IHJPAS. 36 (3) 2023

Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq

P-small Compressible Modules and P-small Retractable Modules

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Article history: Received 27 October 2022, Accepted 19 December 2022, Published in July 2023.

doi.org/10.30526/36.3.3089

Abstract

Let R be a commutative ring with 1 and M be left unitary $R-module$. In this papers we introduced and studied concept P-small compressible $module$ (An $R-module\ M$ is said to be P-small compressible if M can be embedded in every of it is nonzero P-small submodule of M. Equivalently, M is P-small compressible if there exists a monomorphism $:M \longrightarrow N$,$0 \neq N \ll_P M$, $R-module\ M$ is said to be P-small retractable if $Hom(M,K) \neq 0$, for every non-zero P-small submodule $K$of M. Equivalently, M is P-small retractable if there exists a homomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_P M$ as a generalization of compressible $module$ and retractable $module$ respectively and give some of their advantages characterizations and examples.

Keywords: Compressible module, Retractable module, Small submodule, P-small submodule, P-small Compressible module, P-small Retractable module. Hollow module, PS-Hollow module.

1. Introduction

Let R be a commutative ring with 1 and M be left unitary $R-module$. Authors introduced and studied concept small submodules. A proper submodule N of an R −module M is termed a small submodule $(N \ll M)$, if $N + L \neq M$ for every submodule L of M[1]. A proper submodule N of M is said to be prime if whenever $r \in R$, $m \in M$ such that $r.m \in N$ $implies$ either $m \in N$ or $\in [N:M]$; $[N:M] = \{r \in R: rM \subseteq N\}$[2] . In [3] Iman M.A.Hadi and Tammader A.Ibrahiem introduced and studied the concept of P-small submodules , where a submodule N of an $R-module\ M$ is called P-small submodule $N \ll_P M$ if $N + P \neq M$ for any prime submodule P of M. An $R-moduleM$ is called compressible if M can be embedded in every non-zero submodule. An $R-module\ M$ is said to be P-small compressible if M can be embedded in every of it is nonzero P-small submodule of M. Equivalently, M is P-small compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_P M$.

In this paper we introduce and study the concept of P-small compressible as a generalization of compressible module, and we give some properties , characterization and examples. In addition, we see that under condition. P-small compressible, small compressible and compressible are equivalent.some of their advantages characterizations and examples are given. We also study the relation between P-small compressible module, P-small retractable module and some of classes of modules.

2. Preliminaries

Definition (2.1): Let M be an $R-$module and $N \leq M$:

1. N is called small submodule of M ,$(N \ll M)$ if $N + K = M$ implies $K = M$, for any submodule K of M[1].
2. An $R-$module$M$ is called hollow if every proper submodule is small in$M$[4].
3. A proper submodule N of M is called prime if whenever $r \in R$, $m \in M$ implies either $m \in N$ or $r \in [N{:}M]$: $[N{:}M] = \{r \in R{:} rM \subseteq N\}$ [2]
4. A proper submodule N is called P-small submodule of M , $(N \ll_P M)$ if $N + P \neq M$, for any prime submodule P of M, [3].
5. An $R-$module$M$ is called PS-hollow if every proper submodule in M is P-small[3].
6. An $R-module\ M$ is said to be small compressible if M can be embedded in every nonzero small submodule of M. Equivalently, M is small compressible if there exists a monomorphism $f{:}M \longrightarrow N$ whenever $0 \neq N \ll M$[5].
7. An $R-module\ M$ is called quasi-Dedekind $module$ if for all $f \in End_R(M)$, $f \neq 0$ implies $Kerf = 0$, [7].
8. An $R-module\ M$ is called small quasi-Dedekind $module$ if for all $f \in End_R(M)$, $f \neq 0$ implies $Kerf \ll M$, [7].

Remark(2.2) :[3](1) $(\bar{2})$ is P-small sub$module$ of Z_6 as $Z-module$.

(2) $(\bar{2})$ and $(\bar{3})$ are not P-small submodule of Z_6.

(3) If M is semi-simple $module$, then (0) is the only P-small submodule.

Remark (2.3): Each small submodule is P-small. But the converse is not true in general for example$(\bar{2})$ is P-small sub$module$ of Z_6 as $Z-module$ but not small.

Lemma (2.4):

1. Let N be a proper submodule of M. If W$\subset N \ll_P M$, then W$\ll_P M$. In particular if W is a direct summand of N and $N \ll_P M$, then W$\ll_P M$.
2. Let N_1 and N_2 be proper submodules of M. If $N_1 + N_2 \ll_P M$, then $N_1 \ll_P M$, $N_2 \ll_P M$, the converse is not true.
3. Let $A \subset B \subset N \subset M$. If B$\ll_P N$, then $A \ll_P M$.
4. Let M, M' be $R-modules$ and $\psi{:}M \longrightarrow M'$ be an $R-homomorphism$. If $A \ll_P M$, then $\psi(A) \ll_P M'$.

3 .P-small Compressible Module

In this section, we introduce the concept of P-small compressible module as a generalization of compressible $module$, give some of basic properties, examples and characterizations of this concept.

Definition (3.1): An $R-module\ M$ is said to be P-small compressible if M can be embedded in every of it is nonzero P-small submodule of M. Equivalently, M is P-small compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_P M$.

Remarks and examples (3.2):

1. It's obvious that every compressible module is P-small compressible $module$, but the converse is not true. For example Z_6 as Z-module is P-small compressible since $(\bar{0})$ is the only P-small submodule, but not compressible.
2. $Z\ as\ Z-module$ is P-small compressible module, because it's compressible $module$.
3. $Z_P\ as\ Z-module$ is P-small compressible $module$; $P\ is\ a\ prime\ number$.
4. Every simple $R-module$ is P-small compressible module but not conversely, because $Z\ as\ Z-module$ is a P-small compressible $module$ but not simple.
5. $Z_4\ as\ Z-module$ is not P-small compressible.(Because Z_4 can't be embedded in $\langle\bar{2}\rangle$ and $\langle\bar{2}\rangle \ll_P Z_4$) .
6. A homomorphic image of a P-small compressible $module$ need not be P-small compressible in general for example $Z\ as\ Z-module$ is a P-small compressible module and $\frac{z}{4z} \simeq z_4$ is not P-small compressible module view remark (5).

Proposition(3.3): A P-small submodule of P-small compressible $module$ is also P-small compressible $module$.

Proof: Let $0 \neq K \ll_P M$ and M be P-small compressible $module$ and let $0 \neq L \leq K \ll_P M$, then $L \ll_P M$ [3]. Since M is P-small compressible, so $\exists$ a monomorphism $f: M \longrightarrow L$ and $i: K \longrightarrow M$ is the inclusion homomorphism, then $f \circ i: K \longrightarrow L$ is a monomorphism. Therefore K is a P-small compressible $module$.

Proposition(3.5): If an $R-module\ M$ has no prime submodule such that $\exists$ a monomorphism $f: M \longrightarrow N$, $\forall N \subsetneq M$, then M is P-small compressible .

Proof: Suppose M has no prime submodule and let $N \subsetneq M$, then $N \ll_P M$ [3]and by assumption M is P-small compressible.

Proposition(3.6): Let M_1 and M_2 be isomorphic $R-modules$. Then $M_1$is P-small compressible if and only if $M_2$is P-small compressible.

Proof: Let $0 \neq N \ll_P M_1$ and suppose that M_2 is P-small compressible. Let $\phi: M_1 \longrightarrow M_2$ be an isomorphism., then by[3] $0 \neq \phi(N) \ll_P M_2$.Put $K = \phi(N) \ll_P M_2$, so $\alpha: M_2 \longrightarrow K$ is a monomorphism (by assumption), let $g = \phi^{-1}\big|_K$, then $g: K \longrightarrow M_1$ is a monomorphism. $g(K) = \phi^{-1}(\phi(N)) = N$. Hence, we have a composition $\psi = g \circ \alpha \circ \phi$, hence $\psi: M_1 \longrightarrow N$ is a monomorphism. Therefore $M_1$is P-small compressible $module$.

Remark(3.7): The direct sum of P-small compressible $module$ need not be P-small compressible. Consider the following example let $Z_4 \simeq Z_2 \oplus Z_2$ as Z-module . Z_2 is P-small compressible module, but Z_4 is not P-small compressible module see remarks and examples (2.3) point(5)

Proposition(3.8): Let $M = M_1 \oplus M_2$ be an R −module such that $ann_R M_1 \oplus ann_R M_2 = R$. If M_1 and M_2 are P-small compressible $modules$, then M is P-small compressible.

Proof: Let $0 \neq N = K_1 \oplus K_2 \ll_P M$. Then by theorem (1.12)[3] .$0 \neq K_1 \ll_P M_1 \leq M$ and $0 \neq K_2 \ll_P M_2 \leq M$. But $M_1$and M_2 P-small compressible $modules$, so $\exists$ monomorphisms $f: M_1 \longrightarrow K_1$and $g: M_2 \longrightarrow K_2$. Define $\psi: M \longrightarrow N$ by $\psi(a, b) = (f(a), g(b))$, it can be easily show that $\psi$is a monomorphism. Therefore M is P-small compressible.

Proposition(3.9): Let $M = M_1 \oplus M_2$ be P-small compressible $module$ such that $ann_R M_1 \oplus ann_R M_2 = .\ 0 \neq K_1 \ll_P M_1 \leq M$ and $0 \neq K_2 \ll_P M_2 \leq M$ with $N = K_1 \oplus K_2 \ll_P M$, then M_1 and M_2 are P-small compressible $modules$.

Proof: Let $0 \neq K_1 \leq N = K_1 \oplus K_2 \ll_P M$, then by remarks and examples(1.2)(7)[3] $K_1 \ll_P M$, but M be P-small compressible $module$, so ∃ a monomorphisms $f: M \longrightarrow K_1$ and $J: M_1 \longrightarrow M_1 \oplus M_2 = M$, hence we have a composition . Let $\psi = f \circ J$, thus $\psi : M_1 \longrightarrow K_1$ is a monomorphism . Therefore M_1 is P-small compressible $module$.

The same way we can prove M_2 is P-small compressible $module$.

Remarks and Examples (3.10):

1. Every P-small compressible module is small compressible module.

Proof: Let $0 \neq N \ll M$, then by [3] $N \ll_P M$ and M is P-small compressible $module$, therefor M is small compressible module.

2. Z_6 as $Z - module$ is small compressible, since $(\bar{0})$ the only P-small submodule of Z_6.
3. $Q\ as\ Z - module$ is not P-small compressible module, since $Hom_R(Q, Z) = 0$, where $Z \ll_P Q$.

Proposition(3.11): Let M be an $R - module$ and $0 \neq m \in M$ such that $R_m \subsetneq M$, then M is small compressible if and only if M is P-small compressible.

Proof: Suppose that M is small compressible $module$ and let $N \ll_P M$, then by [3] $N \ll M$ and since M is small compressible $module$, therefore M is P-small compressible. Conversely it's clear by remarks and examples (3.10)point(1)

Corollary(3.12): A small compressible $module$ M is P-small compressible, if every cyclic submodule of M is P-small submodule in M .

Proof: obviously by above proposition.

Proposition(3.13): Let M be a finitely generated (or multiplication) $R - module$. Then M is small compressible if and only if M is P-small compressible.

Proof: Let $N \ll_P M$. We want to show that M is P-small compressible. Since M is finitely generated (or multiplication), then by proposition(1.4)[3], so $N \ll M$, but M is small compressible $R - module$. Therefore M is P-small compressible. Conversely clear by remarks and examples (3.10) point (1).

Corollary(3.14): Let M be a noetherian $R - module$. Then M is small compressible if and only if M is P-small compressible.

Proof: Since M is noetherian, then every submodule is finitely generated, then the result follows by proposition(3.13). Therefore M is small compressible. Conversely clear by remarks and examples (3.10) point (1).

Recall that an $R - module\ M$ is called almost finitely generated if M is not finitely generated and every proper submodule of of M is finitely generated[6].

Proposition(3.15): Let M be an almost finitely generated $R - module$. Then M is P-small compressible if and only if M is small compressible.

Proof: Let $N \ll_P M$. We want to show that M is P-small compressible. Since M is almost finitely generated[6], then by corollary (1.11)[3], we get $N \ll M$, but M is small compressible $R - module$. Therefore M is P-small compressible. Conversely clear by remarks and examples (3.10) point (1).

Proposition(3.16): Let M be a hollow $module$. Then the following statements are equivalent:

(1) M is compressible $module$.

(2) M is P-small compressible $module$.

(3) small compressible $module$.

Proof: (1) ⟹ (2) It's clear by remarks and examples (3.2) point (1).

(2) ⟹ (3) It's clear by remarks (3.10) point (1)

(3) ⟹ (1)Let$K \leq M$. Since M is $hollow\ module$ and small compressible $module$, then ∃ a monomorphism $f: M \longrightarrow K$. Therefor M is compressible $module$.

We introduce the following

Definition (3.17): An $R - module\ M$ is called P-small quasi-Dedekind $module$ if for all ∈ (M) , $f \neq 0$ implies $Kerf \ll_P M$.

Remark (3.18): It's clear that every quasi-Dedekind is P-small quasi-Dedekind.

Proposition(3.19): If M is P-small quasi-Dedekind $module$, then M can't be compressible .

Proof: Suppose that M is P-small quasi-Dedekind $module$ and let $N = Kerf \leq M$, but M is P-small quasi-Dedekind, then $Kerf \ll_P M$, $f \neq 0$, thus can't be embedded M in $Kerf$, because $Hom(M, Kerf) = 0$. Therefore M can't be compressible $module$.

Remark(3.20):

Every small quasi-Dedekind is P-small quasi-Dedekind.

Proof: Let $0 \neq f \in End_R(M)$, where M is an $R - module$ since M is a small quasi-Dedekind, then $Kerf \ll M$, hence $Kerf \ll_P M$. Thus M is a P-small quasi-Dedekind module.

4. P-small Retractable Module

In this section, we introduce the concept of P-small retractable module as a generalization of retractable $module$, give some of basic properties, examples and characterizations of this concept.

Definition (4.1): An $R - module\ M$ is said to be P-small retractable if $M\ Hom(M, K) \neq 0$, for every non-zero P-small submodule $K$of M. Equivalently, M is P-small retractable if there exists a homomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_P M$.

Remarks and Examples(4.2):

1. It's obvious that every P-small compressible module is P-small retractable $module$, but the converse is not true for instanceZ_4 is P-small retractable but not P-small compressible $module$ see remarks and examples (3.2) point(5).
2. $Z\ as\ Z - module$ is P-small retractable $module$, because it's P-small compressible $module$.
3. Every simple $R - module$ is P-small retractable module but not conversely, because $Z\ as\ Z - module$ is a P-small retractable $module$ but not simple.
4. Every retractable $R - module$ is P-small retractable $R - module$, but the converse is not true.
5. Every semi-simple $R - module$ is P-small retractable because it is retractable.
6. Every compressible $module$ is P-small retractable $module$, but the converse is not true for instance Z_4 is P-small retractable but not P-small compressible $module$ see remarks and examples (3.2)point(5).
7. A homomorphic image of a P-small retractable $module$ is a P-small retractable $module$.

Remark(4.3): The direct sum of P-small retractable $module$ is P-small retractable $module$.

Proposition(4.4): A P-small submodule of P-small retractable $module$ is also P-small retractable $module$.

Proof: Let $0 \neq K \ll_P M$ and M be P-small retractable $module$ and let $0 \neq L \leq K \ll_P M$, by remarks and examples (1.2)point(3), [3]. $L \ll_P M$. Since M is P-small retractable, so ∃ a homomorphism $f: M \longrightarrow L$ and $i: K \longrightarrow M$ is the inclusion homomorphism, then $f \circ i: K \longrightarrow L$ be a homomorphism. Therefore K is a P-small retractable $module$.

Proposition(4.5): Let M_1 and M_2 be isomorphic $R - modules$. Then $M_1$is P-small retractable if and only if $M_2$is P-small retractable

Proof: Let $0 \neq N \ll_P M_1$ and suppose that M_2 is P-small retractable. Let $f: M_1 \longrightarrow M_2$ be an isomorphism. Then by[3] $0 \neq f(N) \ll_P M_2$. Put $K = f(N) \ll_P M_2$, we get $h: M_2 \longrightarrow K$ is a homomorphism (by assumption), let $g = f^{-1} \big|_K$, then $g: K \longrightarrow M_1$ is a monomorphism. $g(K) = f^{-1}(f(N)) = N$. Hence we have a composition
H $= g \circ h \circ f$. Hence H: $M_1 \longrightarrow N$ is a monomorphism. Therefore $M_1$is P-small retractable $module$.

Proposition(4.7): Let M be $PS - hollow\ module$, then the following are equivalent

(1) M is retractable $module$.

(2) M is P-small retractable $module$.

Proposition(4.8): If M is P-small quasi-Dedekind $R - module$, then M can't be P-small retractable .

Proof: Suppose that M is P-small quasi-Dedekind $module$ and let $N = Kerf \leq M$, but M is P-small quasi-Dedekind, then $Kerf \ll_P M,\ f \neq 0$, thus $Hom(M, Kerf) = 0$. Therefore M can't be P-small retractable $module$.

Recall that an $R - module\ M$ is called monoform if for each non-zero submodule N of M and for each $f \in Hom_R(N, M), f \neq 0$ implies $Kerf = 0$, [5].

Definition(4.9): An $R - module\ M$ is called P-small monoform if for each non-zero submodule N of M and for each $f \in Hom_R(N, M), f \neq 0$ implies $Kerf \ll_P N$.

Remark(4.10): Every P-small compressible $R - module$ is P-small monoform, but not conversely. For example , Z_8 as $Z - module$ is P-small monoform but not P-small compressible.

Proposition(4.11): Let M be a quasi-Dedekind $R - module$. Then M is P-small monoform if and only if M is P-small compressible.

Proof: Suppose that M is P-small monoform. Let $0 \neq N \ll_P M$, then $0 \neq f \in Hom_R(N, M)$. Since M is quasi-Dedekind , then $f \circ g: M \longrightarrow N \longrightarrow M$ is a monomorphism, hence $g: M \longrightarrow N$ is a monomorphism. Thus M is P-small compressible. Conversely it is clear by remark (4.10).

5. Conclusion

In this work, the class of compressible and retractable modules have been generalized to a new concepts called P-small compressible and P-small retractable modules. Several characteristics of this type of modules have been studied. Sufficient conditions under which these modules with compressible and retractable are discuss

Also we see relations between P-small compressible modules and other related modules as P-small retractable module P-small quasi-Dedekind, P-small monoform.

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