) + J(W)$ or $F_2F_3F_4 \subseteq F + soc(W) + J(W)$or $F_1F_2F_4 \subseteq F + soc(W) + J(W)$. Since W is multiplication, then $F = [F:_R W]W$, and since W is faithful multiplication, then by **Lemma 2.6** $soc(W) = soc(R)W$ and by **Lemma 2.7** $J(W) = J(R)W$. Thus either $P_1P_3P_4W \subseteq [F:_R W]W + soc(R)W + J(R)W$ or $P_2P_3P_4W \subseteq [F:_R W]W + soc(R)W + J(R)W$ or $P_1P_2P_4W \subseteq [F:_R W]W + soc(R)W + J(R)W$. Hence either $P_1P_3P_4 \subseteq [F:_R W] + soc(R) + J(R)$ or $P_2P_3P_4 \subseteq [F:_R W] + soc(R) + J(R)$ or $P_1P_2P_4 \subseteq [F:_R W] + soc(R) + J(R)$. Therefore $[F:_R W]$ is EXNPQ2AB ideal of R.

$(\Leftarrow)$ *S*uppose that $[F:_R W]$ is EXNPQ2AB ideal of R, and $rstA \subseteq F$ for r, s, $t \in R$ and A is a submodule of W, since W is a multiplication, then $A = PW$ for some ideal P of R, that is $rstPW \subseteq F$, implies that $rstP \subseteq [F:_R W]$, but $[F:_R W]$ is EXNPQ2AB ideal of R, then by **Proposition 2.24** either $rsP \subseteq [F:_R W] + soc(R) + J(R)$ or $rtP \subseteq [F:_R W] + soc(R) + J(R)$ or $stP \subseteq [F:_R W] + soc(R) + J(R)$. Thus either $rsPW \subseteq [F:_R W]W + soc(R)W + J(R)W$ or $rtPW \subseteq [F:_R W]W + soc(R)W + J(R)W$ or $stPW \subseteq [F:_R W]W + soc(R)W + J(R)W$. Since W is a faithful multiplication, then $[F:_R W]W = F$ and by **Lemma 2.6** and **Lemma 2.7** either $rsA \subseteq F + soc(W) + J(W)$ or $rtA \subseteq F + soc(W) + J(W)$ or $stA \subseteq F + soc(W) + J(W)$. Thus by **Proposition 2.24** F is EXNPQ2AB submodule of W.

Proposition 3.6

Let $F \neq W$ and W is multiplication projective R-module. Then F is EXNPQ2AB submodule of W if and only if $[F:_R W]$ is EXNPQ2AB ideal of R.

Proof.

$(\Rightarrow)$ Assume that F is EXNPQ2AB submodule of W, and $P_1P_2P_3b \subseteq [F:_R W]$ for some ideals P_1, P_2, P_3 of R and $b \in R$, then $P_1P_2P_3(bW) \subseteq F$. But F is EXNPQ2AB submodule of W, then by **Proposition 2.25** either $P_1P_3bW \subseteq F + soc(W) + J(W)$ or $P_2P_3bW \subseteq F + soc(W) + J(W)$ or $P_1P_2bW \subseteq F + soc(W) + J(W)$. Since W is multiplication, then$F = [F:_R W]W$, and since W is projective R-module W, then by **Lemma 2.10** $soc(W) = soc(R)W$ and by **Lemma 2.9** $J(W) = J(R)W$. Thus either $P_1P_3bW \subseteq [F:_R W]W + soc(R)W + J(R)W$ or $P_2P_3bW \subseteq [F:_R W]W + soc(R)W + J(R)W$ or $P_1P_2bW \subseteq [F:_R W]W + soc(R)W + J(R)W$. Hence $P_1P_3b \subseteq [F:_R W] + soc(R) + J(R)$or $P_2P_3b \subseteq [F:_R W] + soc(R) + J(R)$ or $P_1P_2b \subseteq [F:_R W] + soc(R) + J(R)$. Therefore by **Proposition 2.27** $[F:_R W]$ is EXNPQ2AB ideal of R.

($\Longleftarrow$) Suppose that $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ, and $rs\text{Ƥ}A \subseteq \text{Ƒ}$ for r, $s \in$ Ɽ and some submodule A of W and for some ideal Ƥ of Ɽ since W is a multiplication, then $A = J\text{W}$ for some ideal J of Ɽ, that is $rs\text{Ƥ}J\text{W} \subseteq \text{Ƒ}$, implies that $rs\text{Ƥ}J \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]$, but $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB $ideal$ of Ɽ, then by **Proposition 2.28** either $rsJ \subseteq [\text{Ƒ}:_{\text{R}} \text{W}] + soc(\text{Ɽ}) + J(\text{Ɽ})$ or $r\text{Ƥ}J \subseteq [\text{Ƒ}:_{\text{R}} \text{W}] + soc(\text{Ɽ}) + J(\text{Ɽ})$ or $s\text{Ƥ}J \subseteq [\text{Ƒ}:_{\text{R}} \text{W}] + soc(\text{Ɽ}) + J(\text{Ɽ})$. Thus either $rsJ\text{W} \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W}$ or $r\text{Ƥ}J\text{W} \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W}$ or $s\text{Ƥ}J\text{W} \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W}$. Hence by **Lemma 2.10** and **Lemma 2.9** either $rsA \subseteq \text{Ƒ} + soc(\text{W}) + J(\text{W})$ or $r\text{Ƥ}A \subseteq \text{Ƒ} + soc(\text{W}) + J(\text{W})$ or $s\text{Ƥ}A \subseteq \text{Ƒ} + soc(\text{W}) + J(\text{W})$. Thus by **Proposition 2.28** Ƒ is EXNPQ2AB submodule of W.

Proposition 3.7

Let Ƒ $\neq$ W and W is non-singular multiplication Ɽ-module W over an a good $ring$ Ɽ. Then Ƒ is EXNPQ2AB submodule of W if and only if $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ.

Proof.

($\Longrightarrow$) Let $abct \in [\text{Ƒ}:_{\text{R}} \text{W}]$ for a, b, $c, t \in$ Ɽ, then $abc(t\text{W}) \subseteq \text{Ƒ}$. But Ƒ is EXNPQ2AB submodule of W, then by **Proposition 2.24** either $ab(t\text{W}) \subseteq \text{Ƒ} + (soc(\text{W}) + J(\text{W}))$ or $ac(t\text{W}) \subseteq \text{Ƒ} + (soc(\text{W}) + J(\text{W}))$ or $bc(t\text{W}) \subseteq \text{Ƒ} + (soc(\text{W}) + J(\text{W}))$. Since W is multiplication, then $\text{Ƒ} = [\text{Ƒ}:_{\text{R}} \text{W}]\text{W}$ and since W is non-singular multiplication, then by **Lemma 2.17** $soc(\text{W}) = soc(\text{Ɽ})\text{W}$ and Ɽ is a good ring then by **Remark 2.11** $J(\text{W}) = J(\text{Ɽ})\text{W}$. Thus either $ab(t\text{W}) \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + (soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W})$ or $(t\text{W}) \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + (soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W})$ or $ac(t\text{W}) \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + (soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W})$, then either $abt \in [\text{Ƒ}:_{\text{R}} \text{W}] + \left(soc(\text{Ɽ}) + J(\text{Ɽ})\right)$ or $bct \in [\text{Ƒ}:_{\text{R}} \text{W}] + \left(soc(\text{Ɽ}) + J(\text{Ɽ})\right)$ or $act \in [\text{Ƒ}:_{\text{R}} \text{W}] + \left(soc(\text{Ɽ}) + J(\text{Ɽ})\right)$. Hence by **Proposition 2.24** $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ.

($\Longleftarrow$) Suppose that $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ, and $abcx \in \text{Ƒ}$ for $a, b, c \in$ Ɽ, $x \in$ W, hence $abc(x) \subseteq \text{Ƒ}$. Since W is a multiplication, then $(x) = J\text{W}$ for some ideal J of Ɽ, that is $abcJ\text{W} \subseteq \text{Ƒ}$, implies that $abcJ \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]$, but $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ, then by definition either $abJ \subseteq [\text{Ƒ}:_{\text{R}} \text{W}] + soc(\text{Ɽ}) + J(\text{Ɽ})$ or $acJ \subseteq [\text{Ƒ}:_{\text{R}} \text{W}] + soc(\text{Ɽ}) + J(\text{Ɽ})$ or $bcJ \subseteq [\text{Ƒ}:_{\text{R}} \text{W}] + soc(\text{Ɽ}) + J(\text{Ɽ})$. Thus either $abJ\text{W} \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W}$ or $acJ\text{W} \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W}$ or $bcJ\text{W} \subseteq [\text{Ƒ}:_{\text{R}} \text{W}]\text{W} + soc(\text{Ɽ})\text{W} + J(\text{Ɽ})\text{W}$. Hence by **Lemma 2.17** and **Remark 2.11** either $ab(x) \subseteq \text{Ƒ} + soc(\text{W}) + J(\text{W})$ or $ac(x) \subseteq \text{Ƒ} + soc(\text{W}) + J(\text{W})$ or $bc(x) \subseteq \text{Ƒ} + soc(\text{W}) + J(\text{W})$. Next, follows either $abx \in \text{Ƒ} + soc(\text{W}) + J(\text{W})$ or $acx \in \text{Ƒ} + soc(\text{W}) + J(\text{W})$ or $bcx \in \text{Ƒ} + soc(\text{W}) + J(\text{W})$.Therefore Ƒ is EXNPQ2AB submodule of W.

As a direct application of **Proposition 3.7**, we get the following corollary:

Corollary 3.8

Let Ƒ $\neq$ W and W is non-singular multiplication Ɽ-module W over Artinian $ring$ Ɽ. Then Ƒ is EXNPQ2AB submodule of W if and only if $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ.

By Proof of **Proposition 3.7** and using **Lemma 2.15** we get:

Proposition 3.9

Let Ƒ $\neq$ W and W is non-singular multiplication Ɽ-module W over local ring Ɽ. Then Ƒ is EXNPQ2AB submodule of W if and only if $[\text{Ƒ}:_{\text{R}} \text{W}]$ is EXNPQ2AB ideal of Ɽ.

Proposition 3.10

Let $Ƒ \neq Ŵ$ and $Ŵ$ is Z-regular multiplication Ʀ-module $Ŵ$ over an a good ring Ʀ. Then $Ƒ$ is EXNPQ2AB submodule of $Ŵ$ if and only if $[Ƒ:_{R} Ŵ]$ is EXNPQ2AB ideal of Ʀ. Ƥ

Proof.

($\Longrightarrow$) Let $rstƤ \subseteq [Ƒ:_{R} Ŵ]$ for $r,\ s,\ t \in Ʀ$ and Ƥ is an ideal of Ʀ, then $rstƤŴ \subseteq Ƒ$. But $Ƒ$ is EXNPQ2AB submodule of $Ŵ$, then either $rsƤŴ \subseteq Ƒ + (soc(Ŵ) + J(Ŵ))$ or $rtƤŴ \subseteq Ƒ + (soc(Ŵ) + J(Ŵ))$ or $stƤŴ \subseteq Ƒ + (soc(Ŵ) + J(Ŵ))$. Since $Ŵ$ is multiplication, then $Ƒ = [Ƒ:_{R} Ŵ]Ŵ$ and since $Ŵ$ is a Z-regular, then by **Lemma 2.21** $soc(Ŵ) = soc(Ʀ)Ŵ$ and Ʀ is a good ring then by **Remark 1.11** $J(Ŵ) = J(Ʀ)Ŵ$. Thus either $rsƤŴ \subseteq [Ƒ:_{R} Ŵ]Ŵ + (soc(Ʀ)Ŵ + J(Ʀ)Ŵ)$ or $rtƤŴ \subseteq [Ƒ:_{R} Ŵ]Ŵ + (soc(Ʀ)Ŵ + J(Ʀ)Ŵ)$ or $stƤŴ \subseteq [Ƒ:_{R} Ŵ]Ŵ + (soc(Ʀ)Ŵ + J(Ʀ)Ŵ)$, it follows that either $rsƤ \subseteq [Ƒ:_{R} Ŵ] + \big(soc(Ʀ) + J(Ʀ)\big)$ or $rtƤ \subseteq [Ƒ:_{R} Ŵ] + \big(soc(Ʀ) + J(Ʀ)\big)$ or $stƤ \subseteq [Ƒ:_{R} Ŵ] + \big(soc(Ʀ) + J(Ʀ)\big)$. Hence by **Proposition 2.24** $[Ƒ:_{R} Ŵ]$ is EXNPQ2AB ideal of Ʀ.

($\Longleftarrow$) Suppose that $[Ƒ:_{R} Ŵ]$ is EXNPQ2AB ideal of Ʀ, and $rstA \subseteq Ƒ$ for $r, s, t \in Ŵ$ and A is a submodule of $Ŵ$. Since $Ŵ$ is a multiplication, then $A = ƤŴ$, that is $rstA = rstƤŴ \subseteq Ƒ$, implies that $rstƤ \subseteq [Ƒ:_{R} Ŵ]$, but $[Ƒ:_{R} Ŵ]$ is EXNPQ2AB $ideal$ of Ʀ, then by **Proposition 2.24** either $rsƤ \subseteq [Ƒ:_{R} Ŵ] + soc(Ʀ) + J(Ʀ)$ or $rtƤ \subseteq [Ƒ:_{R} Ŵ] + soc(Ʀ) + J(Ʀ)$ or $stƤ \subseteq [Ƒ:_{R} Ŵ] + soc(Ʀ) + J(Ʀ)$.Thus either $rsƤŴ \subseteq [Ƒ:_{R} Ŵ]Ŵ + soc(Ʀ)\,Ŵ + J(Ʀ)\,Ŵ$ or $rtƤŴ \subseteq [Ƒ:_{R} Ŵ]\,Ŵ + soc(Ʀ)Ŵ + J(Ʀ)Ŵ$ or $stƤŴ \subseteq [Ƒ:_{R} Ŵ]Ŵ + soc(Ʀ)Ŵ + J(Ʀ)Ŵ$. Hence by **Lemma 2.21** and **Remark 2.11** either $rsA \subseteq Ƒ + soc(Ŵ) + J(Ŵ)$ or $rtA \subseteq Ƒ + soc(Ŵ) + J(Ŵ)$ or $stA \subseteq Ƒ + soc(Ŵ) + J(Ŵ)$. Therefore $Ƒ$ is EXNPQ2AB submodule of $Ŵ$.

As a direct application of **Proposition 3.10**, we get the following corollary:

Corollary 3.11

Let $Ƒ \neq Ŵ$ and $Ŵ$ is Z-regular multiplication Ʀ-module $Ŵ$ over Artinian ring Ʀ. Then $Ƒ$ is EXNPQ2AB submodule of $Ŵ$ if and only if $[Ƒ:_{R} Ŵ]$ is EXNPQ2AB ideal of Ʀ.

By Proof of **Proposition 3.10** and using **Lemma 2.15** we get:

Proposition 3.12

Let $Ƒ \neq Ŵ$ and $Ŵ$ Z-regular multiplication Ʀ-module $Ŵ$ over local ring Ʀ. Then $Ƒ$ is EXNPQ2AB submodule of $Ŵ$ if and only if $[Ƒ:_{R} Ŵ]$ is EXNPQ2AB ideal of Ʀ.

4. More Result of EXNPQ2AB Submodules in Multiplication Modules.

In this part we studied more result of EXNPQ2AB submodules in multiplication modules. And we got the most important results.

Proposition 4.1

Let $Ŵ$ be a finitely generated multiplication projective Ʀ-module, and Ɓ is an ideal of Ʀ with $ann_{R}(Ŵ) \subseteq Ɓ$. Then Ɓ is EXNPQ2AB ideal of Ʀ if and only if $ƁŴ$ is EXNPQ2AB submodule of $Ŵ$.

Proof.

($\Longrightarrow$) Let $H_1H_2H_3A \subseteq \mathcal{B}W$ for some submodules H_1,H_2,H_3, A of W. Since W is a multiplication, then $H_1 = J_1W$, $H_2 = J_2W$, $H_3 = J_3W$ and $A = J_4W$ for some ideals J_1, J_2, J_3 and $J_4$of R. That is $H_1H_2H_3A = J_1J_2J_3J_4W \subseteq \mathcal{B}W$. But W is a finitely generated multiplication R-module then by **Lemma 2.18** $J_1J_2J_3J_4 \subseteq \mathcal{B} + ann_R(W)$, but $ann_R(W) \subseteq \mathcal{B}$, implies that $\mathcal{B} + ann_R(W) = \mathcal{B}$, thus $J_1J_2J_3J_4 \subseteq \mathcal{B}$. Now, by assumption $\mathcal{B}$ is EXNPQ2AB ideal of R then by **Proposition 3.2** either $J_1J_3J_4 \subseteq \mathcal{B} + (soc(R) + J(R))$ or $J_2J_3J_4 \subseteq \mathcal{B} + (soc(R) + J(R))$ or $J_1J_2J_4 \subseteq \mathcal{B} + (soc(R) + J(R))$, hence either $J_1J_3J_4W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $J_2J_3J_4W \subseteq \mathcal{B}W + (soc(R)W + J(R)W)$ or $J_1J_2J_4W \subseteq \mathcal{B}W + soc(R)W + J(R)W$. Since W is a projective then by **Lemma 2.10** and **Lemma 2.9** $\big(soc(W) + J(W)\big) = (soc(R)W + J(R)W)$, thus either $H_1H_3A \subseteq \mathcal{B}W + \big(soc(W) + J(W)\big)$ or $H_2H_3A \subseteq \mathcal{B}W + \big(soc(W) + J(W)\big)$ or $H_1H_2A \subseteq \mathcal{B}W + \big(soc(W) + J(W)\big)$. Hence by **Proposition 3.2** $\mathcal{B}W$ is EXNPQ2AB submodule of W.

($\Longleftarrow$) Let $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4 \subseteq \mathcal{B}$, for $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ are ideals in R, implies that $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3(\mathcal{P}_4W) \subseteq \mathcal{B}W$. But $\mathcal{B}W$ is EXNPQ2AB submodule of W, then by **Proposition 2.25** either $\mathcal{P}_1\mathcal{P}_2(\mathcal{P}_4W) \subseteq \mathcal{B}W + (soc(W) + J(W))$or $\mathcal{P}_1\mathcal{P}_3(\mathcal{P}_4W) \subseteq \mathcal{B}W + (soc(W) + J(W))$ or $\mathcal{P}_2\mathcal{P}_3(\mathcal{P}_4W) \subseteq \mathcal{B}W + (soc(W) + J(W))$. But W is a projective then $\big(soc(W) + J(W)\big) = (soc(R)W + J(R)W)$. Thus either $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_4W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $\mathcal{P}_1\mathcal{P}_3\mathcal{P}_4W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4W \subseteq \mathcal{B}W + soc(R)W + J(R)W$, hence either $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_4 \subseteq \mathcal{B} + soc(R) + J(R)$ or $\mathcal{P}_1\mathcal{P}_3\mathcal{P}_4 \subseteq \mathcal{B} + soc(R) + J(R)$ or $\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4 \subseteq \mathcal{B} + soc(R) + J(R)$. Then by **Proposition 2.25** $\mathcal{B}$ is EXNPQ2AB ideal of R.

Proposition 4.2

Let W be a faithful finitely generated multiplication R-module, and $\mathcal{B}$ is an ideal of R. Then $\mathcal{B}$ is EXNPQ2AB ideal of R if and only if $\mathcal{B}W$ is EXNPQ2AB submodule of W.

Proof.

($\Longrightarrow$) Let $r\mathcal{P}Jx \subseteq \mathcal{B}W$ for any $r \in R$, $x \in W$ and $\mathcal{P}, J$ are ideals of R. Next, follows $r\mathcal{P}J(x) \subseteq \mathcal{B}W$. Since W is a multiplication, then $(x) = \mathcal{P}_1W$ for some ideal $\mathcal{P}_1$ of R, that is $r\mathcal{P}J\mathcal{P}_1W \subseteq \mathcal{B}W$. Thus by **Lemma 2.18** we get $r\mathcal{P}J\mathcal{P}_1 \subseteq \mathcal{B} + ann(W)$, but W is faithful, then $ann(W) = \{0\}$, that is $r\mathcal{P}J\mathcal{P}_1 \subseteq \mathcal{B}$. Since $\mathcal{B}$ is EXNPQ2AB ideal of R, then by **Proposition 2.27** either $r\mathcal{P}\mathcal{P}_1 \subseteq \mathcal{B} + soc(R) + J(R)$ or $rJ\mathcal{P}_1 \subseteq \mathcal{B} + soc(R) + J(R)$ or $\mathcal{P}J\mathcal{P}_1 \subseteq \mathcal{B} + soc(R) + J(R)$, hence either $r\mathcal{P}\mathcal{P}_1W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $rJ\mathcal{P}_1W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $\mathcal{P}J\mathcal{P}_1W \subseteq \mathcal{B}W + soc(R)W + J(R)W$, hence by **Lemma 2.6** and **Lemma 2.7** either $r\mathcal{P}(x) \subseteq \mathcal{B}W + soc(W) + J(W)$ or $rJ(x) \subseteq \mathcal{B}W + soc(W) + J(W)$ or $\mathcal{P}J(x) \subseteq \mathcal{B}W + soc(W) + J(W)$. That is either $r\mathcal{P}x \subseteq \mathcal{B}W + soc(W) + J(W)$ or $rJx \subseteq \mathcal{B}W + soc(W) + J(W)$ or $\mathcal{P}Jx \subseteq \mathcal{B}W + soc(W) + J(W)$. Hence by **Proposition 2.26** $\mathcal{B}W$ is an EXNPQ2AB submodule of W.

($\Longleftarrow$) Let $rst\mathcal{P} \subseteq \mathcal{B}$ for $r, s, t \in R$ and $\mathcal{P}$ ideal of R, hence $rst(\mathcal{P}W) \subseteq \mathcal{B}W$, but $\mathcal{B}W$ is an EXNPQ2AB submodule of W, then either $rs(\mathcal{P}W) \subseteq \mathcal{B}W + soc(W) + J(W)$ or $rt(\mathcal{P}W) \subseteq \mathcal{B}W + soc(W) + J(W)$ or $st(\mathcal{P}W) \subseteq \mathcal{B}W + soc(W) + J(W)$. Thus by **Lemma 2.6** and **Lemma 2.7** either $rs\mathcal{P}W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $rt\mathcal{P}W \subseteq \mathcal{B}W + soc(R)W + J(R)W$ or $st\mathcal{P}W \subseteq \mathcal{B}W + soc(R)W + J(R)W$, hence either $rs\mathcal{P} \subseteq \mathcal{B} + soc(R) + J(R)$ or $rt\mathcal{P} \subseteq \mathcal{B} + soc(R) + J(R)$ or $st\mathcal{P} \subseteq \mathcal{B} + soc(R) + J(R)$. Therefore $\mathcal{B}$ is EXNPQ2AB ideal of R.

Proposition 4.3

Let $\mathbb{W}$ be a finitely generated non-singular multiplication module over good ring $\mathbb{R}$ and $\mathcal{B}$ is an ideal of $\mathbb{R}$ with $ann_{\mathbb{R}}(\mathbb{W}) \subseteq \mathcal{B}$. Then $\mathcal{B}$ is EXNPQ2AB ideal of $\mathbb{R}$ if and only if $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

($\Longrightarrow$) Let $rs\mathcal{P}A \subseteq \mathcal{B}\mathbb{W}$, for $r, s \in \mathbb{R}$, $\mathcal{P}$ is an ideal of $\mathbb{R}$ and A is a submodule of $\mathbb{W}$. Since $\mathbb{W}$ is a multiplication, then $A = \mathcal{P}_1\mathbb{W}$, for some ideal $\mathcal{P}_1$ of $\mathbb{R}$, then $rs\mathcal{P}\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W}$. But $\mathbb{W}$ is a finitely generated multiplication $\mathbb{R}$-module then by **Lemma 2.18** $rs\mathcal{P}\mathcal{P}_1 \subseteq \mathcal{B} + ann_{\mathbb{R}}(\mathbb{W})$, since $ann_{\mathbb{R}}(\mathbb{W}) \subseteq \mathcal{B}$, implies that $\mathcal{B} + ann_{\mathbb{R}}(\mathbb{W}) = \mathcal{B}$, hence $rs\mathcal{P}\mathcal{P}_1 \subseteq \mathcal{B}$. But $\mathcal{B}$ is EXNPQ2AB ideal of $\mathbb{R}$ then by **Proposition 2.28** either $rs\mathcal{P}_1 \subseteq \mathcal{B} + (soc(\mathbb{R}) + J(\mathbb{R}))$ or $r\mathcal{P}\mathcal{P}_1 \subseteq \mathcal{B} + (soc(\mathbb{R}) + J(\mathbb{R}))$ or $s\mathcal{P}\mathcal{P}_1 \subseteq \mathcal{B} + (soc(\mathbb{R}) + J(\mathbb{R}))$. Thus either $rs\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$ or $r\mathcal{P}\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$ or $s\mathcal{P}\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$. Since $\mathbb{W}$ is non-singular, then by **Lemma 2.17** $soc(\mathbb{R})\mathbb{W} = soc(\mathbb{W})$ and $\mathbb{R}$ is good ring then $J(\mathbb{R})\mathbb{W} = J(\mathbb{W})$. Hence either $rs\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $r\mathcal{P}\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $s\mathcal{P}\mathcal{P}_1\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$. That is either $rsA \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $r\mathcal{P}A \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $s\mathcal{P}A \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$. Therefore by **Proposition 2.28** $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

($\Longleftarrow$) Let $r\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 \subseteq \mathcal{B}$, for $r \in \mathbb{R}$, and $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are ideals of $\mathbb{R}$, implies that $r\mathcal{P}_1\mathcal{P}_2(\mathcal{P}_3\mathbb{W}) \subseteq \mathcal{B}\mathbb{W}$. Since $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$, then by **Proposition 2.29** either $r\mathcal{P}_1(\mathcal{P}_3\mathbb{W}) \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $r\mathcal{P}_2(\mathcal{P}_3\mathbb{W}) \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $\mathcal{P}_1\mathcal{P}_2(\mathcal{P}_3\mathbb{W}) \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$. But $\mathbb{W}$ is non-singular and $\mathbb{R}$ is good ring then $\big(soc(\mathbb{W}) + J(\mathbb{W})\big) = (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$. Thus either $r\mathcal{P}_1\mathcal{P}_3\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$ or $r\mathcal{P}_2\mathcal{P}_3\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$ or $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{R})\mathbb{W} + J(\mathbb{R})\mathbb{W})$, then either $r\mathcal{P}_1\mathcal{P}_3 \subseteq \mathcal{B} + (soc(\mathbb{R}) + J(\mathbb{R}))$ or $r\mathcal{P}_2\mathcal{P}_3 \subseteq \mathcal{B} + (soc(\mathbb{R}) + J(\mathbb{R}))$ or $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 \subseteq \mathcal{B} + soc(\mathbb{R}) + J(\mathbb{R})$. Hence by **Proposition 2.28** $\mathcal{B}$ is EXNPQ2AB ideal of $\mathbb{R}$.

Corollary 4.4

Let $\mathbb{W}$ be a finitely generated non-singular multiplication module over Artinian ring $\mathbb{R}$ and $\mathcal{B}$ is an ideal of $\mathbb{R}$ with $ann_{\mathbb{R}}(\mathbb{W}) \subseteq \mathcal{B}$. Then $\mathcal{B}$ is EXNPQ2AB ideal of $\mathbb{R}$ if and only if $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

Proposition 4.5

Let $\mathbb{W}$ be a finitely generated non-singular multiplication module over local ring $\mathbb{R}$ and $\mathcal{B}$ is an ideal of $\mathbb{R}$ with $ann_{\mathbb{R}}(\mathbb{W}) \subseteq \mathcal{B}$. Then $\mathcal{B}$ is EXNPQ2AB ideal of $\mathbb{R}$ if and only if $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

Similarly to the Proof of **Proposition 4.3** by using **Lemma 2.15**.

Proposition 4.6

Let $\mathbb{W}$ be a finitely generated multiplication Z-regular module over good ring $\mathbb{R}$ and $\mathcal{B}$ is an ideal of $\mathbb{R}$ with $ann_{\mathbb{R}}(\mathbb{W}) \subseteq \mathcal{B}$. Then $\mathcal{B}$ is EXNPQ2AB ideal of $\mathbb{R}$ if and only if $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

$(\Longrightarrow)$ Let $rstx \in \mathcal{B}\mathbb{W}$ for $r, s, t \in \mathrm{R}$ and $x \in \mathbb{W}$, that is $rst\langle x\rangle \subseteq \mathcal{B}\mathbb{W}$. Since $\mathbb{W}$ is a multiplication, then $\langle x\rangle = \mathcal{P}\mathbb{W}$ for some ideal $\mathcal{P}$ of R, that is $rst\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W}$. But $\mathbb{W}$ is a finitely generated multiplication R-module then by **Lemma 2.18** $rst\mathcal{P} \subseteq \mathcal{B}$. But $\mathcal{B}$ is EXNPQ2AB ideal of R then by **Proposition 2.24** either $rs\mathcal{P} \subseteq \mathcal{B} + (soc(\mathrm{R}) + J(\mathrm{R}))$ or $rt\mathcal{P} \subseteq \mathcal{B} + (soc(\mathrm{R}) + J(\mathrm{R}))$ or $st\mathcal{P} \subseteq \mathcal{B} + (soc(\mathrm{R}) + J(\mathrm{R}))$. Thus either $rs\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W})$ or $rt\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W})$ or $st\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W})$. Since $\mathbb{W}$ is $Z\text{-}regular$ then by **Lemma 2.21** $soc(\mathrm{R})\mathbb{W} = soc(\mathbb{W})$ and R is good ring then $J(\mathrm{R})\mathbb{W} = J(\mathbb{W})$. Hence either $rs\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $rt\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $st\mathcal{P}\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$. That is either $rs\langle x\rangle \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $rt\langle x\rangle \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $st\langle x\rangle \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$, thus either $rsx \in \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $rtx \in \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $stx \in \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$. Therefore $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

$(\Longleftarrow)$ Let $abcd \in \mathcal{B}$, for $a, b, c, d \in \mathrm{R}$, implies that $abc(d\mathbb{W}) \subseteq \mathcal{B}\mathbb{W}$. Since $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$, then by **Proposition 2.24** either $ab(d\mathbb{W}) \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $ac(d\mathbb{W}) \subseteq \mathcal{B}\mathbb{W} + (soc(\mathbb{W}) + J(\mathbb{W}))$ or $bc(d\mathbb{W}) \subseteq \mathcal{B}\mathbb{W} + \big(soc(\mathbb{W}) + J(\mathbb{W})\big)$. But $\mathbb{W}$ is Z-regular and R is good ring, then $\big(soc(\mathbb{W}) + J(\mathbb{W})\big) = (soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W})$. Thus either $abd\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W}$ or $acd\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W}$ or $bcd\mathbb{W} \subseteq \mathcal{B}\mathbb{W} + soc(\mathrm{R})\mathbb{W} + J(\mathrm{R})\mathbb{W}$, then either $abd \in \mathcal{B} + soc(\mathrm{R}) + J(\mathrm{R})$ or $acd \in \mathcal{B} + soc(\mathrm{R}) + J(\mathrm{R})$ or $bcd \in \mathcal{B} + soc(\mathrm{R}) + J(\mathrm{R})$. Hence $\mathcal{B}$ is EXNPQ2AB ideal of R.

Corollary 4.7

Let $\mathbb{W}$ be a finitely generated multiplication Z-regular module over Artinian ring R and $\mathcal{B}$ is an ideal of R with $ann_{\mathrm{R}}(\mathbb{W}) \subseteq \mathcal{B}$. Then $\mathcal{B}$ is EXNPQ2AB ideal of R if and only if $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

Proposition 4.8

Let $\mathbb{W}$ be a finitely generated multiplication Z-regular module over local ring R and $\mathcal{B}$ is an ideal of R with $ann_{\mathrm{R}}(\mathbb{W}) \subseteq \mathcal{B}$. Then $\mathcal{B}$ is EXNPQ2AB ideal of R if and only if $\mathcal{B}\mathbb{W}$ is EXNPQ2AB submodule of $\mathbb{W}$.

Proof.

Similar to the Proof of **Proposition 4.6** by using **Lemma 2.15**.

Proposition 4.9

Let $\mathbb{W}$ be a faithful finitely generated multiplication R-module and $V \neq \mathbb{W}$, in which case the following claims are equivalent:

1. V is EXNPQ2AB submodule of $\mathbb{W}$.

2. $[V :_{\mathrm{R}} \mathbb{W}]$ is EXNPQ2AB ideal of R.

3. $V = \mathcal{B}\mathbb{W}$ for some EXNPQ2AB ideal $\mathcal{B}$ of R.

Proof.

$(\mathbf{1} \Leftrightarrow \mathbf{2})$ By **Proposition 3.6**.

($\mathbf{2 \Rightarrow 3}$) Since $[V:_{\text{Ʀ}} \text{Ɯ}]$ is EXNPQ2AB ideal of Ʀ and Ɯ is a faithful, that is $(0) = ann_{\text{Ʀ}}(\text{Ɯ}) = [0:_{\text{Ʀ}} \text{Ɯ}] \subseteq [V:_{\text{Ʀ}} \text{Ɯ}]$ and Ɯ is a multiplication, so $\text{V} = [V:_{\text{Ʀ}} \text{Ɯ}]\text{Ɯ}$, implies that $\text{V} = \text{J}\text{Ɯ}$ for some EXNPQ2AB ideal $\text{J} = [V:_{\text{Ʀ}} \text{Ɯ}]$ of Ʀ.

($\mathbf{3 \Rightarrow 2}$) Suppose that $V = J\text{Ɯ}$ for some EXNPQ2AB ideal J of Ʀ. Since Ɯ is multiplication, then $\text{V} = [\text{V}:_{\text{Ʀ}} \text{Ɯ}]\text{Ɯ}$. That is $J\text{Ɯ} = [\text{V}:_{\text{Ʀ}} \text{Ɯ}]\text{Ɯ}$, but Ɯ is faithful finitely generated multiplication then by **Lemma 2.23** we get $[\text{V}:_{\text{Ʀ}} \text{Ɯ}] = \text{J}$. Thus $[\text{V}:_{\text{Ʀ}} \text{Ɯ}]$ is EXNPQ2AB ideal of Ʀ.

Proposition 4.10
Let Ɯ be a finitely generated multiplication projective Ʀ-module and $V \neq \text{Ɯ}$, in which case the following claims are equivalent:

1. V is EXNPQ2AB submodule of Ɯ.

2. $[V:_{\text{Ʀ}} \text{Ɯ}]$ is EXNPQ2AB ideal of Ʀ.

3. $V = \text{Ɓ}\text{Ɯ}$ for some EXNPQ2AB ideal Ɓ of Ʀ.

Proof.

($\mathbf{1 \Leftrightarrow 2}$) By **Proposition 3.7**.

($\mathbf{2 \Rightarrow 3}$) Clear.

($\mathbf{3 \Rightarrow 2}$) Assume that $V = \text{Ɓ}\text{Ɯ}$ …..(1) for some EXNPQ2AB ideal Ɓ of Ʀ with $ann_{\text{Ʀ}}(\text{Ɯ}) \subseteq \text{Ɓ}$. while Ɯ is a multiplication, then $V = [V:_{\text{Ʀ}} \text{Ɯ}]\text{Ɯ}$…..(2), from (1) and (2) we have $[V:_{\text{Ʀ}} \text{Ɯ}]\text{Ɯ} = \text{Ɓ}\text{Ɯ}$. Since Ɯ is a finitely generated, then by **Lemma 2.22** Ɯ is weak cancellation, then $[V:_{\text{Ʀ}} \text{Ɯ}] + ann_{\text{Ʀ}}(\text{Ɯ}) = \text{Ɓ} + ann_{\text{Ʀ}}(\text{Ɯ})$, but $ann_{\text{Ʀ}}(\text{Ɯ}) \subseteq \text{Ɓ}$, and $ann_{\text{Ʀ}}(\text{Ɯ}) \subseteq [V:_{\text{Ʀ}} \text{Ɯ}]$**,** implies that $ann_{\text{Ʀ}}(\text{Ɯ}) + \text{Ɓ} = \text{Ɓ}$ and $[V:_{\text{Ʀ}} \text{Ɯ}] + ann_{\text{Ʀ}}(\text{Ɯ}) = [V:_{\text{Ʀ}} \text{Ɯ}]$. Thus $\text{Ɓ} = [V:_{\text{Ʀ}} \text{Ɯ}]$, but Ɓ is EXNPQ2AB ideal of Ʀ, hence $[V:_{\text{Ʀ}} \text{Ɯ}]$ is EXNPQ2AB ideal of Ʀ.

5. Conclusion.

In this paper, we introduced the some characterizations in class of multiplication modules. And, we show by example the residual of Extend Nearly Pseudo Quasi-2-Absorbing submodule need not to be Extend Nearly Pseudo Quasi-2-Absorbing ideal; we gave an example of that. Under a certain condition it is equivalent. Also, we studied the characterized Extend Nearly Pseudo Quasi-2-Absorbing ideals by Extend Nearly Pseudo Quasi-2-Absorbing submodules. In the end, we got a lot of important results.

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