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Direct Method for Variational Problems Using Boubaker Wavelets

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Abstract

The wavelets have many applications in engineering and the sciences, especially mathematics. Recently, in 2021, the wavelet Boubaker (WB) polynomials were used for the first time to study their properties and applications in detail. They were also utilized for solving the Lane-Emden equation. The aim of this paper is to show the truncated Wavelet Boubaker polynomials for solving variation problems. In this research, the direct method using wavelets Boubaker was presented for solving variational problems. The method reduces the problem into a set of linear algebraic equations. The fundamental idea of this method for solving variation problems is to convert the problem of a function into one that involves a finite number of variables. Different numerical examples were given to demonstrate the applicability and validity of this method using the Matlab program. Also, the results of this technique were compared with the exact solution, and graphs were added to these examples to test the convergence of Wavelet Boubaker polynomials using this method.

Keywords: Boubaker wavelets, Calculus of variation problems, Nonlinear programming, Numerical methods.

1. Introduction

Many problems arising in mathematical physics and geometry are connected with the calculus of variations, which is determined by finding the maximal and minimal functional functions. The functionals are defined by definite integrals, which include boundary conditions and appear in the mathematical formula, see [2].

Wavelet theory is an emerging field in mathematical research and is applied in a broad range of engineering disciplines. Wavelets are very successful in accurately solving numerical problems. [3] Utilized Legendre wavelets to solve variational problems. [4] Used the Haar wavelet to solve the same problems. [5] Applied direct restarted Pell to solve the problem of variation, then used the spectral method with Chebyshev wavelets in another paper to solve the calculus of variations

[6]. [7-8] using a Spline polynomial with collocation method and [9] used a new technique to find the numerical solution for solving isoperimetric problems.

Many researchers have utilized different procedures to solve calculus of variational problems. [10] Developed the new functions for solving the problems of variational, then [11] used a combination of many functions with Bernoulli polynomials. [12] Found the approximate solution for boundary value problems using the wavelet function. [13] Studied moving or fixed boundary Muntz wavelets for solving variation problems.

This paper is arranged as follows: in Section 2, Orthogonal Boubaker polynomials and their properties with recurrence relations In Section 3, Boubaker wavelets and their properties In Section 4, the application of Boubaker wavelet polynomials for solving variational problems with some numerical examples has been presented. In Section 5, the convergence test for the introduced method has been studied, and at last the conclusion has been reached.

2. Orthogonal Boubaker polynomials and their properties:

Boubaker polynomials haven't been orthogonal, so by applying the Gram-Schmit process to Boubaker polynomials, one can obtain orthogonal Boubaker polynomials, $OB_m(t)$. Several papers have been applied with different applications in physics, applied sciences, etc. (see [14], [15]).

Orthogonal Boubaker polynomials (OB) of m^{th} degree were presented on the interval [0, 1] by the following equation: [1]

$$OB_m(t) = \frac{(m!)^2}{(2m)!}\sum_{k=0}^{m}(-1)^{m+k}\frac{(m+k)!}{(m-k)!(k!)^2}t^k \qquad \dots (1)$$

A recursive relation of the orthogonal Boubaker polynomial on the interval [0, 1] has given as follows:

$$OB_{m+1}(t) = \frac{((m+1)!)^2}{(2(m+1))!}\left[\frac{(2m+1)(2m)!}{(m+1)(m!)^2}(2t-1)OB_m(t) - \frac{(m)}{(m+1)}\frac{2(m-1)!}{((m-1)!)^2}OB_{m-1}(t)\right], m \geq 2$$

with $OB_0(t) = 1 \text{ and } OB_1(t) = \frac{1}{2}(2t-1)$.

3. Boubaker wavelet and their properties:

The Boubaker wavelets can be defined as below (see [1])

$$WB_{n,m}(t) = \{\sqrt{2m+1}\ \ 2^{\frac{k}{2}}\ \ 0 \quad other\ wise\ \ \frac{(2m)!}{(m!)^2}OB_m(2^k t - n),\ \ \frac{n}{2^{k-1}} \leq t < \frac{n+1}{2^{k-1}} \qquad \dots (2)$$

where $WB_{nm}(t) = WB(m,n,t)$ as Boubaker wavelets so n =0,1, 2, …, 2^{k-1} ,k is any positive integer , m =0,1,2…,M and t is normalized time .

The first six $WB_{nm}(t)$ with k=1, are given as follows:

$WB_0(t) = 1,$

$WB_1(t) = \frac{\sqrt{3}}{2}(4t - 3),$

$WB_2(t) = \frac{\sqrt{5}}{6}(24t^2 - 36t + 13)$,

$WB_3(t) = \frac{\sqrt{7}}{20}(160t^3 - 360t^2 + 264t - 63),$

$WB_4(t) = \frac{\sqrt{9}}{70}(1120t^4 - 3360t^3 + 3720t^2 - 1800t + 321),$

$WB_5(t) = \frac{\sqrt{11}}{252}(8064t^5 - 30240t^4 + 44800t^3 - 32760t^2 + 11820t - 1683),$

$WB_6(t) = \frac{\sqrt{13}}{924}(59136t^6 - 266112t^5 + 493920t^4 - 483840t^3 + 263760t^2 - 75852t + 8989)$.

The differential with respect to *t* of Boubaker wavelet polynomials $W\dot{B}_m(t)$ is given as follows:

$W\dot{B}_0(t) = 0$,

$W\dot{B}_1(t) = 2\sqrt{3}$,

$W\dot{B}_2(t) = \frac{\sqrt{5}}{6}(48t - 36)$,

$W\dot{B}_3(t) = \frac{\sqrt{7}}{20}(480t^2 - 720t + 264),$

$W\dot{B}_4(t) = \frac{\sqrt{9}}{70}(4480t^3 - 10080t^2 + 7440t - 1800),$

$\dot{W}B_5(t) = \frac{\sqrt{11}}{252}(40320t^4 - 120960t^3 + 134400t^2 - 65520t + 11820),$

$W\dot{B}_6(t) = \frac{\sqrt{13}}{924}(354816t^5 - 1330560t^4 + 1975680t^3 - 1451520t^2 + 527520t - 75852)$.

4. Boubaker wavelet polynomials for solving variational problems:

We demonstrate the application of wavelet Boubaker polynomials to solve some variational problems.

A function $f(t)$ is defined over $L^2[0,1]$, the function approximate by wavelet Boubaker polynomials is defined as follows:

$$f(t) = \sum_{i=0}^{\infty} c_i WB_i(t) = c^T WB(t) \qquad \text{... (3)}$$

where $c_i = \langle f(t), WB_i(t)\rangle$ and $\langle\ ,\ \rangle$ is inner product on $L^2[0,1]$.

If the series in equation (3) is truncated, then equation (3) can be written as

$$f(t)=\sum_{i=0}^{N}c_i WB_i(t)=c^T WB(t) \qquad \text{... (4)}$$

where $c=[c_0, c_1, \ldots, c_N]^T$ and $WB_i(t)=[WB_0, WB_1, \ldots, WB_N]^T$

Differentiating equation (4) with respect t, to obtain $\dot{f}= c^T\dot{W}B(t)$

where $c=[c_0, c_1, \ldots, c_N]^T$ and $\dot{W}B(t)=[\dot{W}B_0, \dot{W}B_1, \ldots, \dot{W}B_N]^T$

The matrix of derivatives D is given as $\frac{dWB(t)}{dt}=\dot{W}B\,(t)=DWB(t)$

Where $\dot{W}B\,(t)$ derivative of wavelet Boubaker functions

To demonstrate this procedure, we consider this example of finding the minimum of functional [14]

Example1:

$$J(x)=\int_0^1[\dot{x}^2(t)+t\dot{x}(t)+x^2(t)]dt, \qquad \ldots (5)$$

with two conditions $x(0)=0$, $x(1)=\frac{1}{4}$ … (6)

and the exact solution $x(t)=\frac{-e^{-t}[(-1+e^t)(e-2e^2-2e^t+e^{1+t})]}{4(-1+e^2)}$

Now, we use wavelet Boubaker polynomials of $M=4$ and $M=6$ to approximate the function $x(t)$

Suppose $x(t)=\sum_0^5 c_i WB_i(t)$ or $x(t)=c^T WB(t)$ … (7)

And $x^2(t)=c^T WB(WB)^T c$ … (8)

where $c=[c_0, c_1, \ldots, c_5]^T$ and $WB(t)=[WB_0, WB_1, \ldots, WB_5]$,

Differentiating equation (7), we get $\dot{x}(t)=c^T D\,WB(t)$ … (9)

and $\dot{x}^2(t)=c^T D\,(WB)(D\,(WB))^T c$ …(10)

Substituting equation (7) - (10) into equation (5), we obtain

$$J(x)=\int_0^1[c^T D\,(WB)\big(D\,(WB)\big)^T c+tc^T D\,(WB)+c^T WB(WB)^T c]dt \qquad \ldots (11)$$

We can simplify equation (11) to $J(x)=\frac{1}{2}c^T Hc+q^T c$ … (12)

where $H=2\int_0^1[D\,(WB)\big(D\,(WB)\big)^T+WB(WB)^T]dt$, and $q^T=\int_0^1 t\,D\,(WB)^T dt$,

Equation (7) with boundary conditions (6), can imply

$$x(0)=c^T WB(0)=0 \text{ and } x(1)=c^T WB(1)=\frac{1}{4} \qquad \ldots (13)$$

We can rewrite equations (12) & (13), as follows

Min $J(x)=\frac{1}{2}c^T Hc+q^T c$

subject to *Fc*-*b*=0 where $F=[WB^T(0) \quad WB^T(1)]$, $b=\begin{bmatrix}0 & \frac{1}{4}\end{bmatrix}$

we can find the parameter *c* using Lagrange equation as

$$c^*=-H^{-1}c+H^{-1}F^T(FH^{-1}F^T)^{-1}(FH^{-1}c+b)$$

When *M*=4 the approximate value of *x* is

x(*t*)= [0.21463728 ,0.04727207 ,-0.01564479, 0.00192278] *WB* (*t*)

and when *M*=6 an approximate value of *x* is

x(*t*)= [0.21464181, 0.04746694,-0.01585894, 0.00129175, -0.00023926 ,0.00001933] *WB* (*t*)

Table (1) shows the numerical results for example (1) with *M*=4 and *M*=6 compared with exact solution, and graphically in figure (1).

Table 1. Results for Example1

t	x_{exact}	$x_{app.}(t)$ M=4	M=4 Absolute error	$x_{app.}(t)$ M=6	M=6 Absolute error
0	0.00000000	0.00000000	0.00000000	0.00000000	0.0000000
0.1	0.04195072	0.04180602	0.00014470	0.04195070	0.00000002
0.2	0.07931714	0.07922621	0.00009093	0.07931742	0.00000027
0.3	0.11247322	0.11250475	0.00003152	0.11247334	0.00000011
0.4	0.14175081	0.14188583	0.00013502	0.14175057	0.00000023
0.5	0.16744291	0.16761363	0.00017071	0.16744257	0.00000034
0.6	0.18980668	0.18993234	0.00012566	0.18980660	0.00000007
0.7	0.20906592	0.20908615	0.00002022	0.20906621	0.00000029
0.8	0.22541340	0.22531923	0.00009416	0.22541370	0.00000030
0.9	0.23901272	0.23887579	0.00013693	0.23901258	0.00000014
1.0	0.2500000000	0.25000000	0.00000000	0.25000000	0.00000000

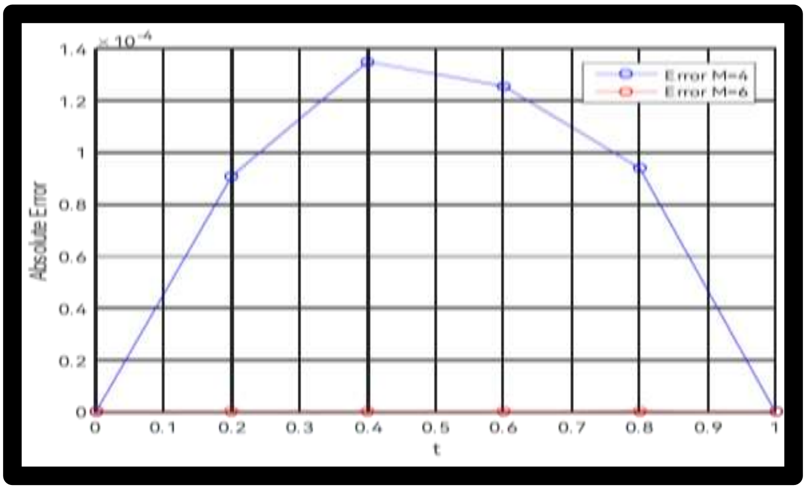

Figure 1. Shows that the Absolute Error when *M*=6 is very less than *M*=4 for Example1

Example 2:

Min $J = \int_0^1 (\dot{x}^2 + x^2)\, dt$

with boundary conditions $x(0) = 0\,,\ \ x(1) = 1\,,$

and exact solution is $x(t) = \frac{e^{t} - e^{-t}}{e^{1} - e^{-1}}$

In the same procedure, we can solve example2. Table (2) shows the numerical results for example (2) with *M*=4 and *M*=6 compared with exact solution, and graphically in figure (2).

When *M*=4 the approximate value of *x* is
x(*t*)= [0.70703400,0.32030428,0.03906027, 0.00769113] *WB* (*t*)
and when *M*=6 the approximate value of *x* is
x(*t*)= [0.70703595,0.32001609,0.03929145,0.00870795,0.00060523,0.00007777] *WB* (*t*)

Table 2. Results for Example2

t	x_{exact}	M=4 $x_{app.}(t)$	M=4 Absolute error	M=6 $x_{app.}(t)$	M=6 Absolute error
0	0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.08523370	0.08540591	0.00017221	0.08523414	0.00000043
0.2	0.17132045	0.17145031	0.00012986	0.17131995	0.00000049
0.3	0.25912183	0.25910993	0.00001190	0.25912108	0.00000075
0.4	0.34951660	0.34936152	0.00015507	0.34951638	0.00000021
0.5	0.44340944	0.44318181	0.00022762	0.44340990	0.00000046
0.6	0.54174007	0.54154756	0.00019250	0.54174071	0.00000063
0.7	0.64549262	0.64543551	0.00005710	0.64549284	0.00000021
0.8	0.75570548	0.75582241	0.00011693	0.75570520	0.00000027
0.9	0.87348169	0.87368498	0.00020329	0.87348147	0.00000021
1.0	1.00000000	1.00000000	0.00000000	1.00000000	0.00000000

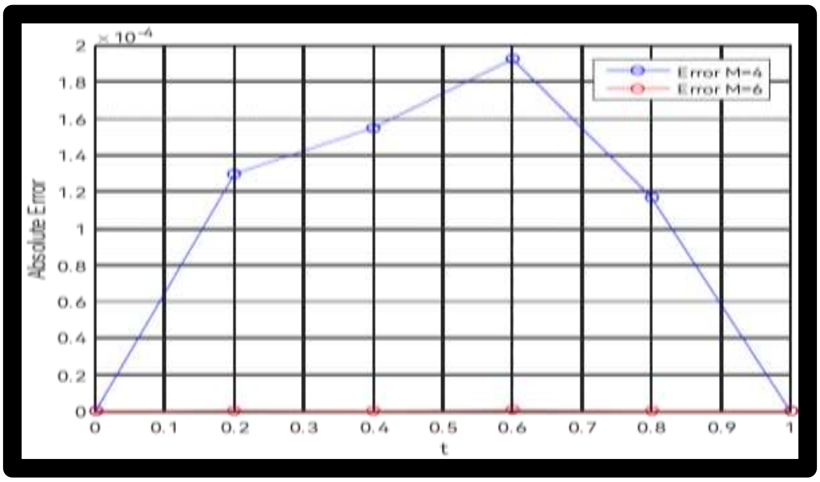

Figure2. Shows that the Absolute Error when *M*=6 is very less than *M*=4 for Example2 .

Example 3:

Min $j= \int_0^1 (\dot{x}^2 - x^2)\, dt$

with boundary conditions $x(0) = 0,\ x(1) = 1$,

and exact solution is $x(t) = \frac{sint}{sin1}$

Table (3) and figure (3) illustrate the results of example (3).

When M=4 the approximate value of x is

$x(t)$= [0.80163166, 0.24981351, 0.04537278, 0.00806631] WB (t)

and when M =6 the approximate value of x is

$x(t)$= [0.80164433, 0.24944837, 0.04508324, 0.00682505, 0.00070902, 0.00008006] WB (t)

Table 3. Results for Example3

t	u_{exact}	M= 4 u_{appr} (t)	M=4 Absolute Error	M= 6 u_{appr}.(t)	M=6 Absolute Error
0	0.00000000	0.00000000	0.00000000	0.00000000	0.0000000
0.1	0.11864154	0.11885365	0.00021211	0.11862523	0.00001631
0.2	0.23609766	0.23624932	0.00015166	0.23610543	0.00000777
0.3	0.35119476	0.35116260	0.00003216	0.35121956	0.00002480
0.4	0.46278285	0.46256910	0.00021374	0.46280277	0.00001992
0.5	0.56974696	0.56944444	0.00030251	0.56974637	0.00000058
0.6	0.67101835	0.67076422	0.00025412	0.67099789	0.00002045
0.7	0.76558514	0.76550406	0.00008108	0.76556103	0.00002411
0.8	0.85250246	0.85263956	0.00013709	0.85249567	0.00000678
0.9	0.93090186	0.93114634	0.00024447	0.93091791	0.00001604
1.0	1.00000000	1.00000000	0.00000000	1.00000000	0.00000000

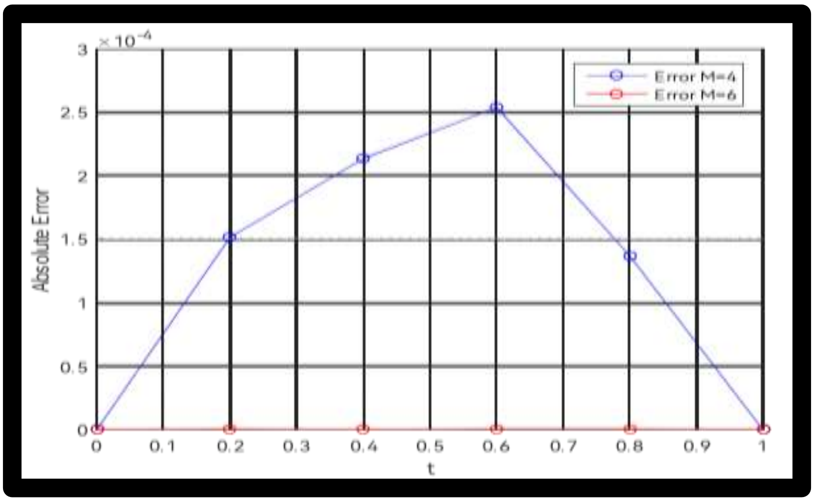

Figure 3. Shows that the Absolute Error when M=6 is very less than M=4 for Example3

5. The Convergence Test of wavelet Boubaker polynomials:

By (theorem (1), see [1]) if $x(t)$ is continually defined on [0,1] and $\alpha(t)$ is the approximate of $\alpha^*(t)$ by applying Boubaker wavelet. Also, suppose that $x(t)$ is bounded by a positive constant that is $|x(t)| < \epsilon$. Then, the coefficients of $x(t)$ are bounded.

The state can be expanded using Wavelet Boubaker polynomials, as:

$$x_{iN}(t) = \sum_{k=1}^{N} c_{ik} WB_k(t)$$

$$x_i(t) = x_{iN}(t) + \sum_{k=N+1}^{\infty} c_{ik} WB_k(t)$$

or $x_i(t) = x_{iN}(t) + e_i(t)$ …(14)

The coefficients in equation (14) is limited by [1]

Such that the residual $\|e(t)\|$ is less than some ε

Where $e(t) = max\,\{e_1(t), e_2(t), \dots, e_N(t)\}$,

Then, using the convergence test for the technique to the variable x in terms of N proposed L^2norm of x_i , i=1,2,…,n

$$\left[\int_0^1 (x(t) - x_{iN}(t))^2 dt\right]^{\frac{1}{2}} < \varepsilon_i\,,\quad i=1,2,\dots,n$$

$$\varepsilon = max\{\varepsilon_1(t), \varepsilon_2(t), \dots, \varepsilon_n(t)\},$$

$$\left[\int_0^1 (x(t) - x_N(t))^2 dt\right]^{\frac{1}{2}} < \varepsilon,$$

Using the Boubaker wavelet polynomials for approximating x variables, we get

$$\left[\int_0^1 \Big(\sum_{i=0}^{N+M} c_i WB_i(t) - \sum_{i=0}^{N} c_i WB_i(t)\Big)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$= \left[\int_0^1 \Big(\sum_{i=N+1}^{N+M} c_i WB_i(t)\Big)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$= \left[\int_0^1 \Big(\sum_{i=N+1}^{N+M} c_i WB_i(t)\Big)\Big(\sum_{i=N+1}^{N+M} c_i WB_i(t)\Big) dt\right]^{\frac{1}{2}} < \varepsilon$$

$$= \sum_{i=N+1}^{N+M} \sum_{j=N+1}^{N+M} c_i c_j \int_0^1 WB_i(t)\, WB_j(t)\; dt < \varepsilon$$

Hence, the Wavelet Boubaker polynomials can be reduced to the form

$$\sum_{i=N+1}^{N+M} c_i^2 < \varepsilon$$

when the squares of the remaining coefficients becomes neglected, a favorable approximation to the solution is achieved.

6. Conclusion

In this paper, a direct method with Wavelet Boubaker polynomials was achieved as an evaluation solution for reducing the variational problems into quadratic programming problems. Only a few terms of Wavelet Boubaker polynomials were needed to obtain an accurate solution. The numerical results were compared with the exact solution, which proved the capability and validity of such problems with easy steps. Matlab plotting was also used to demonstrate the results.

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