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Pure Maximal Submodules and Related Concepts

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Abstract

In this work we discuss the concept of pure-maximal denoted by (Pr-maximal) submodules as a generalization to the type of R- maximal submodule, where a proper submodule K of an R-module W is called Pr- maximal if $K< H \leq W$,for any submodule H of W is a pure submodule of W, We offer some properties of a Pr-maximal submodules, and we give Definition of the concept, near-maximal, a proper submodule
H of an R-module W is named near (N-maximal) whensoever K is pure submodule of W such that $H < K \leq W$ then K=W.Al so we offer the concept Pr-module, An R-module W is named Pr-module, if every proper submodule of W is Pr-maximal. A ring R is named Pr-ring if whole proper ideal of R is a Pr-maximal ideal, we offer the concept pure local (Pr-local) module an R-module W is named pure local (Pr-local) module. If it has only a Pr-maximal submodule which includes all proper submodule of W. A ring R is named pure local (Pr-local) ring, if R is a Pr-local R-module. We give some relatio among Pr-maximal submodules and others related concept.

Keywords: R-module, R-submodule, Pr-module, Pr-maximal, Pr-local, N-maximal.

Introduction

In this work R is commutative ring with identity, and all R-modules are left until. A proper submodule P of an R-module $W$is named a pure submodule "if for every ideal I of R, $P \cap IW = IP$" [1]. A proper submodule K of an R-module W, is
named maximal in [3] "if whenever H is a submodule of R-Module W with $K < H \leq W$ Implies $H = W$". Abduljaleel and Yaseen in [2] offer the concept of large maximal submodules as a generalization of the concept maximal submodules, "where a proper submodule K of an R-module W is named large-maximal(L-maximal) if $K < H \leq W$ implies H is an essential submodule of W, where a submodule H of R-module W is named essential, if for every non-zero

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submodule L of W, $H \cap L \neq (0)$" [3]. Many authors studies module and submodule for example see [4] and [5]. In [11] B.H.AL-Bahrani generalization of the type of a purely extending modules, defined using Y-closed submodules, In,this discuss, we introduce, the concept of pure-maximal (Pr-maximal) submodules as a generalization of maximal submodule

, where a proper submodule K of an R-module W is named Pr-maximal, if

$K < H \leq W$, for any submodule H of W, implies $H$is a pure a submodule of W, In section two we give several properties of this type of submodules as every multiplication module contains a Pr-maximal submodule. Also, if N, K are non-zero submodule of such that $N \leq K$ if N is Pr-maximal in W then K is Pr-maximal in W and if N is Pr-maximal submodule of an R-module W and I be ideal of R, if $[N_W: I]$ is a proper submodule of W then $[N_W: \mathrm{I}]$ is Pr-maximal submodule. we study the relation, among Pr-maximal (submodules and other related module), In section three we study Pr-maximal submodule, under the multiplication module and we check some condition under which Pr-maximal submodules and maximal submodules are equivalent. Every multiplication module contains a Pr-maximal submodule. Also, we have every cyclic R-module has Pr-maximal submodule. We found if W is a F-regular module then every submodule of W is Pr-maximal.

2. Preliminaries

This section is going to review some well-known definitions in a algebraic theory.

Definition 2.1 [3]

A proper submodule K of an R-module$W$i s named maximal if $H \leq W$ such that $K < H \leq W$ namely H=W

Definition 2.2 [1]

A submodule K of an R-module W is named pure R-submodule if $K \cap IW = IK$ for each ideal I of R

Definition 2.3 [6]

"A submodule $H$of an R-module $W$is called weak maximal if $\frac{W}{}$is F-regular R-module".

Lemma (2.4) [7]

"If f:$W_1 \rightarrow W_2$is an epimorphism and K is pure submodule of W_2, then $f^{-1}(K)$ is pure in W_1."

Definition 2.5 [8]

"An R-module $W$is named pure simple if $W \neq (0)$ and it has no pure, submodule except (0) and W.

Definition 2.6 [4]

An R-module W is called faithful if $ann_R W$=$(\bar{0})$

Definition 2.7 [9,10]

"An R-module M is said to be multiplication if for each submodule $H$of M, there exists an ideal I of R such that H=IM". Equivalently "M is a multiplication R-module if and only if for each submodule $H$of M, H= $[H:_R M]M$" .

Proposition 2.8 [10]

If W is an R-module and has an, unique maximal submodule K, then W is called local module.

3. Pr-Maximal Submodules:

In this section, the basic definitions and facts related to this work are recalled, which starts with the following definition**.**

Definition (3.1)

A sound R-submodule K of an R-module W is named pure maximal (Pr-maximal) submodule of Wif there exists$H \leq W$with $K < H \leq W$ then $H$is pure R- submodule of W ($H \leq_P W$).

Remark and Example (3.2)

1. Every maximal submodule of R-module W is Pr-maximal.

Proof: Impose K maximal R- submodule of R-module Wthere exist 0≠H submodule of W such that $K < H \leq W$ since K is maximal then $H = W$ but W is pure of W therefore is Pr-maximal the convers is not true as the following example in W=Z_4+ Z_2 as a Z-module Let, K=2$Z_4 \oplus (\bar{0})$ and H=$Z_4 \oplus (\bar{0})$ such that 2$Z_4 \oplus (\bar{0}) < Z_4 \oplus (\bar{0}) \leq Z_4 \oplus Z_2$ since H=$Z_4 \oplus (\bar{0})$ is a summand of W =$Z_4 \oplus Z_2$, hence is pure in W, then K is Pr-maximal submodule of W but k is not maximal since $Z_4 \oplus (\bar{0}) \neq Z_4 \oplus Z_2$.

2. A subset of Pr-maximal R-submodule need not be Pr-maximal R-submodule as the breech example in Z_{12}as Z-module impose $3Z_{12} < Z_{12} \leq Z_{12}$ implies 3Z_{12} is Pr-maximal submodule of Z_{12} since $Z_{12} \leq_P Z_{12}$, but 6$Z_{12}$is not Pr-maximal submodule of Z_{12}since 6 $Z_{12} < 2Z_{12} < Z_{12}$ and $2Z_{12}$ is not pure in Z_{12}.
3. Z_4 as Z-module we have $\{\bar{0}, \bar{2}\}$is Pr-maximal since Z_4 is pure of Z_4 and $\{\bar{0},\bar{2}\} < Z_4 \leq Z_4$.
4. $Z_6 = \{\bar{0},\bar{3}\} \oplus \{\bar{0},\bar{2},\bar{4}\}$, $3Z_6$is Pr-maximal of Z_6 since $\{\bar{0}, \bar{3}\} < Z_6 \leq Z_6$ and $2Z_6$ is pr-maximal of Z_6, also since $\{\bar{0},\bar{2},\bar{4}\} < Z_6 \leq Z_6$
5. If $\frac{W}{H}$ is simple, then H is Pr-maximal.

Proof: clearly since $\frac{W}{H}$ is simple implies H is maximal then assist remark (2.2) every, $H$is Pr-maximal.

6. Let H and K are nonzero submodule of Wsuch that $H \leq K \leq W$if H is Pr-maximal of K and K is Pr-maximal of W then H is not Pr-maximal of $W$for example let W=2$Z_{24}$and H=6Z_{24},K=2Z_{24} ,$H$is Pr-maximal in Ksince 62$Z_{24} \leq 2Z_{24}$ and 2$Z_{24} \leq_p$2Z_{24} and K is Pr-maximal in Wsince $K$is maximal but $H$is not Pr-maximal in Wsince 6Z_{24} <2$Z_{24} \leq Z_{24}$ and2$Z_{24}$is not pure in Z_{24}.

7. If W=Z_{12} as, Z-module and H=4Z_{12}a R-submodule of W,then $H$is not Pr-maximal in W since 4Z_{12} <2$Z_{12} \leq Z_{12}$and 2$Z_{12}$is not pure in W.
8. If $W$is a semi simple R-module, thence every sound R-submodule of $W$is R- maximal if and only if it is Pr-maximal for example 2$Z_6$and 3$Z_6$are Pr-maximal submodule of Z_6 as Z-module.

9. Every proper submodule contain in a pure submodule is, Pr-maximal.

Definition (3.3)

A proper submodule H of an R-module $W$is named near-maximal(N-maximal) whenever $K$is pure submodule of Wsuch that $H < K \leq W$then $K = W$.

Remark (3.4)

Pr-maximal submodule need not be N-maximal as the following example Show: In $Z_4 \oplus Z_2$ as a Z-module $2Z_4 \oplus (\bar{0})$ is Pr-maximal since, $2Z_4 \oplus (\bar{0}) < Z_4 \oplus (\bar{0}) \leq Z_4 \oplus Z_2$and $Z_4 \oplus (\bar{0}) \leq_P Z_4 \oplus Z_2$ but $Z_4 \oplus (\bar{0}) \neq Z_4 \oplus Z_2$ so not N-maximal

$4Z_{12}$= {$\bar{0}$,$\bar{4}$,$\bar{8}$} is N-maximal in Z_{12} since $4Z_{12} < Z_{12} \leq Z_{12}$ and Z_{12}=Z_{12}, but not Pr-maximal since $4Z_{12} < 2Z_{12} \leq Z_{12}$ and $2Z_{12}$ is not pure in Z_{12}

Proposition (3.5)

If $W$is an F-regular module then every submodule of W is, Pr-maximal.

Proof

Let H, K be submodule of W such that $H < K \leq W$since W is F-regular then respective submodule of W is pure hence K is pure submodule of Wthen H is Pr-maximal.

Remark (3.6)

If $W$is simple R-module thence every R-submodule of W is Pr-maximal.

Proof: Since every simple$R$- module is regular, thence every maximal is Pr-maximal. **Proposition (3.7)**

If $\frac{W}{H}$ is F-regular module then any pure submodule of W is Pr-maximal.

Proof

Let H is a pure R-submodule of W such that H$< K \leq W$ and since $\frac{W}{H}$ is F-regular then $\frac{K}{H}$ is pure submodule of $\frac{W}{H}$ since H is pure of W then K is pure in W [4] implies H is Pr-maximal submodule of W.

Corollary (3.8)

If H is a weak R-maximal submodule of an R-module W thence every pure submodule of W is P r-maximal.

Proposition (3.9)

If N, K are non- zero submodule of W such that $N \leq K$ if N is Pr-maximal in W then K is Pr-maximal in W.

Proof

Let H submodule of W such that $K < H \leq W$ since $N < K$ and N is pr-maximal of W then H is pure namely K is pr-maximal assist definition (3.1).

Corollary (3.10)

If N and K are sound submodules of R-module W and $N \cap K$is pr-maximal of W, then both N and K are pr-maximal submodule of W.

Proof

Since $N \cap K \leq N$ and $N \cap K$ is Pr-maximal of W, thence assist proposition (3.9) N is Pr-maximal similarity we prove K is Pr-maximal.

Corollary (3.11)

If N, K are nonzero submodules of R-module W if N or K are Pr-maximal, then $N + K$ is Pr-maximal.

Proof: Since N is, Pr-maximal and $N <$ N +K ≤W assist proposition (3.9) implies $N + K$ is Pr-maximal.

Corollary (3.12)

If N is, Pr-maximal submodule of an R-module W and I is an ideal of R, if $[N_W: I]$ is a Proper submodule of W then, $[N_W: I]$ is pr-maximal R-submodule of W. Then [$[N_W: I]$ is Pr-maximal submodule of W.

Proof

Since $N < [N_W: I]$ assist [4] and N is pr-maximal submodule of W then, $[N_W: I]$ is Pr-maximal assist proposition (2.10).

Remark (3.13)

The convers of corollary (2.12) is not true since submodule of Pr-maximal not Pr-maximal as the following example let $W=Z_{12}$as Z-module the ideal of Z-module the ideal I=2Z of Z, and N=$6Z_{12}$ so $[N_W: I]$=$\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ is Pr-maximal since $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\} < Z_{12} \leq Z_{12}$ but $N = 6Z_{12}$ is not Pr-maximal of Z_{12}, since $6Z_{12} < 2Z_{12} \leq Z_{12}$ and $2Z_{12}$ not pure in Z_{12},

Proposition (3.15)

Let f: $W_1 \to W_2$ be an epimorphism, where W_1 *and* W_2 be R-modules. If H is Pr-maximal submodule of W_2, then $f^{-1}(H)$ is Pr-maximal R-submodule of W_1. **Proof**

Suppose $f^{-1}(H) < S \leq W_1$, so $f(f^{-1}(H)) < f(s) \leq W_2$then $H \leq f(s) \leq W_2$, since H is Pr-maximal R-submodule of W_2, thence $f(s) \leq_p W_2$, implies S=$f^{-1}(f(S)) \leq_p W_1$, assist lemma (2.14), so $f^{-1}(H)$ is a Pr-maximal R-submodule of W_1.

Theorem (3.16)

If W is a pure simple R-module and K be a proper submodule of W. Then let the following:

1. K submodule is maximal.
2. K submodule is Pr-maximal.
3. K submodule is N-maximal.

Proof: (1)↔ (2) clearly

(2)→(3) let $K < S \leq W$, since K is, Pr-maximal then$S <_P W$, but W is semi simple, hence K=W, so that K is N-maximal submodule of W.

(1)→(3) clear.

If W is regular, then (3)→(1).

Proposition (3.17)

If W is a semi simple R-module and N be a proper submodule of W. thence consider the following of W

1. N submodule is a maximal.
2. N submodule is a Pr-maximal.
3. N is a N-maximal.

Proof: (1)↔(2) clear

(2)→(3) Let $N < K \leq W$, since N is, Pr-maximal then$K <_P W$, but W is semi simple, hence $K = W$, so that N is N-maximal R-submodule of W (1)→(3) clear.

If W is regular, then (3)→(1)

Remark (3.18): In Z_{12} as Z-module consider the following.

$K < Z_{12}$	maximal	N-maximal	Pr-maximal
$2Z_{12}$	✓	✓	✓
$3Z_{12}$	✓	✓	✓
$4Z_{12}$	×	✓	×
$6Z_{12}$	×	×	×

4. Pure Maximal Submodules and Related Concepts

Proposition (4.1)

Whole multiplication R- module deemed a Pr-maximal R-submodule.

Proof:

Since whole multiplication module breakpoint maximal R-submodule thence assist remark (3.2) we have respective multiplication module contains a Pr-maximal submodule.

Corollary (4.2)

Every cyclic R-module breakpoint Pr-maximal submodule.

Proof

Since respective cyclic R-module is a multiplication R-module so assist proposition (4.1), it breakpoint Pr-maximal.

Theorem (4.3)

If $W$is faithful finitely generated and multiplication R-module and H be a submodule of W thence, the following, are equivalent

1. H submodule is, Pr-maximal of W
2. $[H_R{:}W]$ ideal is Pr-maximal of R.
3. $H{=}IW$ Pr-maxima for some ideal I of R.

Proof: (1) → (2)

Let $H_R{:}W] < J \leq R$ and J is an ideal of $R\ is\ J,H < JW \leq RW$ since W is R-module multiplication, then $H = [H_R{:}W]W \leq JW \leq RW = W$, so $H < JW \leq RW$, since $H$is, Pr-maximal then $JW \leq_P RW = W$, so $J \leq_P R$ [9] and hance $[H_R{:}W]$ is Pr-maximal ideal of R.

(2)→(3)

Since W R-module is multiplication thence for H a R-submodule of W thence H $[H_R{:}W]W$ assist (2) each $[H_R{:}W]$ is ideal Pr-maximal of R then, $H{=}IW$for some ideal Pr-maximal of R.

(3)→(1)

Let $H < K \leq W$ since $H{=}IW$ for some ideal Pr-maximal of R, since W is , multiplication then,$K = JW$ for some Pr-maximal ideal $J$of R since W is multiplication $W = RW, H = IW$, then, $H = JW < JW \leq RW = W$but W faithful finitely generated multiplication W then $I < J \leq R$, by (3) I is Pr-maximal ideal of R, hance $J \leq_P R$ then $JW \leq_P W$ by [8] implies H is Pr-maximal of W.

Now, we introduce the following concept.

Definition (4.4)

An R-module M is named Pr-module, if every proper submodule of M is Pr-maximal. A ring R is named Pr-ring if every proper ideal of R is an Pr-maximal ideal.

Examples (4.5):

1. Z_6 is Pr-module as Z-module
2. Whole semisimpl R-module is Pr-module.
3. Z_{12} as Z-module is not Pr-module, since $6Z_{12}$ is not Pr-maximal submodule of Z_{12}.

Theorem (4.6): If M is a R-module finitely generated faithful and multiplication.
Then, M is a Pr-module iff R is Pr-ring.

Proof: →) Suppose, M is a Pr-module and impose I is a proper ideal of R. Since M is a multiplication R-module, Then, $H = IM$. But M is Pr-module, hence H is a Pr-maximal submodule of M. Assists Theorem (3.3),I is an ideal, Pr-maximal of R.

←) Assume that R is a Pr-ring and let H be a proper submodule of M. Since M is R-module a multiplication, thence $H = IM$, for some ideal I of R. Since I is a Pr-maximal ideal, then assists Theorem (4.3) H is Pr-maximal submodule of M. thus M is Pr-module.

Remark (4.7)

Not all, finite R-module is local for example Z_6= {$\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\ \bar{5}$ } is not local since $2Z_6$ and $3Z_6$ are maximal submodules this not local but Z_4 R-module is local since $2Z_4$is the only proper maximal submodule of Z_4

Proposition (4.8)

Every local R-module has Pr-maximal submodule.

Proof: Let W is local R-module, then W has only one maximal R-submodule implies has Pr-maximal submodule assist remark (2.2).

Remark (4.9)

The convers of proposition (4.8) is not true for example Z_6 has Pr-maximal submodules but not local.

We need to give the following concept

Definition (4.10)

An R-module M is named pure local (Pr-local) module. If it has only one Pr-maximal submodule which contains all proper submodule of M. A ring R is called pure local (Pr-local) ring, if R is a Pr-local R-module

Remark and Examples (4.11)

1. The Z-module $3Z_{24}$ is a Pr-local module, since it has only one R-submodule Pr-maximal that is $6Z_{24}$.
2. Every local is, Pr-local but not conversely for example: The Z-module $2Z_{12}$ in Z_{12} as Z-module is Pr-local since $4Z_{12}$ is the only Pr-maximal submodule of $2Z_{12}$.But not local since $4Z_{12}$ and $6Z_{12}$ are maximal submodules of $2Z_{12}$.
3.If W is a non-zero multiplication and Pr-local R-module and H is a non-zero Pr-maximal R-submodule of W, thence H not necessary, pure submodule for example: If W=Z_4 as Z-module, W is non-zero multiplication and Pr-local Z-module and $2Z_4$ is non-zero, submodule, Pr-maximal of Z_4 but not pure.

Remark (4.12): description of R-module.

R-module	local	Pr-local
Z_4	✓	✓
Z_6	×	×
$3Z_{12}$	×	✓

5. Conclusions

We will try to generalize the concept of Pr-maximal R-submodules to some other concepts in future works.

In this study, the concepts of Pr-maximal R-submodules is studied as a generalization of maximal submodule and some properties of this concept are investigated such as:

1. Every maximal submodule of R-module W is Pr-maximal.
2. A subset of Pr-maximal R-submodule need not be Pr-maximal R-submodule.
3. If $W$is an F-regular module then every submodule of $W$is Pr-maximal.
4. If N, K are non-zero submodule of W such that $N \leq K$ if N is Pr-maximal in W then K is Pr-maximal in W.
5. Let $f: W_1 \rightarrow W_2$ be an epimorphism, where W_1 and W_2 be R-modules. If H is Pr-maximal submodule of W_2, then $f^{-1}(H)$ is Pr-maximal R-submodule of W_1.
6. Every cyclic R-module breakpoint Pr-maximal submodule.
7. Every local R-module breakpoint Pr-maximal R-submodule.

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