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An Analytical Study of the Convergence and Stability of the New Four-Step Iterative Schemes

Zena Hussein Maibed *
Department of Mathematics, College of Education for Pure Sciences Ibn AL-Haitham, University of Baghdad, Baghdad, Iraq.

Omar Mohammed Abbas Joodi
Department of Mathematics, College of Education for Pure Sciences Ibn AL-Haitham, University of Baghdad, Baghdad, Iraq.

Shrooq Bahjat Smeein
Department of Information Section Mathematics,
University of Technology and Applied Sciences-Muscat,
Sultanate of Oman.

***Corresponding Author:** mrs_zena.hussein@yahoo.com

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Abctract

Based on the needs of the scientific community, researchers tended to find new iterative schemes or develop previous iterative schemes that would help researchers reach the fixed point with fewer steps and with stability, will be define in this paper the Multi_Iplicit Four-Step Iterative (MIFSI) which is development to four-step impicit fixed point iterative, to develop the aforementioned iterative scheme, we will use a finite set of projective functions ,nonexpansive function and finite set from a new functions called generalized quasi like contractive which is an amalgamation of quasi contractive function and contractive like function , by the last function and a set of sequential organized steps, we will be able to prove the existence of the fixed point(f-point) of the MIFSI and Fur-Step Iterative(FSI), furthermore, we found MIFSI faster than FSI. On the other hand, the stability of the new iterative is proved.

Keywords: Fixed point, implicit iterative schemes, convergent, projective function, stability, contractive function, four-step iterative.

1. Introduction and preliminary notes

The fixed-point theory is a key component in the solution of numerous issues in a wide range of scientific disciplines. Takahashi who originally introduced the idea of convexity in metric spaces[1]. Metric spaces that are convex are more generalized. Fixed point theorems in

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convex metric spaces (CMS) have been studied by numerous researchers, including Ciric [2], Shimiz, and Takahashi [3], and many more.

Definition 1.1:[1]: Let $\mathcal{X}: S \times S \times [0,1] \to S$ is namely convex structure when the condition is hold:

$$d\big(t, \mathcal{X}(ɼ, s, ƛ)\big) \leq ƛ d(t, ɼ) + (1 - ƛ) d(t, s) \qquad (1.1)$$

When S is metric space, say about the metric space and convex structure $\mathcal{X}$ which is denoted by $(S, d, ƛ)$. Let U be a nonempty, closed subset of metric space S, say U is closed convex subset if $\mathcal{X}(ɼ, s, ƛ) \in U$ for all $ɼ, s \in U$ and $ƛ \in [0,1]$.$Ŧ: S \to S$ be self mapping, while some CMS cannot be embedded into normed spaces, all normed spaces are inherently CMS.

Example1.1. Let $S = \{(r_1, r_2, r_3) \in R^3 : r_1, r_2, r_3 > 0\}$. For $r = (r_1, r_2, r_3)$,
$s = (s_1, s_2, s_3) \in S$ $t = (t_1, t_2, t_3)$, and $\alpha, \beta, \gamma \in I = [0,1]$ with $\alpha + \beta + \gamma = 1$, we define a mapping $\mathcal{X}: S^3 \times I^3 \to S$ by :

$$\mathcal{X}(ɼ, s, t; \alpha, \beta, \gamma) = (\alpha r_1 + \beta s_1 + \gamma t_1, \alpha r_2 + \beta s_2 + \gamma t_2, \alpha r_3 + \beta s_3 + \gamma t_3).$$

And define a metric $d: S \times S \to [0, \infty)$ by :

$$d(ɼ, s) = |r_1 s_1 + r_2 s_2 + r_3 s_3|.$$

Then, it is clear that $(S, d, \mathcal{X})$ is CMS, but not normed space.

When the Banach principle could not be applied, Mann [4] developed the Mann iterative technique, recognized as one-step iterative, so as to demonstrate that the series tends toward the f- points. As a follow-up to Mann's iterative technique, Ishikawa [5] developed a new iterative process recognized as two-step iterative to attain the convergence of a Lipschitzian pseudocontractive operator in 1974, using the techniques of solution iteration and the auxiliary principle.Noor [6] developed the Noor iterative scheme and also recognized the three-step iterative scheme to investigate the approximation of the solutions to the inclusions of variations in Hilbert spaces. Using contractive-like operators, Asaduzzaman studied a four-step fixed-point iterative technique and its convergence in real Banach space [7].

The set for all f-points of Ŧ denoted by$F(Ŧ) = \{ɼ \in U;\ Ŧɼ = ɼ\}$ can be written as the f-point equation: $Ŧɼ = ɼ$ for a wide variety of physical equations. We solve this problem by selecting an initial value, $ɼ_0$, and solving it iteratively. The iterative definition of the sequence $\{ɼ_n\}_{n=0}^{\infty}$ leads to an increasing f-point in the constant U, or a mapping g from U on to U nonexpansive if the next statement holds for any $ɼ, s \in U$

$$d(gɼ, gs) \leq d(ɼ, s) \qquad (1.4)$$

Numerous writers have presented and explored alternative iteration strategies for approximating fixed points for diverse classes of contractive circumstances and studied the convergence, rate of convergence and stability (see[8–23]).

The function $ꝑ_U: S \to U$ is called the projective metric function, such that $d\big(ɼ, ꝑ_U(ɼ)\big) \leq min_{s \in U} d(ɼ, s)$ for all $ɼ \in S$ and $s \in U$. There are three main parts to this paper: first, a study of the convergence of each of MIFSI and FSI; second, a study of the acceleration between the two previous iterative schemes and third, a proof of stability for the new iterative.

Definition 1.2 [14]**:** Any mappingŦ is called quasi-contractive if there exists $\varpi \geq 0$ and $p \in (0,1)$ such that

$$d(Ŧɼ, Ŧs) \leq \varpi d(ɼ, Ŧɼ) + p d(ɼ, s) \quad \forall ɼ, s \in U \qquad (1.5)$$

Definition1.3 [15]**:** Any mapping$Ŧ$ is called contractive-like mapping if the statement below is true:

$$d(Ŧr, Ŧs) \leq \emptyset\big(d(r, Ŧr)\big) + \omega d(r, s) \qquad \forall r, s \in U \tag{1.6}$$

Where $\emptyset: R^{+} \to R^{+}$ strictly increasing with $\emptyset(0) = 0$ and $\omega \in (0,1)$.

Definition1.4 [7]**:** The FSI is defined as follows:

$$\begin{aligned} r_n &= (1 - a_n)r_{n-1} + a_n Ŧs_n \\ s_n &= (1 - b_n)r_{n-1} + b_n Ŧt_n \\ t_n &= (1 - c_n)r_{n-1} + c_n Ŧv_n \\ v_n &= (1 - q_n)r_{n-1} + q_n Ŧr_{n-1} \qquad n \geq 1 \end{aligned} \tag{1.7}$$

$\{a_n\}, \{b_n\}, \{c_n\}$,nd $\{q_n\}$ real sequences in [0,1].

Definition1.5 [16]**:** Let (S, d) be a metric space, then the sequence $\{r_n\}_{n=0}^{\infty}$ convergence to $f \in S$. If for every $\epsilon > 0$, there exists $k \in N$, such that $d(r_n, f) < \epsilon$, for every $n > k$, and write $\lim_{n\to\infty} r_n = f$ or write $r_n \to f$.

Definition 1.6 [17]**:** Let $\{r_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ be two sequences lies in R converge to r and s, respectively, such that $\ell = \lim_{n\to\infty} \frac{|r_n - r|}{|s_n - s|}$:

1- If $\ell = 0$ then $\{r_n\}_{n=0}^{\infty}$ is converge to r faster than $\{s_n\}_{n=0}^{\infty}$ converge to s.

2- If $0 < \ell < \infty$ then $\{r_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ have the same rate.

Some authors have defined new iterative schemes in different spaces and functions,and demonstrated their acceleration compared to the currently leading iterative schemes, see [18–21].

Definition1.7 [22]: suppose $(S, d, \mathcal{X})$ are a CMS and $Ŧ: S \to S$ self-mapping, $f \in F(Ŧ)$. Let $\{r_n\}_{n=0}^{\infty} \subset S$ be the sequence produced by an iterative method of hiring Ŧ with the definition given by:

$$r_{n+1} = \mathfrak{f}_{Ŧ,a_n}^{r_n}, \text{ n=0,1,2,…} \tag{1.8}$$

Some functions $\mathfrak{f}_{Ŧ,\alpha_n}^{r_n}$ have a convex structure with $a_n \in [0,1]$ and $r_0 \in S$ the initial approximation , $r_n \to f$. Let $\{s_n\}_{n=0}^{\infty} \subset S$ and $\epsilon_n = d\left(s_{n+1}, \mathfrak{f}_{Ŧ,a_n}^{s_n}\right), n = 0,1,2, \ldots$. say $r_{n+1} = \mathfrak{f}_{Ŧ,a_n}^{r_n}$ is Ŧ-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} s_n = f$

Lemma 1.1[17]: If $\sigma \in R$, $0 \leq \sigma < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers and $\lim_{n\to\infty} \epsilon_n = 0$, $\{g_n\}_{n=0}^{\infty}$ any sequence of positive number satisfying:

$$g_{n+1} \leq \sigma g_n + \epsilon_n \qquad \text{n=0,1,2,…} \tag{1.9}$$

Then $\lim_{n\to\infty} g_n = 0$.

In this paper, we denote that $\mathfrak{p}_{0U}, \mathfrak{p}_{1U}, \mathfrak{p}_{2U}, \ldots,$and $\mathfrak{p}_{kU}$ are finite projective metric functions, g nonexpansive function. We represent for f-point of $Ŧ, \mathfrak{p}_U$, g by $F(Ŧ), (\mathfrak{p}_U)$and $F(g)$ respectively, f is common f-point if $f \in F(Ŧ) \cap F(g) \cap F(\mathfrak{p}_U)$, and represent for the set of all common f-point by $CF(Ŧ, \mathfrak{p}_U, g)$.

2. Main Results

In this section we define MIFSI and study the convergence, rate of converge and stability with respect to generalized quasi like contractive.

Definition 2.1 Any mapping$Ŧ$ is called generalized quasi like contractive mapping if

$$d(Ŧr, Ŧs) \leq ƛd(r,s) + \xi\emptyset\big(d(Ŧr,r)\big) + \eta \min\{d(Ŧr,r), d(Ŧs,s)\} ,\forall r, s \in U \quad (2.1)$$

Where $\emptyset: R^+ \to R^+$ with $\emptyset(0) = 0$, $ƛ \in [0,1]$ and $\eta, \xi \geq 0$.

Definition 2.2 The multi_implicit four-step iterative (MIFS) is defined as follows:

$$r_n = a_{n0}\mathrm{p}_{0U}r_{n-1} + \sum_{i=1}^{k} a_{ni}\, Ŧ^i s_n$$
$$s_n = b_{n0}gt_n + (1-b_{n0})\mathrm{p}_{iU}Ŧ^0 t_n$$
$$t_n = c_{n0}\mathrm{p}_{0U}v_n + \sum_{i=1}^{k} c_{ni}\, \mathrm{p}_{iU}Ŧ^i v_n$$
$$v_n = q_{n0}r_n + (1-q_{n0})Ŧ^0 r_n \qquad n \geq 1 \quad (2.2)$$

Where $Ŧ^i$ are finite generalized quasi like contractive mapping define by (2.1) such that *i=0,1,2,…,k*. $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{q_n\}$ real sequences in [0,1].

Now we state and prove the convergence and stability theorem,for MIFS iteratives .

Theorem 2.1: Let $Ŧ^i$ be a finite generalized quasi like contractive mappings for all $i = 0,1,2,\dots,k$ and g be a nonexpansive mapping if $f \in C\mathrm{F}(Ŧ^i, \mathrm{p}_{iU}, g)$. Then, the MIFS$\{r_\mathrm{n}\}_{\mathrm{n}=0}^{\infty}$ defined by (2.2) with $\Sigma(1-a_n) = \infty$, converges to the f-point f of $Ŧ^i$:.

Proof: Let $f \in C\mathrm{F}(Ŧ^i, \mathrm{p}_{iU}, g)$, then from $\{\mathrm{r_n}\}_{\mathrm{n}=0}^{\infty}$MIFS

$$\begin{aligned} d(r_n, f) &= d\big(\mathcal{X}\big(\mathrm{p}_{0U}r_{n-1}, Ŧ^i s_n, a_{ni}\big), f\big) \\ &\leq a_{n0}d(\mathrm{p}_{0U}r_{n-1}, f) + \textstyle\sum_{i=1}^{k} a_{ni}d\big(Ŧ^i s_n, f\big) \\ &\leq a_{n0}d(r_{n-1}, f) + \textstyle\sum_{i=1}^{k} a_{ni}\begin{pmatrix} ƛ_i d(s_n, f) + \xi_i\emptyset_i\big(Ŧ^i f, f\big) \\ +\eta_i\, min\{d\big(Ŧ^i s_n, s_n\big), d\big(Ŧ^i f, f\big)\} \end{pmatrix} \\ &\leq a_{n0}d(r_{n-1}, f) + ƛ_i \textstyle\sum_{i=1}^{k} a_{ni}d(s_n, f) \end{aligned} \quad (2.3)$$

$$\begin{aligned} d(s_n, f) &= d(\mathcal{X}(gt_n, \mathrm{p}_{0U}Ŧ^0 t_n, b_{n0}), f) \\ &\leq b_{n0}d(gt_n, f) + (1-b_{n0})d(f_{0U}Ŧ^0 t_n, f) \\ &\leq b_{n0}d(t_n, f) + (1-b_{n0})d(Ŧ^0 t_n, f) \\ &\leq b_{n0}d(t_n, f) + (1-b_{n0})\begin{pmatrix} ƛ_0 d(t_n, f) + \xi_0\emptyset_0\big(Ŧ^i f, f\big) \\ +\eta_0 min\{d(Ŧ^0 t_n, t_n), d\big(Ŧ^i f, f\big)\} \end{pmatrix} \\ &\leq (b_{n0} + (1-b_{n0})ƛ_0)d(t_n, f) \end{aligned} \quad (2.4)$$

$$\begin{aligned} d(t_n, f) &= d\big(\mathcal{X}\big(\mathrm{p}_{0U}v_n, \mathrm{p}_{iU}Ŧ^i v_n, c_{ni}\big), f\big) \\ &\leq c_{n0}d(f_{0U}v_n, f) + \textstyle\sum_{i=1}^{k} c_{ni}d(\, f_{iU}Ŧ^i v_n, f) \\ &\leq c_{n0}d(v_n, f) + \textstyle\sum_{i=1}^{k} c_{ni}d(\, Ŧ^i v_n, f) \\ &\leq c_{n0}d(v_n, f) + \textstyle\sum_{i=1}^{k} c_{ni}\begin{pmatrix} ƛ_i d(v_n, f) + \xi_i\emptyset_i\big(Ŧ^i f, f\big) \\ +\eta_i\, min\{d\big(Ŧ^i v_n, v_n\big), d\big(Ŧ^i f, f\big)\} \end{pmatrix} \\ &\leq d(v_n, f)(c_{n0} + ƛ_i \textstyle\sum_{i=1}^{k} c_{ni}) \end{aligned} \quad (2.5)$$

$$\begin{aligned} d(v_n, f) &= d(\mathcal{X}\big(r_n, Ŧ^0 r_n, q_{n0}\big), f) \\ &\leq q_{n0}d\big(r_n, f\big) + (1-q_{n0})d\big(Ŧ^0 r_n, f\big) \\ &\leq q_{n0}d\big(r_n, f\big) + (1-q_{n0})\begin{pmatrix} ƛ_0 d\big(r_n, f\big) + \xi_0\emptyset_0(Ŧ^0 f, f) \\ +\eta_0\, min\{d\big(Ŧ^0 r_n, r_n\big), d\big(Ŧ^i f, f\big)\} \end{pmatrix} \\ &\leq (q_{n0} + (1-q_{n0})ƛ_0)d(r_n, f) \end{aligned} \quad (2.6)$$

Take $ƛ = max\{\, ƛ_i\, , i = 0,1,2 \dots k\}$.

From (2.3),(2.4),(2.5), and (2.6) we have :

$$d(r_n,f) \le a_{n0}\, d(r_{n-1},f) + \left[\lambda \sum_{i=1}^{k} a_{ni} \begin{pmatrix} q_{n0} + \\ (1-q_{n0})\lambda \end{pmatrix} \begin{pmatrix} c_{n0} + \\ \lambda \sum_{i=1}^{k} c_{ni} \end{pmatrix} \begin{pmatrix} b_{n0} + \\ (1-b_{n0})\lambda \end{pmatrix}\right] d(r_n,f)$$

$$(1-\left[\lambda \sum_{i=1}^{k} a_{ni} \begin{pmatrix} q_{n0} + \\ (1-q_{n0})\lambda \end{pmatrix} \begin{pmatrix} c_{n0} + \\ \lambda \sum_{i=1}^{k} c_{ni} \end{pmatrix} \begin{pmatrix} b_{n0} + \\ (1-b_{n0})\lambda \end{pmatrix}\right] d(r_n,f) \le a_{n0}\, d(r_{n-1},f)$$

$$d(r_n,f) \le \frac{a_{n0}\, d(r_{n-1},f)}{1-[\lambda \sum_{i=1}^{k} a_{ni}\,(q_{n0}+(1-q_{n0})\lambda)\,(c_{n0}+\lambda \sum_{i=1}^{k} c_{ni})\,(b_{n0}+(1-b_{n0})\lambda)]}$$

Let $\frac{R_n}{S_n} = \frac{a_{n0}}{1-[\lambda \sum_{i=1}^{k} a_{ni}(q_{n0}+(1-q_{n0})\lambda)\,(c_{n0}+\lambda \sum_{i=1}^{k} c_{ni})(b_{n0}+(1-b_{n0})\lambda)]}$

$$1-\frac{R_n}{S_n} = 1-\frac{a_{n0}}{1-[\lambda \sum_{i=1}^{k} a_{ni}\,(q_{n0}+(1-q_{n0})\lambda)\,(c_{n0}+\lambda \sum_{i=1}^{k} c_{ni})\,(b_{n0}+(1-b_{n0})\lambda)]}$$

$\frac{R_n}{S_n} \le \lambda \sum_{i=1}^{k} a_{ni}\,(q_{n0}+(1-q_{n0})\lambda)\,(c_{n0}+\lambda \sum_{i=1}^{k} c_{ni})\,(b_{n0}+(1-b_{n0})\lambda) + a_{n0}$

$$d(r_n,f) \le \left[\lambda \sum_{i=1}^{k} a_{ni} \begin{pmatrix} q_{n0} + \\ (1-q_{n0})\lambda \end{pmatrix} \begin{pmatrix} c_{n0} + \\ \lambda \sum_{i=1}^{k} c_{ni} \end{pmatrix} \begin{pmatrix} b_{n0} + \\ (1-b_{n0})\lambda \end{pmatrix} + a_{n0}\right] d(r_{n-1},f)$$

$$d(r_n,f) \le \left[\lambda \begin{pmatrix} 1- \\ a_{n0} \end{pmatrix} \begin{pmatrix} q_{n0} + \\ (1-q_{n0})\lambda \end{pmatrix} \begin{pmatrix} c_{n0} + \\ \lambda(1-c_{n0}) \end{pmatrix} \begin{pmatrix} b_{n0} + \\ (1-b_{n0})\lambda \end{pmatrix} + a_{n0}\right] d(r_{n-1},f)$$

$$\le \left[1-(1-a_{n0})\left(1-\lambda \begin{pmatrix} 1- \\ a_{n0} \end{pmatrix} \begin{pmatrix} q_{n0} + \\ (1-q_{n0})\lambda \end{pmatrix} \begin{pmatrix} c_{n0} + \\ (1-c_{n0})\lambda \end{pmatrix} \begin{pmatrix} b_{n0} + \\ (1-b_{n0})\lambda \end{pmatrix}\right)\right] d(r_{n-1},f)$$

$$\vdots$$

$$\le \prod_{j=1}^{n} \left[1-(1-a_{j0})\left(1-\lambda \begin{pmatrix} 1- \\ a_{j0} \end{pmatrix} \begin{pmatrix} q_{j0} + \\ (1-q_{j0})\lambda \end{pmatrix} \begin{pmatrix} c_{j0} + \\ (1-c_{j0})\lambda \end{pmatrix} \begin{pmatrix} b_{j0} + \\ (1-b_{j0})\lambda \end{pmatrix}\right)\right] d(r_0,f)$$

Take limit as $n \to \infty$ for both side we have $d(r_n,f) = 0$.

■

By the same way, we can prove the four-step iterative converges to the f-point f of Ŧ:.
Now, we prove the MIFSI convergent is faster than FSI.

Theorem 2.2 Let $Ŧ^i$ be a finite generalized quasi like contractive mapping for all $i = 0,1,2,\dots,k$ defined by (2.1) and g be a nonexpansive mapping, If $f \in C\text{F}(Ŧ^i, p_{iU}, g)$. Then, for $r_0 \in$ U, the MIFSI $\{r_n\}_{n=0}^{\infty}$ defined by (2.2) convergence faster than FSI $\{r_n\}_{n=0}^{\infty}$ defined by (1.4):

Proof: Since $f \in C\text{F}(Ŧ^i, p_{iU}, g)$, then from $\{r_n\}_{n=0}^{\infty}$MIFS

$$d(r_n,f) = d(\mathcal{X}(p_{0U}r_{n-1}, Ŧ^i s_n, a_{ni}), f)$$
$$\le a_{n0} d(p_{0U}r_{n-1},f) + \sum_{i=1}^{k} a_{ni} d(Ŧ^i s_n, f)$$
$$\le a_{n0} d(r_{n-1},f) + \lambda_i \sum_{i=1}^{k} a_{ni} d(s_n,f) \tag{2.3}$$

$$d(s_n,f) = d(\mathcal{X}(g t_n, p_{0U}Ŧ^0 t_n, b_{n0}), f)$$
$$\le b_{n0} d(g t_n, f) + (1-b_{n0}) d(f_{0U}Ŧ^0 t_n, f)$$
$$\le (b_{n0} + (1-b_{n0})\lambda_0) d(t_n,f) \tag{2.4}$$

$$d(t_n,f) = d(\mathcal{X}(p_{0U}v_n, p_{iU}Ŧ^i v_n, c_{ni}), f)$$
$$\le c_{n0} d(f_{0U}v_n, f) + \sum_{i=1}^{k} c_{ni} d(f_{iU}Ŧ^i v_n, f)$$
$$\le d(v_n,f)(c_{n0} + \lambda_i \sum_{i=1}^{k} c_{ni}) \tag{2.5}$$

$$d(v_n,f) = d(\mathcal{X}(r_n, Ŧ^0 r_n, q_n), f)$$
$$\le q_{n0} d(r_n,f) + (1-q_{n0}) d(Ŧ^0 r_n, f)$$
$$\le (q_{n0} + (1-q_{n0})\lambda_0) d(r_n,f) \tag{2.6}$$

Take $\lambda = max\{\lambda_i\,, i = 0,1,2\dots k\}$.

From (2.3),(2.4),(2.5) ,and (2.6) we have :

$$d(r_n,f) \le \left[\lambda(1-a_{n0})\left(q_{n0}+(1-q_{n0})\lambda\right)\left(c_{n0}+\lambda(1-c_{n0})\right)\left(b_{n0}+(1-b_{n0})\lambda\right)+a_{n0}\right] d(r_{n-1},f)$$

$$\le \left[1-(1-a_{n0})\left(1-\lambda\left(q_{n0}+(1-q_{n0})\lambda\right)\left(c_{n0}+(1-c_{n0})\lambda\right)\left(b_{n0}+(1-b_{n0})\lambda\right)\right)\right] d(r_{n-1},f)$$

We can write

$$\left[1-(1-a_{n0})\left(1-\lambda\left(q_{n0}+(1-q_{n0})\lambda\right)\left(c_{n0}+(1-c_{n0})\lambda\right)\left(b_{n0}+(1-b_{n0})\lambda\right)\right)\right] \le 1-(1-a_{n0})(1-\lambda)$$

$$d(r_n,f) \le \left(1-(1-a_{n0})(1-\lambda)\right)d(r_{n-1},f)$$

$$\vdots$$

$$\prod_{j=1}^{n}\left(1-\left(1-a_{j0}\right)(1-\lambda)\right) d(r_0,f) \qquad (2.7)$$

Now, to get the $\{r_n\}_{n=0}^{\infty}$ for FSI:

$$d(r_n,f) = d\left(\mathcal{X}\left(r_{n-1},₮^0 s_n,a_{n0}\right),f\right)$$
$$\le (1-a_{n0})d\left(r_{n-1},f\right)+a_{n0}d(₮^0 s_n,f)$$
$$\le (1-a_{n0})d\left(r_{n-1},f\right)+\lambda a_{n0}d(s_n,f) \qquad (2.8)$$

$$d(s_n,f) = d\left(\mathcal{X}\left(r_{n-1},₮^0 t_n,b_{n0}\right),f\right)$$
$$\le (1-b_{n0})d\left(r_{n-1},f\right)+b_{n0}d(₮^0 t_n,f)$$
$$\le (1-b_{n0})d\left(r_{n-1},f\right)+\lambda b_{n0}d(t_n,f) \qquad (2.9)$$

$$d(t_n,f) = d\left(\mathcal{X}\left(r_{n-1},₮^0 v_n,c_{n0}\right),f\right)$$
$$\le (1-c_{n0})d\left(r_{n-1},f\right)+c_{n0}d(₮^0 v_n,f)$$
$$\le (1-c_{n0})d\left(r_{n-1},f\right)+\lambda c_{n0}d(v_n,f) \qquad (2.10)$$

$$d(v_n,f) = d\left(\mathcal{X}\left(r_{n-1},₮^0 r_{n-1},q_n\right),f\right)$$
$$\le (1-q_{n0})d\left(r_{n-1},f\right)+q_{n0}d(₮^0 r_{n-1},f)$$
$$\le (1-q_{n0})d\left(r_{n-1},f\right)+\lambda q_{n0}d(r_{n-1},f) \qquad (2.11)$$

From (2.8),(2.9),(2.10) ,and (2.11) we have:

$$d(r_n,f) \le ((1-a_{n0})+\lambda a_{n0}((1-b_n)+\lambda b_{n0}((1-c_{n0})+\lambda c_{n0}((1-q_{n0})+\lambda q_{n0}))))d(r_{n-1},f)$$
$$= (1-a_{n0}(1-\lambda(1-b_{n0}(1-\lambda(1-c_{n0})-\lambda^2 c_{n0}(1-\lambda)-\lambda^3 c_{n0}q_{n0}))))d(r_{n-1},f)$$
$$= (1-a_{n0}(1-\lambda(1-b_{n0}(1-\lambda(1-c_{n0}(1-\lambda(1-q_{n0}(1-\lambda))))))))d(r_{n-1},f)$$

We can write :

$$(1-a_{n0}(1-\lambda(1-b_{n0}(1-\lambda(1-c_{n0}(1-\lambda(1-q_{n0}(1-\lambda)))))))) \le \left(1-a_{n0}(1-\lambda)\right)$$

$$d(r_n,f) \le \left(1-a_{n0}(1-\lambda)\right)d(r_{n-1},f)$$

$$\vdots$$

$$\le \prod_{j=1}^{n}\left(1-a_{j0}(1-\lambda)\right) d(r_0,f) \qquad (2.12)$$

Take $\lim_{n\to\infty}\frac{\{r_n\}_{n=0}^{\infty}\ MIFSI}{\{r_n\}_{n=0}^{\infty}\ FSI} = \frac{\prod_{j=1}^{n}\left(1-\left(1-a_{j0}\right)(1-\lambda)\right)d(r_0,f)}{\prod_{j=1}^{n}\left(1-a_{j0}(1-\lambda)\right)d(r_0,f)} = 0.$

Then, by definition 1.6 the MIFSI converges faster than FSI to f-point when $1-a_{j0} > a_{j0}$.

■

Theorem 2.3 Let $Ŧ^i$ be a finite generalized quasi-like contractive mapping satisfying (2.1) with $CF(Ŧ^i, \mathfrak{p}_{iU}, g) \neq \emptyset$. Then, for $r_0 \in U$, the sequence $\{r_n\}_{n=0}^{\infty}$ defined by the MIFSI iterative (2.2) with the converging point at $f \in CF(Ŧ^i, \mathfrak{p}_{iU}, g)$,is $Ŧ^i$-stable.

Proof: Let $\{l_n\}_{n=0}^{\infty} \subset U$ be an arbitrary sequence such that $\epsilon_n = d(l_n, \mathcal{X}(\mathfrak{p}_{0U} l_{n-1}, Ŧ^i j_n, a_{ni}))$ where, $j_n = \mathcal{X}(g m_n, \mathfrak{p}_{iU} Ŧ^0 m_n, b_{n0})$, $m_n = \mathcal{X}(\mathfrak{p}_{0U} n_n, \mathfrak{p}_{iU} Ŧ^i n_n, c_{ni})$, $n_n = \mathcal{X}(l_n, Ŧ^0 l_n, q_{n0})$ and

$\lim_{n\to\infty} \epsilon_n = 0.$

$$d(l_n, f) \leq d(l_n, \mathcal{X}(\mathfrak{p}_{0U} l_{n-1}, Ŧ^i j_n, a_{ni})) + d(\mathcal{X}(\mathfrak{p}_{0U} l_{n-1}, Ŧ^i j_n, a_{ni}), f)$$
$$\leq \epsilon_n + a_{n0} d(\mathfrak{p}_{0U} l_{n-1}, f) + \sum_{i=1}^{k} a_{ni} d(Ŧ^i j_n, f)$$
$$\leq \epsilon_n + \leq a_{n0} d(l_{n-1}, f) + \sum_{i=1}^{k} a_{ni} \begin{bmatrix} ƛ_i d(j_n, f) + \xi_i \emptyset_i \left(d(Ŧ^i f, f)\right) \\ + \eta_i \min\{d(j_n, Ŧ^i j_n), d(Ŧ^i f, f)\} \end{bmatrix}$$
$$\leq \epsilon_n + \leq a_{n0} d(l_{n-1}, f) + \sum_{i=1}^{k} a_{ni} ƛ_i d(j_n, f) \tag{2.13}$$

$$d(j_n, f) \leq b_{n0} d(g m_n, f) + (1 - b_{n0}) d(\mathfrak{p}_{0U} Ŧ^0 m_n, f)$$
$$\leq b_{n0} d(m_n, f) + (1 - b_{n0}) d(Ŧ^0 m_n, f)$$
$$\leq b_{n0} d(m_n, f) + (1 - b_{n0}) \begin{bmatrix} ƛ_0 d(m_n, f) + \xi_0 \emptyset_0 (d(Ŧ^0 f, f)) \\ + \eta_0 \min\{d(m_n, Ŧ^0 m_n), d(Ŧ^0 f, f)\} \end{bmatrix}$$
$$\leq (b_{n0} + (1 - b_{n0}) ƛ_0) d(m_n, f) \tag{2.14}$$

$$d(m_n, f) \leq c_{n0} d(\mathfrak{p}_{0U} n_n, f) + \sum_{i=1}^{k} c_{ni} d(\mathfrak{p}_{iU} Ŧ^i n_n, f)$$
$$\leq c_{n0} d(n_n, f) + \sum_{i=1}^{k} c_{ni} d(Ŧ^i n_n, f)$$
$$\leq c_{n0} d(n_n, f) + \sum_{i=1}^{k} c_{ni} \begin{bmatrix} ƛ_i d(n_n, f) + \xi_i \emptyset_i \left(d(Ŧ^i f, f)\right) \\ + \eta_i \min\{d(n_n, Ŧ^i n_n), d(Ŧ^i f, f)\} \end{bmatrix}$$
$$\leq c_{n0} d(n_n, f) + \sum_{i=1}^{k} c_{ni}\, ƛ_i d(n_n, f)$$
$$\leq \left(c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ_i\right) d(n_n, f) \tag{2.15}$$

$$d(n_n, f) \leq q_{n0} d(l_n, f) + (1 - q_{n0}) d(Ŧ^0 l_n, f)$$
$$d(n_n, f) \leq q_{n0} d(l_n, f) + (1 - q_{n0}) \begin{bmatrix} ƛ_0 d(l_n, f) + \xi_0 \emptyset_0 (d(Ŧ^0 f, f)) \\ + \eta_0 \min\{d(l_n, Ŧ^0 L_n), d(Ŧ^0 f, f)\} \end{bmatrix}$$
$$d(n_n, f) \leq (q_{n0} + (1 - q_{n0}) ƛ_0) d(l_n, f) \tag{2.16}$$

Take $ƛ = max\{ ƛ_i , i = 1,2, \dots, k\}$.

From (2.13),(2.14),(2.15), and (2.16) we have .

$$d(l_n, f) \leq \epsilon_n + \leq a_{n0} d(l_{n-1}, f) + \sum_{i=1}^{k} a_{ni} ƛ (b_{n0} + (1 - b_{n0}) ƛ)\left(c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ\right)(q_{n0} + (1 - q_{n0}) ƛ) d(l_n, f)$$

$$\left(1 - \sum_{i=1}^{k} a_{ni}\, ƛ\, (b_{n0} + (1 - b_{n0}) ƛ)\left(\begin{matrix} c_{n0} + \\ \sum_{i=1}^{k} c_{ni}\, ƛ \end{matrix}\right)\left(\begin{matrix} q_{n0} + \\ (1 - q_{n0}) ƛ \end{matrix}\right)\right) d(l_n, f) \leq \epsilon_n + a_{n0} d\,(l_{n-1}, f)$$

$$d(l_n, f) \leq \frac{\epsilon_n}{1 - [\sum_{i=1}^{k} a_{ni} ƛ (b_{n0} + (1 - b_{n0}) ƛ)\left(c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ\right)(q_{n0} + (1 - q_{n0}) ƛ)]} + \frac{a_{n0} d\,(l_{n-1}, f)}{1 - [\sum_{i=1}^{k} a_{ni} ƛ (b_{n0} + (1 - b_{n0}) ƛ)\left(c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ\right)(q_{n0} + (1 - q_{n0}) ƛ)]}$$

$$d(l_n, f) \leq \frac{\epsilon_n a_{n0}}{[1 - \sum_{i=1}^{k} a_{ni} ƛ (b_{n0} + (1 - b_{n0}) ƛ)\left(c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ\right)(q_{n0} + (1 - q_{n0}) ƛ)] a_{n0}} + \frac{a_{n0} d\,(l_{n-1}, f)}{1 - [\sum_{i=1}^{k} a_{ni} ƛ (b_{n0} + (1 - b_{n0}) ƛ)\left(c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ\right)(q_{n0} + (1 - q_{n0}) ƛ)]}.$$

But

$$\frac{a_{n0}}{1-[\sum_{i=1}^{k} a_{ni}ƛ(b_{n0}+(1-b_{n0})ƛ)(c_{n0}+\sum_{i=1}^{k} c_{ni}ƛ)(q_{n0}+(1-q_{n0})ƛ)]} \le 1-(1-a_{n0})(1-ƛ)$$

$$d(l_n,f) \le [1-(1-a_{n0})(1-ƛ)]\frac{\epsilon_n}{a_{n0}} + [1-(1-a_{n0})(1-ƛ)]d\,(l_{n-1},f)$$

But, $1-(1-a_{n0})(1-ƛ) < 1$.

since $\lim_{n\to\infty} \epsilon_n = 0$, and by Lemma(1.1), we have $\lim_{n\to\infty} d(l_n,f) = 0$:

which implies that $\lim_{n\to\infty} l_n = f$.

Conversely, if $\lim_{n\to\infty} l_n = f$

$$\begin{aligned}
\epsilon_n &= d(l_n, \mathcal{X}(\mathfrak{p}_{oU} l_{n-1}, Ŧ^i j_n, a_{ni})\\
&\le d(l_n,f) + d(\mathcal{X}(\mathfrak{p}_{0U} l_{n-1}, Ŧ^i j_n, a_{ni}), f)\\
&\le d(l_n,f) + a_{n0} d(\mathfrak{p}_{0U} l_{n-1}, f) + \textstyle\sum_{i=1}^{k} a_{ni} d(Ŧ^i j_n, f)\\
&\le d(l_n,f) + a_{n0} d(l_{n-1},f) + \textstyle\sum_{i=1}^{k} a_{ni}\begin{bmatrix} ƛ_i d(j_n,f) + \xi_i \emptyset_i\left(d(Ŧ^i f,f)\right)\\ +\eta_i \min\{d(j_n, Ŧ^i j_n), d(Ŧ^i f,f)\}\end{bmatrix}\\
&\le d(l_n,f) + a_{n0} d(l_{n-1},f) + \textstyle\sum_{i=1}^{k} a_{ni} ƛ_i d(j_n,f) \qquad (2.17)
\end{aligned}$$

$$\begin{aligned}
d(j_n,f) &\le b_{n0} d(g m_n, f) + (1-b_{n0}) d(Ŧ^0 m_n, f)\\
&\le b_{n0} d(m_n,f) + (1-b_{n0})\begin{bmatrix} ƛ_0 d(m_n,f) + \xi_0 \emptyset_0\left(d(Ŧ^0 f,f)\right)\\ +\eta_0 \min\{d(m_n, Ŧ^0 m_n), d(Ŧ^0 f,f)\}\end{bmatrix}\\
&\le (b_{n0} + (1-b_n)ƛ_0) d(m_{n0}, f) \qquad (2.18)
\end{aligned}$$

$$\begin{aligned}
d(m_n,f) &\le c_{n0} d(\mathfrak{p}_{0U} n_n, f) + \textstyle\sum_{i=1}^{k} c_{ni} d(\mathfrak{p}_{iU} Ŧ^i n_n, f)\\
&\le c_{n0} d(n_n,f) + \textstyle\sum_{i=1}^{k} c_{ni} d(Ŧ^i n_n, f)\\
&\le c_{n0} d(n_n,f) + \textstyle\sum_{i=1}^{k} c_{ni}\begin{bmatrix} ƛ_i d(n_n,f) + \xi_i \emptyset_i\left(d(Ŧ^i f,f)\right)\\ +\eta_i \min\{d(n_n, Ŧ^i n_n), d(Ŧ^i f,f)\}\end{bmatrix}\\
&\le c_{n0} d(n_n,f) + \textstyle\sum_{i=1}^{k} c_{ni} ƛ_i d(n_n,f)\\
&\le \left(c_{n0} + \textstyle\sum_{i=1}^{k} c_{ni} ƛ_i\right) d(n_n,f) \qquad (2.19)
\end{aligned}$$

$$\begin{aligned}
d(n_n,f) &\le q_{n0} d(L_n,f) + (1-q_{n0}) d(Ŧ^0 L_n, f)\\
&\le q_{n0} d(L_n,f) + (1-q_{n0})\begin{bmatrix} ƛ_0 d(L_n,f) + \xi_0 \emptyset_0\left(d(Ŧ^0 f,f)\right)\\ +\eta_0 \min\{d(L_n, Ŧ^0 L_n), d(Ŧ^0 f,f)\}\end{bmatrix}\\
&\le (q_{n0} + (1-q_{n0})ƛ_0) d(L_n,f) \qquad (2.20)
\end{aligned}$$

Take $ƛ = max\{ƛ_i, i = 1,2,\ldots,k\}$.

From (2.17),(2.18),(2.19) ,and (2.20) we have:

$$\epsilon_n \le d(l_n,f) + a_{n0} d(l_{n-1},f) + \left(\sum_{i=1}^{k} a_{ni}\, ƛ \begin{matrix}(b_{n0} + (1-b_{n0})ƛ)\\ (c_{n0} + \sum_{i=1}^{k} c_{ni}\, ƛ)(q_{n0} + (1-q_{n0})ƛ)\end{matrix}\right) d(l_n,f)$$

$$\epsilon_n \le (1 + \textstyle\sum_{i=1}^{k} a_{ni}\, ƛ\, (b_{n0} + (1-b_{n0})ƛ)\, \left(\begin{matrix} c_{n0} + \\ \sum_{i=1}^{k} c_{ni}\, ƛ\end{matrix}\right) \left(\begin{matrix} q_{n0} + \\ (1-q_{n0})ƛ\end{matrix}\right)) d(l_n,f) + a_{n0} d(l_{n-1},f).$$

Take limit for two sides with $\lim_{n\to\infty} d(l_n,f) = 0$, we have $\lim_{n\to\infty} \epsilon_n = 0$.

■

3. Conclusion

The results of this paper as follows:

1. MIFSI is convergent to the f-point of generalized quasi-like contractive.
2. MIFSI has a rate of convergence faster than FSI.

3. We proved the stability of MIFSI with generalized quasi-contractive.

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