IHJPAS. 36 (4) 2023

Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq

Quasi-Semiprime Modules

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Article history: Received 9 January 2023, Accepted 20 February 2023, Published in October 2023

doi.org/10.30526/36.4.3181

Abstract

Suppose that A is an abelain ring with identity and B is a unitary (left) A-module. In this paper, we introduce a type of module, namely quasi-semiprime. A-module, whenever $\sqrt{[N:B]}$ is a prime ideal for proper submodule N of B, then B is called quasi -semiprime module, which is a generalization of quasi-Prime A-module, whenever ann_AN is a prime ideal for proper submodule N of B, then B is quasi-prime module. A comprehensive study of these modules is given, and we study the relationship between quasi-semiprime modules and quasi-prime. We put the condition coprime over cosemiprime ring for the two cocept quasi-prime modules and quasi-semiprime modules, which are equivalent. The concepts of prime modules and quasi-semiprime modules are equivalent. The condition of anti-hopfain makes quasi-prime is quasi-semiprime A-module. Whenever B is cyclic, coprime C-module, where C is the ring, each ideal is semiprime, which implies quasi-prime, quasi-simepime, and annCB are prime ideals. If F is an epimorphism from $B_1 \rightarrow B_2$, whenever B1 is a quasi-prime module, it implies B2 is a quasi-prime A-module, and the inverse image of quasi-semiprime is a quasi-prime A-module.

Keywords: Prime module, Quasi-prime R-modules, Quasi-semiprime R-modules, Coprime R-modules, Antihopfian R-modules.

1. Introduction

Suppose that W is a left A-module, where A is a ring with unity. An A-module B is said to be prime whenever $ann_AB=ann_AN$ for each non-zero submodule Nof B, where $ann_AB=\{a \in A; bx=0 \text{ for each } b \in B\}$[1,2]. Hasan in [3] introduced the concept of qasi-prime A-modules, which is a generalization of prime A-modules, where an A-module W is called quasi-prime modulles if and only if for each non-zero submodule N of W, $ann_A N$ is a prime ideal. Annin [9] calls an A-module W a coprime (dual notion of prime modules) if $ann_AW=ann_AW/A$ for every proper submodule A of W. In this paper, we study a generalization of the quasi-prime module which we called the Quasi-semiprime A-module if $\sqrt{annB/N} = \sqrt{[N:B]}$ is a prime ideal for each submodule N of B.

This paper consists of two sections. In section one; we study the basic properties of a quasi-semiprime A-module. In section two, we study the relation between quasi-semiprime A-modules and prime A-modules.

2. Materials and Methods

Definition (2.1)

B is said to be a quasi-semiprime A-module if $\sqrt{[N:W]}$ is a prime ideal for the proper submodule N of B.

Examples and Remarks (2.2)

1- It is clear that Zn is a quasi-semiprime A-module if and only if n is a prime number.
2- If n can be written as a product of two prime numbers, then Zn is a quasi-semiprime A-module.
Proof:

Let $n=p_1p_2$; p_1, p_2 be two prime numbers, so $N_1=(p_1)$,$N_2=(p_2)$, then$\sqrt{[(P_1:Z_n]}=\sqrt{(p_1)}=(p_1)$, $N_2=(p_2)$, then $\sqrt{[(p_{2):}Z_n]}=\sqrt{(p_2)}=(p_2)$ is a prime ideal, henc Z_n is a quasi-semiprime A-module.

3- $Z_{p\infty}$ is not a quasi-semiprime module, since we know that every submodule of $Z_{p\infty}$ is of the form ($1/p^n$ +Z), where n is a non-negative integer, so $\sqrt{[\frac{1}{p^n}+Z:Zp^{\infty}]}=\sqrt{[\frac{1}{pn}+Z]}=$ (p^n Z) is not prime ideal.
4- Suppose B is a simple A-module, then B is a Quasi-Semiprime A-Module.
Proof: it is clear.

Proposition (2.3)

Every proper submodule N of quasi-semiprime module is a quasi-semiprime module.

Proof:

Suppose N is a proper submodule of quasi-semiprime A-module W. Let K be a proper submodule of N to show that $\sqrt{[K:N]}$ is a prime ideal if ab$\in\sqrt{[K:N]}$, so $a^nb^n\in$ [K:N], so that $a^nb^nN\subseteq$ K $\subseteq$ W that is $a^nb^n\in$ [N:W], but W is a quasi-semiprime A-modul implies either $a^n\in$ [N:W] or $b^n\in$ [N:W], thus either $a^n\in$ [K:N] or $b^n\in$ [K:N] which means either a$\in\sqrt{[K:N]}$ or b$\in\sqrt{[K:N]}$, so $\sqrt{[K:N]}$ is a prime ideal.

Recall that whenever B ≅ B/N for all proper submodule Nof modules B, then we said that anon-simple A-module B anti-hopfian module [4, 5].

Proposition (2.4)

Suppose that B is an anti-hopfian quasi-prime A-module, then B is quasi-semiprime.

Proof:

Since B is an anti-hopfian module, then B ≅ B/N for N be a proper submodule of B, so there exists an isomorphism function f: B→ B/N; f(b)=b+N for each b ∈ B, so it is easy to check that

$ann_A B = ann_A B/N$, then by [6], every anti-hopfian A-module is a coprime E-module, where E=End(W) and by [5] every f ∈ E, either f=0 or f is subjective, thus f(b)=0 or f(b)=B for every w ∈ W If f(w)=0 implies W=N which is a contradiction, so f(W)=W, which means $ann_A B=[N:B]$, implies $\sqrt{ann_A}B = \sqrt{ann_N^B}$ if $a^n b^n \in [N:B]$, then $a^n b^n \in ann_A B$ but B is a quasi-prime A-module, so by [3] implies $ann_A W$ is a prime ideal, so either $a^n \in ann_A W$ or $b^n \in ann_A B$. Thus, either $a \in \sqrt{[N:B]}$ or $b \in \sqrt{[N:B]}$, which means B is a quasi-semiprime A-module.

The condition anti-hopfian we cannot drop for example: Z_6 is quasi-semiprime A-module by (2.2), while it is not quasi-prime by [3], and Z_6 is not anti-hopfain by [6].

Prorosition (2.5)

Suppose B is a coprime A-module of quasi-prime, then B is quasi-semiprime A-module.

Proof

Let B be a quasi-prime A-module, then by [3], $ann_A B$ is a prime ideal, but W is a coprime A-module, so $ann_A B/N$ is a prime ideal for each non-zero submodule N of B, which means $\sqrt{[N:B]}$ is a prime ideal. Thus, W is a quasi-semiprime A-module.

Recall that an ideal K of the ring A is called nil radical and denoted by $\sqrt{K}$, and is defined by: $\sqrt{K}$ ={a∈ A; $a^n \in$ K , for some n Z+}[8].

Not (2.6)

Suppose C is a ring where every ideal is nil radical, which we call cosemiprime ring.

Theorem M (2.7)

Suppose that B is a coprime C-module. The following statements are equivalent:

1) B is a Quasi-Prime Module.

2) B is a Quasi-Semiprime Module.

Proof:

1) → (2) (by Theorem (2-5))

(2) → (1) for each a,b ∈ C if ab ann_C N,then abN=0 implies ab ∈ [(0):N],which means ab∈ $\sqrt{[(0):N]}$, but W is quasi-semiprime module so either a∈ $\sqrt{[(0):N]}$ or b∈ $\sqrt{[(0):N]}$ implies either a∈ $ann_C N$ or b∈ $ann_C N$. Thus, B is a quasi-prime C-module.

Theorem (2.8)

Let B be a cyclic coprime C-module, then the following statements are equivalent:

1- B is a Quasi-Prime C-Module.
2- B is a quasi-semiprime C-module.
3- $ann_C B$ is a prime ideal.

Proof:

1→ 2 (by Theorem (2.5))

2 → 3 if ab ∈ ann_CB, then ab∈ $\sqrt{annB}$. Thus, ab∈ $\sqrt{[N:B]}$ for every submodule N of B which means a^nb^n ∈ [N:B], but B is a quasi-semiprime C-module, so either a^n∈ [N:B] or b^n ∈ [N:B], but B is coprime by [9] implies either a^nB=0 or b^nB=0, which means either a∈ $\sqrt{annB}$ or b∈ $\sqrt{annB}$.Thus, either a∈ ann_CW or b∈ ann_cB.

3 → 1 by [3] implies the result.

Proposition (2.9)

Suppose that B is an A-Module and J is an Ideal Of A which that is contained in ann_AB/N where N is a submodule of B. Then, B is a quasi-semiprime A-module ⟷ B is a quasi-semiprime A/J-Module.

Proof

To show B is quasi-semiprime A\J-module if $(a_1+J)(a_2+J)$∈ $\sqrt{[N:_{A/J}B]}$, where a_1+J,a_2+J∈ B/J, then $(a_1a_2+J)^n$∈ $[N:_{A/I}B]$. Thus, $(a_1^na_2^n+J)x=0$ for all x∈ $ann_{A/J}$B/N. Hence, $a_1^na_2^nx=0$ for all x∈ ann_AB/N which means $a_1^na_2^n$∈ [N:B], so a_1a_2 ∈ $\sqrt{[N:_A B]}$, but B is quasi-semiprime A-module, which implies either a_1 ∈ $\sqrt{[N:_A B]}$ or a_2∈ $\sqrt{[N:_A B]}$. Thus, either a_1^n ∈ $[N:_A B]$ or a_2^n ∈$[N:_A B]$. However, a_1^n+I ∈ ann_AB/N or $a_2{}^n$+J ∈ ann B/N. Thus, either (a_1+I)∈ $\sqrt{[N:_{A/J}B]}$ or (a_2+I)∈ $\sqrt{[N:_{A/J}B]}$, which means B is a quasi-semiprime B/I module.

Conversely ,if B is quasi-semiprime A/J-module ,let N be a nonzero A-submodule of B,let a_1,a_2∈ $\sqrt{[N:_A B]}$,then $a_1^n\,a_2^nx=0$ for all x ∈ann_AB/N.Hence $(a_1^n+J)(a_2^n+J)x=0$ for all x∈ $ann_{A/J}$W/N ,so$(a_1+J)(a_2+J)$∈ $\sqrt{[N:_{A/J}B]}$,whille is a prime ideal,so either (a_1+I)∈ $\sqrt{[N:_{A/J}B]}$ or (a_2+I)∈ $\sqrt{[N:_{A/J}B]}$ .Then we get either a_1^n x=0 or a_2^nx=0 f0r each x ∈ ann_AB/N,so either a_1∈ $\sqrt{[N:_A B]}$ or a_2 ∈ $\sqrt{[N:_A B]}$.

Theorem (2.10)

Suppose B_1, and B_2 are two A-modules, if f: B_1 → B_2, is an epimorphism function, then if B_1 is a quasi-semiprime module, then B_2 is a quasi-semiprime A-module.

Proof:

Since B_1 is a quasi-semiprime A-module, so if $a^nb^nB_1$⊆ N_1 for each a,b∈ A, then either a^nB_1⊆ N or b^nB_1⊆ N. Thus, $f(a^nb^nB_1)$⊆ $f(N_1)$ since f is a homomorphism implies $f(a^n).f(b^n)$∈ $[f(N):f(W_1)]$. Suppose f(a)=x,f(b)=y. Thus, either $f(a^nB_1)$ ⊆$f(N_1)$ or $f(b^nB_1)$⊆ f(N), so either $f(a^n)f(W_1)$ ⊆f(N) or $f(b^n)f(B_1)$⊆ f(N) implies either $x^nf(B_1)$⊆ f(N) or $y^nf(B_1)$⊆ $f(N_1)$, but f is onto, so $f(B_1)=B_2$,$f(N)=N_2$, which means either x∈ $\sqrt{[N_2{:}B_2}$ or y∈ $\sqrt{N_2: B_2}$, whenever xy∈ $\sqrt{N_2: B_2}$. Thus, $\sqrt{N_2: B_2}$ is a prime ideal, which means B_2 is a quasi-semiprime module.

Corollary (2.11)

The inverse image of the quasi-semiprime module is a quasi –semiprime module.

Theorem (2.12)

Let B_1 and B_2 be two quasi-semiprime A-modules such that for each proper submodule K,T of B_1,B_2, respectively, if $[K\oplus T:W]=[K:W]\cap[T:W]$, then $B=B_1\oplus B_2$ is a quasi-semiprime A-module, where $\sqrt{[K:B]}\subseteq\sqrt{[T:B]}$ or $\sqrt{[T:B]}\subseteq\sqrt{[K:B]}$.

Proof

We must prove $\sqrt{[K\oplus T:B]}$ is a prime ideal for the proper submodules K, T of B_1 and B_2 in the order. Since $\sqrt{[K\oplus T:B]}=\sqrt{[K:B]}\cap\sqrt{[T:B]}$ where either $\sqrt{[K:B]}\subseteq\sqrt{[T:B]}$.

Or $\sqrt{[T:B]}\subseteq\sqrt{[K:B]}$. Thus, either $\sqrt{[K\oplus T:B]}=\sqrt{[K:B]}$ or $\sqrt{[K\oplus T:B]}=\sqrt{[T:B]}$, but W_1, and W_2 are quasi-semi-prime modules. Therefore, $\sqrt{[K:W]}$, and $\sqrt{[T:W]}$ are prime ideals in A. Implies $\sqrt{[K\oplus T:B]}$ is a prime ideal in A. Thus, $B_1\oplus B_2$ is quasi-semiprime A-modules.

The condition $\sqrt{[K:B]}\subseteq\sqrt{T:B]}$ or $\sqrt{[T:B]}\subseteq\sqrt{[K:B]}$ we cannot be dropped, for example, let $B_1=Z_6$, and $B_2=Z_3$ are two quasi-semiprime A-modules by (Examples and Remark (2.2), $\sqrt{[(2):Z_{18}]}\nsubseteq\sqrt{[(3):Z_{18}}$ and $\sqrt{[(3):Z_{18}}\nsubseteq\sqrt{[(2):Z_{18}}$

Since $\sqrt{(3)}\nsubseteq\sqrt{(2)}$ *and* $\sqrt{(2)}\nsubseteq\sqrt{(3)}$ *so* $\sqrt{[Z_2\oplus Z_3:Z_{18}}\neq\sqrt{Z_2:Z_{18}}\cap\sqrt{Z_3:Z_{18}}$

$=\sqrt{9Z}\cap\sqrt{6Z}=(3)\cap(6)=(6)$ is not a prime ideal, implying $W=W_1\oplus W_2$ is not a quasi-semiprime module.

3. Quasi-Semi-Prime A-Module and Prime Module

Now, we turn our attention to the relationship between quasi-semiprime modules and prime modules.

Proposition (3.1)

Suppose B is a coprime A-module, then every prime A-module is a quasi-semiprime A-module.

Proof

It follows directly by from [3] and Propositions (2.5).

The next example shows that the converse of Proposition (3.1) is not valid in general.

Let Z_6 as a Z –module is quasi-semiprime module by Examples and Remarks (2.2), while it is not a prime module [1].

Theorem (3.2)

Suppose B is a coprime C-module, then the following statements are equivalent:

1-B is a prime C-module.

2-B is a quasi-semiprime C-module.

Proof:

1→ 2 by (Proposition (3-1)), 2 → 1 because B is a quasi-semiprime C-module, so $\sqrt{[N:B]}$ is a prime ideal for each N submodule of M, so [N:B] is a prime ideal, but B is a coprime C-module, so [6] implies $ann_C B$ is a prime ideal, which means if rb=0 for b∈ B and c∈ C. suppose that b≠ 0 and Cb≠ o, so cB=N≠ 0, thus there exists that b∈ B and n∈ N such that cb=n, this means N=0, which is a contradiction. So B is a Prime.

Proposition (3.3)

Let B be a coprime C-module, then the following statements are equivalent:

1- Bis a quasi-prime module.
2- B is a quasi-semiprime modul.
3- B is a prime module.

Proof

1 → 2 by Theorem(2.7).

2 → 3 by Theorem (3.2).

3→ 1 by [3].

Corollary (3.4)

If B is a coprime C-module, then B is a quasi-semiprime C-Module ⟷ (0) is a prime C-submodule.

Proof

It is clear.

Conclusion

From this research, we introduced a new definition of quasi-semiprime modules and studied the relationship between quasi-semiprime modules and other modules, such as quasi-prime modules and prime modules. If we put the condition coprime, the cocept quasi-prime module, quasi-semiprime module, and prime module are equivalent.

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