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Semi-Small Compressible Modules and Semi-Small Retractable Modules

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Abstract

Let R be a commutative ring with 1 and M be left unitary $R-module$. In this paper we introduced and studied concept of semi-small compressible module (a $R-module\ M$ is said to be semi-small compressible module if M can be embedded in every nonzero semi-small submodule of M. Equivalently, M is semi-small compressible module if there exists a monomorphism : $M \longrightarrow N$,$0 \neq N \ll_{sem} M$, $R-module\ M$ is said to be semi-small retractable module if $Hom(M,K) \neq 0$, for every non-zero semi-small sub module Kin M. Equivalently, M is semi-small retractable if there exists a homomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_{sem} M$.

In this paper we introduce and study the concept of semi-small compressible $modules$ and semi-small retractable $module$s as a generalization of compressible $module$ and retractable $module$ respectively and give some of their advantages characterizations and examples.

Keywords: compressible module, retractable module, small sub module, semi-small sub module, semi-small compressible module, semi-small retractable module.

1. Introduction

Let R be a commutative ring with 1 and M be a left unitary $R-module$. Authors that introduced and studied the concept of small sub modules where a proper sub module N of an R −module M is termed a small sub module $(N \ll M)$, if $N + L \neq M$ for every sub module L of

M[1]. A proper sub module N of M is said to be primary if whenever $r \in R$, $m \in M$ with $rm \in N$ implies either $m \in N$ or $r^n \in [N:M]$ for some positive integer n , where $[N:M] = \{r \in R: rM \subseteq N\}$[2] . In [3] Mijbas and K. Abdullah introduced and studied the concept of semi-small sub modules , where a sub module N of an $R-module\ M$ is termed semi-small sub module $N \ll_{sem} M$ if $N + B \neq M$ for any primary sub module B of M. An $R-module\ M$ is termed compressible if M can be embedded in every non-zero sub module in M,[4]. An $R-module\ M$ is said to be semi-small compressible if M can be embedded in every non-zero semi-small sub module of$M$. Equivalently, M is semi-small compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever$0 \neq N \ll_{sem} M$.

Under which condition we introduce and study the concept of semi-small compressible as a generalization of compressible module, and we give some properties, characterization and examples. In addition, we see that under condition semi-small compressible, small compressible and compressible are equivalent. An $R-module\ M$ is said to be semi-small retractable module if $(M, K) \neq 0$, for every non-zero semi-small sub module $K$of M, some of their advantages characterizations and examples are given. We also study the relation between semi-small compressible, semi-small retractable module and some of classes of modules.

2. Preliminaries

Definition (2.1): Let M be an $R-$module and $N \leq M$:

(1) N is termed small submodule of M , $(N \ll M)$ if $N + K = M$ implies $K = M$, for any sub module K of M[1].

(2) An $R-$module$M$ is termed hollow if every proper sub module of M is small[5].

(3) A proper submodule$N$ of M is termed primary if whenever $r \in R$, $m \in M$ such that $r.m \in N$ implies either $m \in N$ or $r^n \in [N:M]$ for some positive integer n , where $[N:M] = \{r \in R: rM \subseteq N\}$ [2]

(4) A proper sub module N is termed semi-small sub module of M , $(N \ll_{sem} M)$ if $N + P \neq M$, for any primary submodule P of M, [3].

Remarks and examples (2.2): [3]

(1) $(\bar{6})$ is a semi-small sub$module$ of Z_{12} as $Z-module$.

(2) (0) is the only semi-small sub module If M is a semi-simple $module$.

(3) $(\bar{2})$ and $(\bar{3})$ are not semi-small sub module of Z_6.

(4) Each small sub module is semi-small. However, conversely is true or not in general.

(5) Let N be a proper sub module in M. If W$\subset N \ll_{sem} M$. Therefore W$\ll_{sem} M$

(6) Let M, M' be $R-modules$ and $\psi: M \longrightarrow M'$ be an $R-homomorphism$.

If $A \ll_{sem} M$ with $ker\,\psi \leq A$, then $\psi(A) \ll_{sem} M'$.

3. Semi-Small Compressible Modules

In this section, we introduce the concept of semi-small compressible $module$ as a generalization of compressible $module$. Give some of it is basic properties, examples and characterizations of this concept.

Definition (3.1): An $R-module\ M$ is said to be semi-small compressible if M can be embedded in every non-zero semi-small sub module of M. Equivalently, M is semi-small compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_{sem} M$.

Remarks and Examples (3.2):

1. It is obvious that every compressible module is semi-small compressible $module$, but the converse is not true.
2. Z_6 as Z-module is not semi-small compressible since $(\bar{0})$ is the only semi-small sub module, see, [3].
3. $Z\ as\ Z-module$ is semi-small compressible module, because it is compressible $module$, see[4].
4. If $an\ R-moduleM$ is semi-simple, then M is not semi-small compressible module (Because (0) is the only semi-small sub module in M).
5. Every simple $R-module$ is semi-small compressible module but not conversely, because $Z\ as\ Z-module$ is a semi-small compressible $module$ but not simple.
6. $Z_{12}\ as\ Z-module$ is not semi-small compressible. (Because Z_{12} cannot be embedded in $\langle\bar{6}\rangle$ and $\langle\bar{6}\rangle \ll_{sem} Z_{12}$). In addition $Q\ as\ Z-module$ is not semi-small compressible module, since $Hom_R(Q,Z) = 0$, where $Z \ll_{sem} Q$. (Since every finitely generated sub module of Q is semi-small sub module in Q.
7. A homomorphic image of a semi-small compressible $module$ need not be semi-small compressible in general for example $Z\ as\ Z-module$ is a semi-small compressible module and $\frac{z}{12z} \simeq z_{12}$ is not semi-small compressible module see (5).

Proposition (3.3): A semi-small sub module of semi-small compressible $module$ is also semi-small compressible $module$.

Proof: Let $0 \neq K \ll_{sem} M$ and M be semi-small compressible $module$ and let $0 \neq L \leq K \ll_{sem} M$, then $L \ll_{sem} M$ [3]. Since M is semi-small compressible, so $\exists$ a monomorphism $f: M \longrightarrow L$ and $i: K \longrightarrow M$ is the inclusion homomorphism, then $f \circ i: K \longrightarrow L$ is a monomorphism. Therefore K is a semi-small compressible $module$.

Proposition (3.4): Let M_1 and M_2 be isomorphic $R-modules$. Then $M_1$is semi-small compressible if and only if $M_2$is semi-small compressible.

Proof: Suppose that M_2 is semi-small compressible and let $\phi: M_1 \longrightarrow M_2$ be an isomorphism. Let $0 \neq N \ll_{sem} M_1$, then by[3] $0 \neq \phi(N) \ll_{sem} M_2$. Put $K = \phi(N) \ll_{sem} M_2$, so $\alpha: M_2 \longrightarrow K$ is a monomorphism (by assumption), let $h = \phi^{-1}\big|_K$, then $g: K \longrightarrow M_1$ is a monomorphism. $g(K) = \phi^{-1}(\phi(N)) = N$. Hence, we have a composition. Let $\psi = h \circ \alpha \circ \phi$. Hence, $\psi: M_1 \longrightarrow N$ is a monomorphism. Therefore $M_1$is semi-small compressible $module$.

Remark (3.5):_ The direct sum of semi-small compressible $module$ need not be semi-small compressible. Consider the following example, let$Z_6 = Z_3 \oplus Z_2$ as $Z-module$. Z_3, Z_3 are semi-small compressible modules, but Z_6 is not semi-small compressible module see remarks and examples (3.2) point (2).

An $R-module\ M$ is said to be small compressible if M can be embedded in every nonzero small sub module of M. Equivalently, M is small compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll M$[4].

Remark $(\mathbf{3.6})$:

Every semi-small compressible module is small compressible module.

Proof: Let $0 \neq N \ll M$, then by [3] $N \ll_{sem} M$ and M is semi-small compressible $module$, therefor M is small compressible module. Conversely is not true for instance Z_6 as Z-module is small compressible, [6]. However, not semi-small compressible see remarks and examples (3.2) point (2).

Proposition $(\mathbf{3.7})$: Let M be a finitely generated (or multiplication) $R-module$. Then M is small compressible if and only if $M$is semi-small compressible.

Proof: Let $N \ll_{sem} M$. We want to show that $M$is semi-small compressible. Since M is finitely generated ((or multiplication), then by proposition (1.3)[3], so $N \ll M$, but M is small compressible $R-module$. Therefore M is semi-small compressible. Conversely clear by remark (3.6).

Proposition $(\mathbf{3.8})$: Let M be a hollow $module$. Then the following statements are equivalent:

(1) M is compressible $module$.
(2) M is semi-small compressible $module$.
(3) small compressible $module$.

Proof: (1) $\Longrightarrow$ (2) It is clear by remarks and examples (3.2) point (1).

(2) $\Longrightarrow$ (3) It is clear by remark (3.6).

(3) $\Longrightarrow$ (1)Let$K \leq M$. Since M is $hollow\ module$ and small compressible $module$, then $\exists$ a monomorphism $f: M \longrightarrow K$. Therefor M is compressible $module$.

An $R-module$ M is termed quasi-Dedekind $module$ if for all $f \in End_R(M)$, $f \neq 0$ implies $Kerf = 0.$ [7]

An $R-module$ M is termed small quasi-Dedekind $module$ if for all $f \in End_R(M)$, $f \neq 0$ implies $Kerf \ll M.$ [7].

We introduce the following

Definition (3.9): An $R-module$ M is termed semi-small quasi-Dedekind $module$ if for all $f \in End_R(M)$, $f \neq 0$ implies $Kerf \ll_{sem} M$.

Remarks(3.10):

1. It is clear that every quasi-Dedekind$R-module$ is semi-small quasi-Dedekind, but not conversely.
2. Every small quasi-Dedekind is semi-small quasi-Dedekind, but not conversely.

Proof: Let $0 \neq f \in End_R(M)$, where M be an $R-module$ since M is a small quasi-Dedekind, then $Kerf \ll M$, hence $Kerf \ll_{sem} M$. Thus M is a semi-small quasi-Dedekind $module$.

We introduce an $S-hollow$ $R-module$ as a generalization of a hollow module.

Definition (3.11): An R −module$M$ is termed S-hollow if every proper submodule in M is semi-small.

Remarks (3.12):

(1) Every hollow module is S-hollow module, but the converse is not true.
(2) Every simple module is S-hollow module, but not conversely for example Z_4 as Z −module is S-hollow module, but not simple.
(3) If an $R-moduleM$ is a semi-simple, then M is not S-hollow module (since (0) is the only semi-small sub module).

(4) Z_{12} as Z −module is not S-hollow module, because $(\bar{3})$ is not semi-small sub module in Z_{12} ,[3].

Proposition (3.13): Let M be an S-hollow $R-module$. Then M is compressible module if and only if M is a semi-small compressible module.

Proof: let $0 \neq N \leq M$ and $R-module\ M$ be S-hollow, then $N \leq_{sem} M$, but M is a semi-small compressible module, thus there exists a monomorphism $f: M \longrightarrow N$. Therefore M is compressible module. Conversely clear by remarks and examples (3.2) point (1).

4. Semi-Small Retractable Modules

We introduce the concept of semi-small retractable $module$ as a generalization of retractable $module$. Give some of basic properties, examples and characterizations of this concept.

Definition (4.1): An $R-module\ M$ is said to be semi-small retractable if $(M.K) \neq 0$, for every non-zero semi-small sub module $K$of M. Equivalently, M is semi-small retractable if there exists a homomorphism $f: M \longrightarrow N$ whenever $0 \neq N \ll_{sem} M$.

Remarks and Examples (4.2):

1. It is obvious that every semi-small compressible module is semi-small retractable $module$. However, conversely is not true for instance z_{12} is semi-small retractable but not semi-small compressible $module$ see remarks and examples (3.2) point (7).
2. $Z\ as\ Z-module$ is semi-small retractable $module$, because it is semi-small compressible $module$.
3. Every simple $R-module$ is semi-small retractable module but not conversely, because $Z\ as\ Z-module$ is a semi-small retractable $module$ but not simple.
4. Every retractable $R-module$ is semi-small retractable $R-module$, but the converse is not true ingeneral.
5. Every semi-simple $R-module$ is semi-small retractable because it is retractable.
6. Every compressible $module$ is semi-small retractable $module$, but the converse is not true for instance z_6 is semi-small retractable but not compressible $module$ see,[4].

Proposition (4.3): A semi-small sub module of semi-small retractable $module$ is also semi-small retractable $module$.

Proof: Let $0 \neq K \ll_{sem} M$ and M be semi-small retractable module. Let $0 \neq L \leq K \ll_{sem} M$, by [3] $L \ll_{sem} M$. Since M is semi-small retractable, so $\exists$ a homomorphism $f: M \longrightarrow L$ and $i: K \longrightarrow M$ is the inclusion homomorphism, then $f \circ i: K \longrightarrow L$ be a homomorphism. Therefore K is a semi-small retractable $module$.

Proposition (4.4): Let M_1 and M_2 be isomorphic $R-modules$. Then $M_1$is semi-small retractable if and only if $M_2$is semi-small retractable

Proof: Suppose that M_2 is semi-small retractable and let $f: M_1 \longrightarrow M_2$ be an isomorphism. Let $0 \neq N \ll_{sem} M_1$, then by[3] $0 \neq f(N) \ll_{sem} M_2$. Put $K = f(N) \ll_{sem} M_2$, so $\theta: M_2 \longrightarrow K$ is a homomorphism (by assumption), let $g = f^{-1} \big|_K$, then $g: K \longrightarrow M_1$ is a homomorphism. $g(K) = f^{-1}(f(N)) = N$, hence, we have a composition. Let $\text{H} = g \circ \theta \circ f$. Hence $\text{H}: M_1 \longrightarrow N$ is a homomorphism. Therefore $M_1$is semi-small retractable $module$.

Proposition (4.5): Let M be an $S-hollow\ module$, then the following are equivalent

(1) M is retractable $module$.

(2) M is semi-small retractable $module$.

Proof: (1) $\Longrightarrow$ (2) clearly by remarks and examples (4.2) point (4).

(2) $\Longrightarrow$ (1)Let $0 \neq N \leq M$ and $R-module\ M$ be S-hollow, then $N \leq_{sem} M$, but M is a semi-small retractable module. Thus, $Hom(M,N) \neq 0$ .Therefore M is retractable module.

Proposition (4.6): If M is semi-small quasi-Dedekind $R-module$, then M cannot be semi-small retractable.

Proof: Suppose that M is semi-small quasi-Dedekind $module$ and let$N = Kerf \leq M$, but M is semi-small quasi-Dedekind, then$Kerf \ll_{sem} M$,$f \neq 0$, thus $Hom(M, Kerf) = 0$. Therefore M cannot be semi-small retractable $module$.

Recall that an $R-module\ M$ is termed monoform if for each non-zero sub module N of M and for each $f \in Hom_R(N,M), f \neq 0$ implies $Kerf = 0$, [8].

Definition (4.7): An $R-module\ M$ is termed semi-small monoform if for each non-zero sub module N of M and for each $f \in Hom_R(N,M), f \neq 0$ implies $Kerf \ll_{sem} N$.

Remarks and examples (4.8):

(1) Every semi-small compressible $R-module$ is semi-small monoform, but not conversely. For example, Z_4 as $Z-module$ is semi-small monoform but not semi-small compressible.

(2) Every semi-small quasi-Dedekind $R-module$ is semi-small monoform. However, not conversely. For example, Z_4 as $Z-module$ is semi-small monoform but not semi-small quasi-Dedekind.

Proposition (4.9): Let M be a quasi-Dedekind $R-module$. Then M is semi-small monoform if and only if M is semi-small compressible.

Proof: Suppose that M is semi-small monoform. Let $0 \neq N \ll_{sem} M$, then $0 \neq f \in Hom_R(N,M)$. Since M is quasi-Dedekind , then $f \circ g: M \longrightarrow N \longrightarrow M$ is a monomorphism, hence $g: M \longrightarrow N$ is a monomorphism. Thus M is semi-small compressible. Conversely see remark (4.8) point (1).

5. Conclusion

In this work, the class of compressible and retractable modules have been generalized to new concepts called semi-small compressible and semi-small retractable modules. Several characteristics of this type of modules have been studied. Sufficient conditions under which these modules with compressible and retractable are discussed.

In addition, we see relations between semi-small compressible modules and other related modules as semi-small retractable module semi-small quasi-Dedekind, semi-small monoform.

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