P-small Compressible Modules and P-small Retractable Modules

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Abstract
Let $R$ be a commutative ring with 1 and $M$ be left unitary $R$-module. In this papers we introduced and studied concept P-small compressible module (An $R$-module $M$ is said to be P-small compressible if $M$ can be embedded in every of it is nonzero P-small submodule of $M$. Equivalently, $M$ is P-small compressible if there exists a monomorphism $f: M \rightarrow N \neq 0 \neq N \triangleleft P M$, $R$ - module $M$ is said to be P-small retractable if $\text{Hom}(M, K) \neq 0$, for every nonzero P-small submodule $K$ of $M$. Equivalently, $M$ is P-small retractable if there exists a homomorphism $f: M \rightarrow N$ whenever $0 \neq N \triangleleft P M$ as a generalization of compressible module and retractable module respectively and give some of their advantages characterizations and examples.

Keywords: Compressible module, Retractable module, Small submodule, P-small submodule, P-small Compressible module, P-small Retractable module. Hollow module, PS-Hollow module.

1. Introduction
Let $R$ be a commutative ring with 1 and $M$ be left unitary $R$-module. Authors introduced and studied concept small submodules. A proper submodule $N$ of an $R$-module $M$ is termed a small submodule $(N \triangleleft M)$, if $N + L \neq M$ for every submodule $L$ of $M[1]$. A proper submodule $N$ of $M$ is said to be prime if whenever $r \in R, m \in M$ such that $r \cdot m \in N$ implies either $m \in N$ or $m \in [N : M] : [N : M] = \{r \in R : rM \subseteq N\}[2]$. In [3] Iman M.A.Hadi and Tammader A.Ibrahem introduced and studied the concept of P-small submodules, where a submodule $N$ of an $R$-module $M$ is called a P-small submodule $N \triangleleft P M$ if $N + P \neq M$ for any prime submodule $P$ of $M$. An $R$ - module $M$ is called compressible if $M$ can be embedded in every non-zero submodule. An $R$ - module $M$ is said to be P-small compressible if $M$ can be embedded in every of it is nonzero P-small submodule of $M$. Equivalently, $M$ is P-small compressible if there exists a monomorphism $f: M \rightarrow N$ whenever $0 \neq N \triangleleft P M$.

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In this paper we introduce and study the concept of P-small compressible as a generalization of compressible module, and we give some properties, characterization and examples. In addition, we see that under condition, P-small compressible, small compressible and compressible are equivalent. Some of their advantages characterizations and examples are given. We also study the relation between P-small compressible module, P-small retractable module and some of classes of modules.

2. Preliminaries

**Definition (2.1):** Let $M$ be an $R$ -module and $N \leq M$:

1. $N$ is called small submodule of $M$ , $(N \ll M)$ if $N + K = M$ implies $K = M$ , for any submodule $K$ of $M[1]$.
2. An $R$ -module $M$ is called hollow if every proper submodule is small in $M[4]$.
3. A proper submodule $N$ of $M$ is called prime if whenever $r \in R$, $m \in M$ implies either $m \in N$ or $r \in [N: M] : [N: M] = \{r \in R : rM \subseteq N\}$ [2].
4. A proper submodule $N$ is called P-small submodule of $M$ , $(N \ll_p M)$ if $N + P \neq M$ , for any prime submodule $P$ of $M$, [3].
5. An $R$ -module $M$ is called PS-hollow if every proper submodule in $M$ is P-small[3].
6. An $R$ -module $M$ is said to be small compressible if $M$ can be embedded in every nonzero small submodule of $M$. Equivalently, $M$ is small compressible if there exists a monomorphism $f: M \rightarrow N$ whenever $0 \neq N \ll M [5]$. 
7. An $R$ -module $M$ is called quasi-Dedekind module if for all $f \in \text{End}_R(M)$, $f \neq 0$ implies $\text{Ker} f = 0$, [7].
8. An $R$ -module $M$ is called small quasi-Dedekind module if for all $f \in \text{End}_R(M)$, $f \neq 0$ implies $\text{Ker} f \ll M$, [7].

**Remark (2.2):** [3](1) $Z_6$ is P-small submodule of $Z_6$ as $Z$ - module.

(2) $Z_2$ and (3) are not P-small submodule of $Z_6$.

(3) If $M$ is semi-simple module, then (0) is the only P-small submodule.

**Remark (2.3):** Each small submodule is P-small. But the converse is not true in general for example(2) is P-small submodule of $Z_6$ as $Z$ - module but not small.

**Lemma (2.4):**

1. Let $N$ be a proper submodule of $M$. If $W \subset N \ll_p M$, then $W \ll_p M$. In particular if $W$ is a direct summand of $N$ and $N \ll_p M$, then $W \ll_p M$.
2. Let $N_1$ and $N_2$ be proper submodules of $M$. If $N_1 + N_2 \ll_p M$, then $N_1 \ll_p M, N_2 \ll_p M$, the converse is not true.
3. Let $A \subset B \subset N \subset M$. If $B \ll_p N$, then $A \ll_p M$.
4. Let $M, M'$ be $R$ -modules and $\psi: M \rightarrow M'$ be an $R$ - homomorphism. If $A \ll_p M$, then $\psi(A) \ll_p M'$.

3. P-small Compressible Module

In this section, we introduce the concept of P-small compressible module as a generalization of compressible module, give some of basic properties, examples and characterizations of this concept.
Definition (3.1): An $R$–module $M$ is said to be $P$-small compressible if $M$ can be embedded in every of it is nonzero $P$-small submodule of $M$. Equivalently, $M$ is $P$-small compressible if there exists a monomorphism $f: M \to N$ whenever $0 \neq N \ll_{\rho} M$.

Remarks and examples (3.2):
1. It’s obvious that every compressible module is $P$-small compressible module, but the converse is not true. For example $Z_{6}$ as $Z$-module is $P$-small compressible since $(0)$ is the only $P$-small submodule, but not compressible.
2. $Z$ as $Z$–module is $P$-small compressible module, because it’s compressible module.
3. $Z_{\rho}$ as $Z$–module is $P$-small compressible module; $P$ is a prime number.
4. Every simple $R$–module is $P$-small compressible module but not conversely, because $Z$ as $Z$–module is a $P$-small compressible module but not simple.
5. $Z_{4}$ as $Z$–module is not $P$-small compressible. (Because $Z_{4}$ can’t be embedded in $(2)$ and $(2) \ll_{\rho} Z_{4}$).
6. A homomorphic image of a $P$-small compressible module need not be $P$-small compressible in general for example $Z$ as $Z$–module is a $P$-small compressible module and $\frac{Z}{2Z} \simeq Z_{4}$ is not $P$-small compressible module view remark (5).

Proposition (3.3): A $P$-small submodule of $P$-small compressible module is also $P$-small compressible module.

Proof: Let $0 \neq K \ll_{\rho} M$ and $M$ be $P$-small compressible module and let $0 \neq L \leq K \ll_{\rho} M$, then $L \ll_{\rho} M$ [3]. Since $M$ is $P$-small compressible, so $\exists$ a monomorphism $f: M \to L$ and $i: K \to M$ is the inclusion homomorphism, then $f \circ i: K \to L$ is a monomorphism. Therefore $K$ is a $P$-small compressible module.

Proposition (3.5): If an $R$–module $M$ has no prime submodule such that $\exists$ a monomorphism $f: M \to N$, $\forall N \subseteq M$, then $M$ is $P$-small compressible.

Proof: Suppose $M$ has no prime submodule and let $N \subseteq M$, then $N \ll_{\rho} M$ [3] and by assumption $M$ is $P$-small compressible.

Proposition (3.6): Let $M_{1}$ and $M_{2}$ be isomorphic $R$–modules. Then $M_{1}$ is $P$-small compressible if and only if $M_{2}$ is $P$-small compressible.

Proof: Let $0 \neq N \ll_{\rho} M_{1}$ and suppose that $M_{2}$ is $P$-small compressible. Let $\phi: M_{1} \to M_{2}$ be an isomorphism, then by [3] $0 \neq \phi(N) \ll_{\nu} M_{2}$. Put $K = \phi(N) \ll_{\nu} M_{2}$, so $\alpha: M_{2} \to K$ is a monomorphism (by assumption), let $g = \phi^{-1} \mid_{K}$, then $g: K \to M_{1}$ is a monomorphism. $g(K) = \phi^{-1}(\phi(N)) = N$. Hence, we have a composition $\psi = g \circ \alpha \circ \phi$, hence $\psi: M_{1} \to N$ is a monomorphism. Therefore $M_{1}$ is $P$-small compressible module.

Remark (3.7): The direct sum of $P$-small compressible module need not be $P$-small compressible.

Consider the following example let $Z_{4} \cong Z_{2} \oplus Z_{2}$ as $Z$-module. $Z_{2}$ is $P$-small compressible module, but $Z_{4}$ is not $P$-small compressible module see remarks and examples (2.3) point (5).

Proposition (3.8): Let $M = M_{1} \oplus M_{2}$ be an $R$–module such that $\text{ann}_{R} M_{1} \cap \text{ann}_{R} M_{2} = R$.

If $M_{1}$ and $M_{2}$ are $P$-small compressible modules, then $M$ is $P$-small compressible.

Proof: Let $0 \neq N = K_{1} \oplus K_{2} \ll_{\rho} M$. Then by theorem (1.12) [3] $0 \neq K_{1} \ll_{\rho} M_{1} \leq M$ and $0 \neq K_{2} \ll_{\rho} M_{2} \leq M$. But $M_{1}$ and $M_{2}$ $P$-small compressible modules, so $\exists$ monomorphisms $f: M_{1} \to K_{1}$ and $g: M_{2} \to K_{2}$. Define $\psi: M \to N$ by $\psi(a, b) = (f(a), g(b))$, it can be easily show that $\psi$ is a monomorphism. Therefore $M$ is $P$-small compressible.
Proposition (3.9): Let $M = M_1 \oplus M_2$ be P-small compressible module such that
$\text{ann}_RM_1 \oplus \text{ann}_RM_2 = 0 \neq K_1 \ll_p M_1 \leq M$ and $0 \neq K_2 \ll_p M_2 \leq M$ with
$N = K_1 \oplus K_2 \ll_p M$, then $M_1$ and $M_2$ are P-small compressible modules.

**Proof:** Let $0 \neq K_1 \leq N = K_1 \oplus K_2 \ll_p M$, then by remarks and examples (1.2) (7) [3] $K_1 \ll_p M$, but $M$ be P-small compressible module, so $\exists$ a monomorphisms $f: M \rightarrow K_1$ and $j: M_1 \rightarrow M_1 \oplus M_2 = M$, hence we have a composition. Let $\psi = f \circ j$, thus $\psi : M_1 \rightarrow K_1$ is a monomorphism. Therefore $M_1$ is P-small compressible module.

The same way we can prove $M_2$ is P-small compressible module.

Remarks and Examples (3.10):
1. Every P-small compressible module is small compressible module.

**Proof:** Let $0 \neq N \ll M$, then by [3] $N \ll_p M$ and $M$ is P-small compressible module, therefore $M$ is small compressible module.

2. $Z_6$ as $Z$ module is small compressible, since $(0)$ the only P-small submodule of $Z_6$.

3. $Q$ as $Z$ module is not P-small compressible module, since $\text{Hom}_Q(Q,Z) = 0$, where $Z \ll_p Q$.

Proposition (3.11): Let $M$ be an $R$ module and $0 \neq m \in M$ such that $R_m \not\subseteq M$, then $M$ is small compressible if and only if $M$ is P-small compressible.

**Proof:** Suppose that $M$ is small compressible module and let $N \ll_p M$, then by [3] $N \ll M$ and since $M$ is small compressible module, therefore $M$ is P-small compressible. Conversely it's clear by remarks and examples (3.10) point (1).

Corollary (3.12): A small compressible module $M$ is P-small compressible, if every cyclic submodule of $M$ is P-small submodule in $M$.

**Proof:** obviously by above proposition.

Proposition (3.13): Let $M$ be a finitely generated (or multiplication) $R$ module. Then $M$ is small compressible if and only if $M$ is P-small compressible.

**Proof:** Let $N \ll_p M$. We want to show that $M$ is P-small compressible. Since $M$ is finitely generated (or multiplication), then by proposition (1.4) [3], so $N \ll M$, but $M$ is small compressible $R$ module. Therefore $M$ is P-small compressible. Conversely clear by remarks and examples (3.10) point (1).

Corollary (3.14): Let $M$ be a noetherian $R$ module. Then $M$ is small compressible if and only if $M$ is P-small compressible.

**Proof:** Since $M$ is noetherian, then every submodule is finitely generated, then the result follows by proposition (3.13). Therefore $M$ is small compressible. Conversely clear by remarks and examples (3.10) point (1).

Recall that an $R$ module $M$ is called almost finitely generated if $M$ is not finitely generated and every proper submodule of $M$ is finitely generated [6].

Proposition (3.15): Let $M$ be an almost finitely generated $R$ module. Then $M$ is P-small compressible if and only if $M$ is small compressible.

**Proof:** Let $N \ll_p M$. We want to show that $M$ is P-small compressible. Since $M$ is almost finitely generated [6], then by corollary (1.11) [3], we get $N \ll M$, but $M$ is small compressible $R$ module. Therefore $M$ is P-small compressible. Conversely clear by remarks and examples (3.10) point (1).

Proposition (3.16): Let $M$ be a hollow module. Then the following statements are equivalent:

1. $M$ is compressible module.
(2) $M$ is $P$-small compressible module.
(3) small compressible module.

**Proof:**
(1) $\Rightarrow$ (2) It's clear by remarks and examples (3.2) point (1).
(2) $\Rightarrow$ (3) It's clear by remarks (3.10) point (1)
(3) $\Rightarrow$ (1) Let $K \leq M$. Since $M$ is hollow module and small compressible module, then $\exists$ a monomorphism $f: M \rightarrow K$. Therefore $M$ is compressible module.

We introduce the following

**Definition (3.17):** An $R$-module $M$ is called $P$-small quasi-Dedekind module if for all $\in (M)$, $f \neq 0$ implies $\ker f \ll_P M$.

**Remark (3.18):** It's clear that every quasi-Dedekind is $P$-small quasi-Dedekind.

**Proposition (3.19):** If $M$ is $P$-small quasi-Dedekind module, then $M$ can't be compressible.

**Proof:** Suppose that $M$ is $P$-small quasi-Dedekind module and let $N = \ker f \leq M$, but $M$ is $P$-small quasi-Dedekind, then $\ker f \ll_P M$, $f \neq 0$, thus can't be embedded $M$ in $\ker f$, because $\text{Hom}(M, \ker f) = 0$. Therefore $M$ can't be compressible module.

**Remark (3.20):**
Every small quasi-Dedekind is $P$-small quasi-Dedekind.

**Proof:** Let $0 \neq f \in \text{End}_R(M)$, where $M$ is an $R$-module since $M$ is a small quasi-Dedekind, then $\ker f \ll M$, hence $\ker f \ll_P M$. Thus $M$ is a $P$-small quasi-Dedekind module.

4. **P-small Retractable Module**

In this section, we introduce the concept of $P$-small retractable module as a generalization of retractable module, give some of basic properties, examples and characterizations of this concept.

**Definition (4.1):** An $R$-module $M$ is said to be $P$-small retractable if $M \text{Hom}(M, K) \neq 0$, for every non-zero $P$-small submodule $K$ of $M$. Equivalently, $M$ is $P$-small retractable if there exists a homomorphism $f: M \rightarrow N$ whenever $0 \neq N \ll_P M$.

**Remarks and Examples (4.2):**
1. It's obvious that every $P$-small compressible module is $P$-small retractable module, but the converse is not true for instance $\mathbb{Z}_4$ is $P$-small retractable but not $P$-small compressible module see remarks and examples (3.2) point(5).
2. $Z$ as $Z$-module is $P$-small retractable module, because it's $P$-small compressible module.
3. Every simple $R$-module is $P$-small retractable module but not conversely, because $Z$ as $Z$-module is a $P$-small retractable module but not simple.
4. Every retractable $R$-module is $P$-small retractable $R$-module, but the converse is not true.
5. Every semi-simple $R$-module is $P$-small retractable because it is retractable.
6. Every compressible module is $P$-small retractable module, but the converse is not true for instance $\mathbb{Z}_4$ is $P$-small retractable but not $P$-small compressible module see remarks and examples (3.2)point(5).
7. A homomorphic image of a $P$-small retractable module is a $P$-small retractable module.

**Remark (4.3):** The direct sum of $P$-small retractable module is $P$-small retractable module.

**Proposition (4.4):** A $P$-small submodule of $P$-small retractable module is also $P$-small retractable module.
Proof: Let \( 0 \neq K \ll_p M \) and \( M \) be P-small retractable module and let \( 0 \neq L \leq K \ll_p M \), by remarks and examples (1.2)point(3), [3]. \( L \ll_p M \). Since \( M \) is P-small retractable, so \( \exists \) a homomorphism \( f: M \rightarrow L \) and \( i: K \rightarrow M \) is the inclusion homomorphism, then \( f \circ i: K \rightarrow L \) be a homomorphism. Therefore \( K \) is a P-small retractable module.

Proposition 4.5: Let \( M_1 \) and \( M_2 \) be isomorphic \( R - \)modules. Then \( M_1 \) is P-small retractable if and only if \( M_2 \) is P-small retractable.

Proof: Let \( 0 \neq N \ll_p M_1 \) and suppose that \( M_2 \) is P-small retractable. Let \( f: M_1 \rightarrow M_2 \) be an isomorphism. Then by[3] \( 0 \neq f(N) \ll_p M_2 \). Put \( K = f(N) \ll_p M_2 \), we get \( h: M_2 \rightarrow K \) is a homomorphism (by assumption), let \( g = f^{-1} \mid_K \), then \( g: K \rightarrow M_1 \) is a monomorphism. \( g(K) = f^{-1}(f(N)) = N \). Hence we have a composition \( H = g \circ h \circ f \). Hence \( H: M_1 \rightarrow N \) is a monomorphism. Therefore \( M_1 \) is P-small retractable module.

Proposition 4.7: Let \( M \) be PS - hollow module, then the following are equivalent

(1) \( M \) is retractable module.

(2) \( M \) is P-small retractable module.

Proposition 4.8: If \( M \) is P-small quasi-Dedekind \( R - \)module, then \( M \) can't be P-small retractable.

Proof: Suppose that \( M \) is P-small quasi-Dedekind module and let \( N = Ker f \leq M \), but \( M \) is P-small quasi-Dedekind, then \( Ker f \ll_p M, f \neq 0 \), thus \( Hom(M, Ker f) = 0 \). Therefore \( M \) can't be P-small retractable module.

Recall that an \( R - \)module \( M \) is called monoform if for each non-zero submodule \( N \) of \( M \) and for each \( f \in Hom_R(N, M), f \neq 0 \) implies \( Ker f = 0 \), [5].

Definition 4.9: An \( R - \)module \( M \) is called P-small monoform if for each non-zero submodule \( N \) of \( M \) and for each \( f \in Hom_R(N, M), f \neq 0 \) implies \( Ker f \ll_p N \).

Remark 4.10: Every P-small compressible \( R - \)module is P-small monoform, but not conversely. For example, \( Z \) as \( Z - \)module is P-small monoform but not P-small compressible.

Proposition 4.11: Let \( M \) be a quasi-Dedekind \( R - \)module. Then \( M \) is P-small monoform if and only if \( M \) is P-small compressible.

Proof: Suppose that \( M \) is P-small monoform. Let \( 0 \neq N \ll_p M \), then \( 0 \neq f \in Hom_R(N, M) \). Since \( M \) is quasi-Dedekind, \( f \circ g: M \rightarrow N \rightarrow M \) is a monomorphism, hence \( g: M \rightarrow N \) is a monomorphism. Thus \( M \) is P-small compressible. Conversely it is clear by remark (4.10).

5. Conclusion

In this work, the class of compressible and retractable modules have been generalized to a new concepts called P-small compressible and P-small retractable modules. Several characteristics of this type of modules have been studied. Sufficient conditions under which these modules with compressible and retractable are discuss

Also we see relations between P-small compressible modules and other related modules as P-small retractable module P-small quasi-Dedekind, P-small monoform.
References