On S*g-α-Open Sets In Topological Spaces

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Abstract

In this paper, we introduce a new class of sets, namely , s*g-α-open sets and we show that the family of all s*g-α-open subsets of a topological space (X,τ) from a topology on X which is finer than τ . Also , we study the characterizations and basic properties of s*g-α-open sets and s*g-α-closed sets . Moreover, we use these sets to define and study a new class of functions, namely , s*g-α -continuous functions and s*g-α -irresolute functions in topological spaces . Some properties of these functions have been studied .

Keywords: s*g-α-open sets , s*g-α-closed sets , s*g-α-clopen sets , s*g-α-continuous functions , s*g-α-irresolute functions .

Introduction

Levine, N. [1,2] introduced and studied semi-open sets and generalized open sets respectively . Njastad, O. [3] , Mashhour, A.S. and et.al. [4] , Andrijevic, D. [5] and Abd El-Monsef,M.E. and et.al [6] introduced α-open sets, pre-open sets ,b-open sets and β -open sets respectively . Also, Arya, S.P. and Nour, T.M . [7] , Maki, H. and et.al [8,9], Khan, M. and et.al [10] introduced and investigated generalized semi open sets, generalized α-open sets , α-generalized open sets and s*g-open sets respectively . In this paper, we introduce a new class of sets, namely , s*g-α -open sets and we show that the family of all s*g-α -open subsets of a topological space (X,τ) from a topology on X which is finer than τ . This class of open sets is placed properly between the class of open sets and each of semi-open sets, α-open sets, pre-open sets, b-open sets, β -open sets, generalized semi open sets, generalized α-open sets and α-generalized open sets respectively . Also , we study the characterizations and basic properties of s*g-α -open sets and s*g-α -closed sets . Moreover, we use these sets to define and study a new class of functions, namely , s*g-α -continuous functions and s*g-α -irresolute functions in topological spaces . Some properties of these functions have been studied . Throughout this paper (X,τ) , (Y,σ) and (Z,η) (or simply X , Y and Z) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned .

1.Preliminaries

First we recall the following definitions and Theorems .

Definition(1.1): A subset A of a topological space (X,τ) is said to be :

i) An semi-open (briefly s-open) set [1] if $A \subseteq cl(int(A))$.

ii) An α-open set [3] if $A \subseteq int(cl(int(A)))$.

iii) An pre-open set [4] if $A \subseteq int(cl(A))$.

iv) An b-open set [5] if $A \subseteq int(cl(A)) \cup cl(int(A))$.

v) An β -open set [6] if $A \subseteq cl(int(cl(A)))$.

The semi-closure (resp. α-closure) of a subset A of a topological space (X,τ) is the intersection of all semi-closed (resp. α-closed) sets which contains A and is denoted by $cl_s(A)$ (resp . $cl_\alpha(A)$) . Clearly $cl_s(A) \subseteq cl_\alpha(A) \subseteq cl(A)$.

Definition(1.2): A subset A of a topological space (X,τ) is said to be :

i) A generalized closed (briefly g-closed) set [2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

ii) A generalized semi-closed (briefly gs-closed) set [7] if $cl_s(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

iii) A generalized α-closed (briefly gα-closed) set [8] if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X .

iv) An α-generalized closed (briefly αg-closed) set [9] if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

v) An s*g-closed set [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X .

The complement of a g-closed (resp. gs-closed , gα-closed ,αg-closed , s*g-closed) set is called a g-open (resp. gs-open ,gα-open, αg-open , s*g-open) set .

Definition(1.3): A function $f:(X,\tau)\to(Y,\sigma)$ is called :

i) semi-continuous (briefly s-continuous)[1] if $f^{-1}(V)$ is s-open set in X for every open set V in Y

ii) α-continuous [11] if $f^{-1}(V)$ is α-open set in X for every open set V in Y.

iii) pre-continuous [4] if $f^{-1}(V)$ is pre-open set in X for every open set V in Y.

iv) b-continuous [12] if $f^{-1}(V)$ is b-open set in X for every open set V in Y.

v) β-continuous [6] if $f^{-1}(V)$ is β-open set in X for every open set V in Y.

vi) generalized continuous (briefly g-continuous) [13] if $f^{-1}(V)$ is g-open set in X for every open set V in Y.

vii) generalized semi continuous (briefly gs-continuous)[14] if $f^{-1}(V)$ is gs-open set in X for every open set V in Y.

viii) generalized α-continuous (briefly gα-continuous) [8] if $f^{-1}(V)$ is gα-open set in X for every open set V in Y.

ix) α-generalized continuous (briefly αg-continuous) [15] if $f^{-1}(V)$ is αg-open set in X for every open set V in Y.

x) s*g-continuous [16] if $f^{-1}(V)$ is s*g-open set in X for every open set V in Y.

Definition(1.4)[10],[17]: Let (X,τ) be a topological space and $A\subseteq X$. Then:-

i) The s*g-closure of A , denoted by $cl_{s^*g}(A)$ is the intersection of all s*g-closed subsets of X which contains A .

ii) The s*g-interior of A , denoted by $int_{s^*g}(A)$ is the union of all s*g-open subsets of X which are contained in A .

Theorem(1.5)[17]: Let (X,τ) be a topological space and $A,B\subseteq X$. Then:-

i) $A\subseteq cl_{s^*g}(A)\subseteq cl(A)$.

ii) $int(A)\subseteq int_{s^*g}(A)\subseteq A$.

iii) If $A\subseteq B$, then $cl_{s^*g}(A)\subseteq cl_{s^*g}(B)$.

iv) A is s*g-closed iff $cl_{s^*g}(A)=A$.

v) $cl_{s^*g}(cl_{s^*g}(A))=cl_{s^*g}(A)$.

vi) $X-int_{s^*g}(A)=cl_{s^*g}(X-A)$.

vii) $x\in cl_{s^*g}(A)$ iff for every s*g-open set U containing x , $U\cap A\neq\phi$.

viii) $\bigcup_{\alpha\in\wedge} cl_{s^*g}(U_\alpha)\subseteq cl_{s^*g}(\bigcup_{\alpha\in\wedge} U_\alpha)$.

Theorem(1.6)[18]: Let $X\times Y$ be the product space of topological spaces X and Y . If $A\subseteq X$ and $B\subseteq Y$. Then $cl_{s^*g}(A)\times cl_{s^*g}(B)=cl_{s^*g}(A\times B)$.

2. Basic Properties Of s*g-α-open Sets

In this section we introduce a new class of sets, namely , s*g-α-open sets and we show that the family of all s*g-α-open subsets of a topological space (X,τ) from a topology on X which is finer than τ .

Definition(2.1): A subset A of a topological space (X,τ) is called an s*g-α-open set if $A \subseteq int(cl_{s*g}(int(A)))$. The complement of an s*g-α-open set is defined to be s*g-α-closed . The family of all s*g-α-open subsets of X is denoted by $\tau^{s*g-\alpha}$.

Clearly, every open set is an s*g-α-open, but the converse is not true . Consider the following example .

Example(2.2): Let $X=\{a,b,c\}$ and $\tau=\{X,\phi,\{a\}\}$ be a topology on X . Then $\{a,b\}$ is an s*g-α-open set in X, since $\{a,b\} \subseteq int(cl_{s*g}(int(\{a,b\}))) = int(cl_{s*g}(\{a\}) = int(X) = X$. But $\{a,b\}$ is not open in X .

Remark(2.3): s*g-open sets and s*g-α-open sets are in general independent . Consider the following examples:-

Example(2.4): Let $X=\{a,b,c\}$ and $\tau=\{X,\phi\}$ be a topology on X . Then $\{b\}$ is an s*g-open set in X , but is not s*g-α-open set , since $\{b\} \not\subset int(cl_{s*g}(int(\{b\}))) = int(cl_{s*g}(\phi)) = \phi$. Also, in example (2.2) $\{a,b\}$ is an s*g-α-open set in X , but is not s*g-open , since $\{a,b\}^c = \{c\}$ is not s*g-closed set in X , since $\{a,c\}$ is an semi-open set in X and $\{c\} \subseteq \{a,c\}$, but $cl(\{c\}) = \{b,c\} \not\subset \{a,c\}$.

Theorem(2.5): Every s*g-α-open set is α-open (resp. αg-open, gα-open , pre-open, b-open , β-open) set .

Proof: Let A be any s*g-α-open set in X , then $A \subseteq int(cl_{s*g}(int(A)))$. Since $int(cl_{s*g}(int(A))) \subseteq int(cl(int(A)))$, thus $A \subseteq int(cl(int(A)))$. Therefore A is an α-open set in X . Since every α-open set is αg-open (resp. gα-open , pre-open , b-open , β-open) set .Thus every s*g-α-open set is α-open (resp. αg-open, gα-open , pre-open, b-open , β-open) set .

Remark(2.6): The converse of Theorem (2.5) may not be true in general as shown in the following example .

Example(2.7): Let $X=\{a,b,c\}$ & $\tau=\{X,\phi\}$ be a topology on X . Then the set $\{b,c\}$ is pre-open (resp. αg-open , gα-open , b-open , β-open) in X , but is not s*g-α-open set in X , since $\{b,c\} \not\subset int(cl_{s*g}(int(\{b,c\}))) = int(cl_{s*g}(\phi)) = \phi$.

Theorem(2.8): Every s*g-α-open set is semi-open and gs-open set .

Proof: Let A be any s*g-α-open set in X , then $A \subseteq int(cl_{s*g}(int(A)))$. Since $int(cl_{s*g}(int(A))) \subseteq cl_{s*g}(int(A)) \subseteq cl(int(A))$, thus $A \subseteq cl(int(A))$. Therefore A is a semi-open set in X . Since every semi-open set is gs-open set . Thus every s*g-α-open set is semi-open and gs-open set .

Remark(2.9): The converse of Theorem (2.8) may not be true in general as shown in the following example .

Example(2.10): Let $X=\{a,b,c\}$ & $\tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$ be a topology on X . Then the set $\{a,c\}$ is semi-open and gs-open set in X , but is not an s*g-α-open set in X , since $\{a,c\} \not\subset int(cl_{s*g}(int(\{a,c\}))) = int(cl_{s*g}(\{a\}))) = int(\{a,c\}) = \{a\}$.

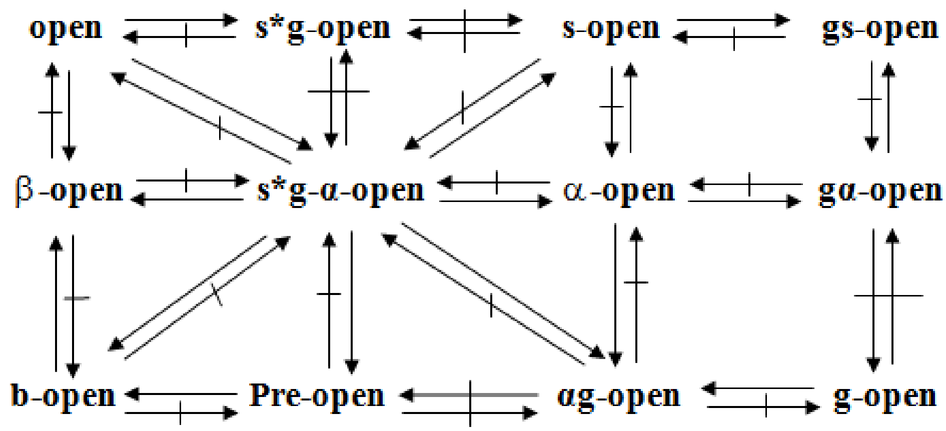
Remark(2.11): pre-open sets and αg-open sets are in general independent . Consider the following examples:-

Example(2.12): Let (R,μ) be the usual topological space . Then the set of all rational numbers Q is a pre-open set , but is not an αg-open set . Also , in Example (2.2) $\{b\}$ is an αg-open set , since , $\{b\}^c = \{a,c\}$ is an αg-closed set , but is not a pre-open set , since $\{b\} \not\subset int(cl(\{b\})) = int(\{c,b\}) = \phi$.

Remark(2.13): g-open sets and gα-open sets are in general independent . Consider the following examples:-

Example(2.14): Let $X=\{a,b,c\}$ & $\tau=\{X,\phi,\{a\},\{a,c\}\}$ be a topology on X . Then the set $\{a,b\}$ is a gα-open set in X , since $\{a,b\}^c=\{c\}$ is gα-closed , but is not a g-open set in X , since $\{a,b\}^c=\{c\}$ is not g-closed . Also, in Example (2.2) $\{c\}$ is a g-open set in X , since , $\{c\}^c=\{a,b\}$ is g-closed , but is not a gα-open set in X , since $\{c\}^c=\{a,b\}$ is not gα-closed .

The following diagram shows the relationships between s*g-α-open sets and some other open sets:



Proposition(2.15): A subset A of a topological space (X,τ) is s*g-α-open if and only if there exists an open subset U of X such that $U\subseteq A\subseteq int(cl_{s*g}(U))$.

Proof: $\Rightarrow$ Suppose that A is a s*g-α-open set in X , then $A\subseteq int(cl_{s*g}(int(A)))$. Since $int(A)\subseteq A$, thus $int(A)\subseteq A\subseteq int(cl_{s*g}(int(A)))$. Put $U=int(A)$, hence there exists an open subset U of X such that $U\subseteq A\subseteq int(cl_{s*g}(U))$.

Conversely, suppose that there exists an open subset U of X such that $U\subseteq A\subseteq int(cl_{s*g}(U))$. Since $U\subseteq A\Rightarrow U\subseteq int(A)\Rightarrow cl_{s*g}(U)\subseteq cl_{s*g}(int(A))\Rightarrow int(cl_{s*g}(U))\subseteq int(cl_{s*g}(int(A)))$. Since $A\subseteq int(cl_{s*g}(U))$, then $A\subseteq int(cl_{s*g}(int(A)))$. Thus A is an s*g-α-open set in X .

Lemma(2.16): Let (X,τ) be a topological space . If U is an open set in X , then $U\cap cl_{s*g}(A)\subseteq cl_{s*g}(U\cap A)$ for any subset A of X .

Proof: Let $x\in U\cap cl_{s*g}(A)$ and V be any s*g-open set in X s.t $x\in V$. Since $x\in cl_{s*g}(A)$, then by Theorem ((1.5),vii) , $V\cap A\neq\phi$. Since $U\cap V$ is an s*g-open set in X and $x\in V\cap U$, then $(V\cap U)\cap A=V\cap(U\cap A)\neq\phi$. Therefore $x\in cl_{s*g}(U\cap A)$. Thus $U\cap cl_{s*g}(A)\subseteq cl_{s*g}(U\cap A)$ for any subset A of X .

Theorem(2.17): Let (X,τ) be a topological space. Then the family of all s*g-α-open subsets of X from a topology on X .

Proof:(i). Since $\phi\subseteq int(cl_{s*g}(int(\phi)))$ and $X\subseteq int(cl_{s*g}(int(X)))$, then $\phi,X\in\tau^{s*g-\alpha}$.

(ii). Let $A,B\in\tau^{s*g-\alpha}$. To prove that $A\cap B\in\tau^{s*g-\alpha}$. By Proposition (2.15) , there exists $U,V\in\tau$ such that $U\subseteq A\subseteq int(cl_{s*g}(U))$ and $V\subseteq B\subseteq int(cl_{s*g}(V))$. Notice that $U\cap V\in\tau$ and $U\cap V\subseteq A\cap B$. Now ,

$$A\cap B\subseteq int(cl_{s*g}(U))\cap int(cl_{s*g}(V))=int(int(cl_{s*g}(U))\cap cl_{s*g}(V))$$

$\subseteq \text{int}(\text{cl}_{s*g}(\text{int}(\text{cl}_{s*g}(U)) \cap V))$ (by Lemma (2.16)) .

$\subseteq \text{int}(\text{cl}_{s*g}(\text{cl}_{s*g}(U) \cap V))$

$\subseteq \text{int}(\text{cl}_{s*g}(\text{cl}_{s*g}(U \cap V))$ (by Lemma (2.16)) .

$= \text{int}(\text{cl}_{s*g}(U \cap V))$ (by Theorem (1.5),v) .

Thus $U \cap V \subseteq A \cap B \subseteq \text{int}(\text{cl}_{s*g}(U \cap V))$. Therefore by Proposition (2.15) , $A \cap B \in \tau^{s*g-\alpha}$.

(iii). Let $\{U_\alpha : \alpha \in \wedge\}$ be any family of s*g-α-open subsets of X , then $U_\alpha \subseteq \text{int}(\text{cl}_{s*g}(\text{int}(U_\alpha)))$ for each $\alpha \in \wedge$. Therefore by Theorem ((1.5) viii) , we get :

$$\bigcup_{\alpha \in \wedge} U_\alpha \subseteq \bigcup_{\alpha \in \wedge} \text{int}(\text{cl}_{s*g}(\text{int}(U_\alpha))) \subseteq \text{int}(\bigcup_{\alpha \in \wedge} \text{cl}_{s*g}(\text{int}(U_\alpha))) \subseteq \text{int}(\text{cl}_{s*g}(\bigcup_{\alpha \in \wedge} \text{int}(U_\alpha)))$$

$\subseteq \text{int}(\text{cl}_{s*g}(\text{int}(\bigcup_{\alpha \in \wedge} U_\alpha)))$. Hence $\bigcup_{\alpha \in \wedge} U_\alpha \in \tau^{s*g-\alpha}$.

Thus $\tau^{s*g-\alpha}$ is a topology on X .

Propositions(2.18): Let (X,τ) be a topological space and B be a subset of X . Then the following statements are equivalent:

i) B is s*g-α-closed .

ii) $\text{cl}(\text{int}_{s*g}(\text{cl}(B))) \subseteq B$.

iii) There exists a closed subset F of X such that $\text{cl}(\text{int}_{s*g}(F)) \subseteq B \subseteq F$.

Proof: $(i) \Rightarrow (ii)$. Since B is an s*g-α-closed set in X $\Rightarrow X - B$ is an s*g-α-open set in X $\Rightarrow X - B \subseteq \text{int}(\text{cl}_{s*g}(\text{int}(X-B))) \Rightarrow X - B \subseteq \text{int}(\text{cl}_{s*g}(X - \text{cl}(B)))$. By Theorem ((1.5), vi) , we get $X - \text{int}_{s*g}(\text{cl}(B)) = \text{cl}_{s*g}(X - \text{cl}(B))$. Hence $X - B \subseteq \text{int}(X - \text{int}_{s*g}(\text{cl}(B))) \Rightarrow X - B \subseteq X - \text{cl}(\text{int}_{s*g}(\text{cl}(B))) \Rightarrow \text{cl}(\text{int}_{s*g}(\text{cl}(B))) \subseteq B$.

$(ii) \Rightarrow (iii)$.

Since $\text{cl}(\text{int}_{s*g}(\text{cl}(B))) \subseteq B$ and $B \subseteq \text{cl}(B)$, then $\text{cl}(\text{int}_{s*g}(\text{cl}(B))) \subseteq B \subseteq \text{cl}(B)$. Put $F = \text{cl}(B)$, thus there exists a closed subset F of X such that $\text{cl}(\text{int}_{s*g}(F)) \subseteq B \subseteq F$.

$(iii) \Rightarrow (i)$.

Suppose that there exists a closed subset F of X such that $\text{cl}(\text{int}_{s*g}(F)) \subseteq B \subseteq F$. Hence $X - F \subseteq X - B \subseteq X - \text{cl}(\text{int}_{s*g}(F)) = \text{int}(X - \text{int}_{s*g}(F))$. Since $X - \text{int}_{s*g}(F) = \text{cl}_{s*g}(X - F)$, then $X - F \subseteq X - B \subseteq \text{int}(\text{cl}_{s*g}(X - F))$. Hence $X - B$ is an s*g-α-open set in X . Thus B is an s*g-α-closed set in X .

Definition(2.19): A subset A of a topological space (X,τ) is called an s*g-α-neighborhood of a point x in X if there exists an s*g-α-open set U in X such that $x \in U \subseteq A$.

Remark(2.20): Since every open set is an s*g-α-open set , then every neighborhood of x is an s*g-α-neighborhood of x , but the converse is not true in general . In example (2.2), $\{a,b\}$ is an s*g-α-neighborhood of a point b , since $b \in \{a,b\} \subseteq \{a,b\}$. But $\{a,b\}$ is not a neighborhood of a point b .

Propositions(2.21): A subset A of a topological space (X,τ) is s*g-α-open if and only if it is an s*g-α-neighborhood of each of its points .

Proof: $\Rightarrow$ If A is s*g-α-open in X , then $x \in A \subseteq A$ for each $x \in A$. Thus A is an s*g-α-neighborhood of each of its points .

Conversely , suppose that A is an s*g-α-neighborhood of each of its points . Then for each $x \in A$, there exists an s*g-α-open set U_x in X such that $x \in U_x \subseteq A$. Hence $\bigcup_{x\in A} U_x \subseteq A$. Since $A \subseteq \bigcup_{x\in A} U_x$, therefore $A = \bigcup_{x\in A} U_x$. Thus A is an s*g-α-open set in X, since it is a union of s*g-α-open sets .

Proposition(2.22): If A is an s*g-α-open set in a topological space (X,τ) and $A \subseteq B \subseteq int(A)$, then B is an s*g-α-open set in X .

Proof: Since A is an s*g-α-open set in X , then by Proposition (2.15), there exists an open subset U of X such that $U \subseteq A \subseteq int(cl_{s*g}(U))$. Since $A \subseteq B \Rightarrow U \subseteq B$. But $int(A) \subseteq int(cl_{s*g}(U)) \Rightarrow U \subseteq B \subseteq int(cl_{s*g}(U))$. Thus B is an s*g-$\alpha$-open set in X .

Proposition(2.23): If A is an s*g-α-closed set in a topological space (X,τ) and $cl(A) \subseteq B \subseteq A$, then B is an s*g-α-closed set in X .

Proof: Since $X - A \subseteq X - B \subseteq X - cl(A) = int(X - A)$, then by Proposition (2.22) $X - B$ is an s*g-α-open set in X . Thus B is an s*g-α-closed set in X .

Theorem(2.24): A subset A of a topological space (X,τ) is clopen (open and closed) if and only if A is s*g-α-clopen (s*g-α-open and s*g-α-closed) .

Proof: ($\Rightarrow$) . It is a obvious .

($\Leftarrow$) . Suppose that A is an s*g-α-clopen set in X , then A is s*g-α-open and s*g-α-closed in X .

Hence $A \subseteq int(cl_{s*g}(int(A)))$ and $cl(int_{s*g}(cl(A))) \subseteq A$. But by Theorem ((1.5) , i, ii) we get ,

$cl_{s*g}(A) \subseteq cl(A)$ and $int(A) \subseteq int_{s*g}(A)$, thus :

$A \subseteq int(cl(int(A)))$ and $cl(int(cl(A))) \subseteq A$.

Since $int(A) \subseteq A \Rightarrow cl(int(A)) \subseteq cl(A)$ ----------- (1)

Since $int(cl(int(A))) \subseteq cl(int(A))$, thus

$A \subseteq int(cl(int(A))) \subseteq cl(int(A)) \Rightarrow cl(A) \subseteq cl(int(A))$ ----------- (2)

Therefore from (1) and (2) , we get $cl(int(A)) = cl(A)$ ------------ (a)

Similarly, since $A \subseteq cl(A) \Rightarrow int(A) \subseteq int(cl(A))$ ----------- (3)

Now, $int(cl(A)) \subseteq cl(int(cl(A))) \subseteq A$, thus

$int(cl(A)) \subseteq int(A)$ ------------ (4)

Therefore from (3) and (4) , we get $int(cl(A)) = int(A)$ ----------- (b)

Since $int(cl(A)) = int(A) \Rightarrow cl(int(cl(A))) = cl(int(A)) = cl(A)$ (by (a)) .

Since $cl(int(cl(A))) \subseteq A$, then $cl(A) \subseteq A$, but $A \subseteq cl(A)$, therefore $A = cl(A)$, hence A is a closed set in X .

Similarly, since $cl(int(A)) = cl(A) \Rightarrow int(cl(int(A))) = int(cl(A)) = int(A)$ (by (b)) .

Since $A \subseteq int(cl(int(A)))$, then $A \subseteq int(A)$, but $int(A) \subseteq A$, therefore $A = int(A)$, hence A is an open set in X . Thus A is a clopen set in X .

Definition(2.25): Let (X,τ) be a topological space and $A \subseteq X$. Then

i) The s*g-α-closure of A , denoted by $cl_{s*g\alpha}(A)$ is the intersection of all s*g-α-closed subsets of X which contains A .

ii) The s*g-α-interior of A , denoted by $int_{s*g\alpha}(A)$ is the union of all s*g-α-open sets in X which are contained in A .

Theorem(2.26): Let (X,τ) be a topological space and $A,B \subseteq X$. Then:-

i) $int(A) \subseteq int_{s*g\alpha}(A) \subseteq A$ and $A \subseteq cl_{s*g\alpha}(A) \subseteq cl(A)$.

ii) $int_{s*g\alpha}(A)$ is an s*g-α-open set in X and $cl_{s*g\alpha}(A)$ is an s*g-α-closed set in X .

iii) If $A \subseteq B$, then $int_{s*g\alpha}(A) \subseteq int_{s*g\alpha}(B)$ and $cl_{s*g\alpha}(A) \subseteq cl_{s*g\alpha}(B)$.

iv) A is s*g-α-open iff $int_{s*g\alpha}(A) = A$ and A is s*g-α-closed iff $cl_{s*g\alpha}(A) = A$.

v) $int_{s*g\alpha}(A \cap B) = int_{s*g\alpha}(A) \cap int_{s*g\alpha}(B)$ and $cl_{s*g\alpha}(A \cup B) = cl_{s*g\alpha}(A) \cup cl_{s*g\alpha}(B)$.

vi) $int_{s*g\alpha}(int_{s*g\alpha}(A)) = int_{s*g\alpha}(A)$ and $cl_{s*g\alpha}(cl_{s*g\alpha}(A)) = cl_{s*g\alpha}(A)$.

vii) $x \in int_{s*g\alpha}(A)$ iff there is an s*g-α-open set U in X s.t $x \in U \subseteq A$.

viii) $x \in cl_{s*g\alpha}(A)$ iff for every s*g-α-open set U containing x , $U \cap A \neq \phi$.

Proof: It is obvious .

Proposition(2.27): Let X and Y be topological spaces . If $A \subseteq X$ and $B \subseteq Y$. Then $A \times B$ is an s*g-α-open set in $X \times Y$ if and only if A and B are s*g-α-open sets in X and Y respectevely . **Proof:** $\Leftarrow$ Since A and B are s*g-α-open sets in X and Y respectevely , then by definition (2.1), we get $A \subseteq int(cl_{s*g}(int(A)))$ and $B \subseteq int(cl_{s*g}(int(B)))$. Hence $A \times B \subseteq int(cl_{s*g}(int(A))) \times int(cl_{s*g}(int(B))) = int(cl_{s*g}(int(A)) \times cl_{s*g}(int(B)))$. Since $cl_{s*g}(A) \times cl_{s*g}(B) = cl_{s*g}(A \times B)$, then $A \times B \subseteq int(cl_{s*g}(int(A \times B)))$. Thus $A \times B$ is an s*g-α open set in $X \times Y$. By the same way, we can prove that A and B are s*g-α-open sets in X and Y respectevely if $A \times B$ is an s*g-α-open set in $X \times Y$.

3 . s*g-α - Continuous Functions and s*g-α - Irresolute Functions

In this section , we introduce a new class of functions , namely , s*g-α-continuous functions and s*g-α-irresolute functions in topological spaces and study some of their properties.

Definition(3.1): A function $f:(X,\tau) \to (Y,\sigma)$ is called s*g-α-continuous if $f^{-1}(V)$ is an s*g-α-open set in X for every open set V in Y .

Proposition(3.2): A function $f:(X,\tau) \to (Y,\sigma)$ is s*g-α-continuous iff $f^{-1}(V)$ is an s*g-α-closed set in X for every closed set V in Y .

Proof: It is Obvious .

Proposition(3.3): Every continuous function is s*g-α-continuous .

Proof: Follows from the definition (3.1) and the fact that every open set is s*g-α-open .

Remark(3.4): The converse of Proposition (3.3) may not be true in general as shown in the following example:

Example(3.5): Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\}\}$ & $\sigma = \{Y,\phi,\{a\},\{a,c\}\} \Rightarrow$ $\tau^{s*g-\alpha} = \{X,\phi,\{a\},$ $\{a,b\},\{a,c\}\}$. Define $f:(X,\tau) \to (Y,\sigma)$ by : $f(a) = a$, $f(b) = b$ & $f(c) = c$ $\Rightarrow$ f is not continuous , but f is s*g-α-continuous , since $f^{-1}(Y) = X$, $f^{-1}(\phi) = \phi$, $f^{-1}(\{a,c\}) = \{a,c\}$, and $f^{-1}(\{a\}) = \{a\}$ are s*g-α-open sets in X .

Remark(3.6): s*g-continuous functions and s*g-α-continuous functions are in general independent . Consider the following examples:-

Example(3.7): Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi\}$ & $\sigma = \{Y,\phi,\{a\}\}$ $\Rightarrow$ $\tau^{s^*g-\alpha} = \tau$ and $\tau^{s^*g} = \{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by : $f(a) = a$, $f(b) = b$ & $f(c) = c \Rightarrow$ f is s*g-continuous , but f is not s*g-α-continuous , since $\{a\}$ is open set in Y , but $f^{-1}(\{a\}) = \{a\}$ is not s*g-α-open in X . Also, in Example (3.5) f is s*g-α-continuous, but is not s*g-continuous , since $\{a,c\}$ is open set in Y , but $f^{-1}(\{a,c\}) = \{a,c\}$ is not s*g-open in X .

Theorem(3.8): Every s*g-α-continuous function is α-continuous (resp. αg-continuous , gα-continuous , pre-continuous , b-continuous ,β-continuous) function .

Proof: Follows from the Theorem (2.5) .

Remark(3.9): The converse of Theorem (3.8) may not be true in general . Observe that in Example (3.7) f is pre-continuous (resp. b-continuous ,β-continuous , gα-continuous , αg-continuous) function , but f is not s*g-α-continuous .

Theorem(3.10): Every s*g-α-continuous function is semi-continuous function and gs-continuous function .

Proof: Follows from the Theorem (2.8) .

Remark(3.11): The converse of Theorem (3.10) may not be true in general as shown in the following example:

Example(3.12): Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$ & $\sigma = \{Y,\phi,\{a\},\{a,c\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by : $f(a) = a$, $f(b) = b$ & $f(c) = c \Rightarrow$ f is semi-continuous and gs-continuous, but f is not s*g-α-continuous, since $\{a,c\}$ is open in Y, but $f^{-1}(\{a,c\}) = \{a,c\}$ is not s*g-α-open in X , since $\{a,c\} \not\subset int(cl_{s^*g}(int(\{a,c\}))) = int(cl_{s^*g}(\{a\})) = int(\{a,c\}) = \{a\}$.

Remark(3.13): Pre-continuous functions and αg-continuous functions are in general independent . Consider the following examples:-

Example(3.14): Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{a,c\}\}$ & $\sigma = \{Y,\phi,\{a\},\{b\},\{a,b\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by : $f(a) = a$, $f(b) = c$ & $f(c) = b$ $\Rightarrow$ f is αg-continuous , but f is not pre-continuous , since $\{b\}$ is open set in Y , but $f^{-1}(\{b\}) = \{c\}$ is not pre-open set in X , since
$\{c\} \not\subset int(cl(\{c\})) = int(\{b,c\}) = \phi$.

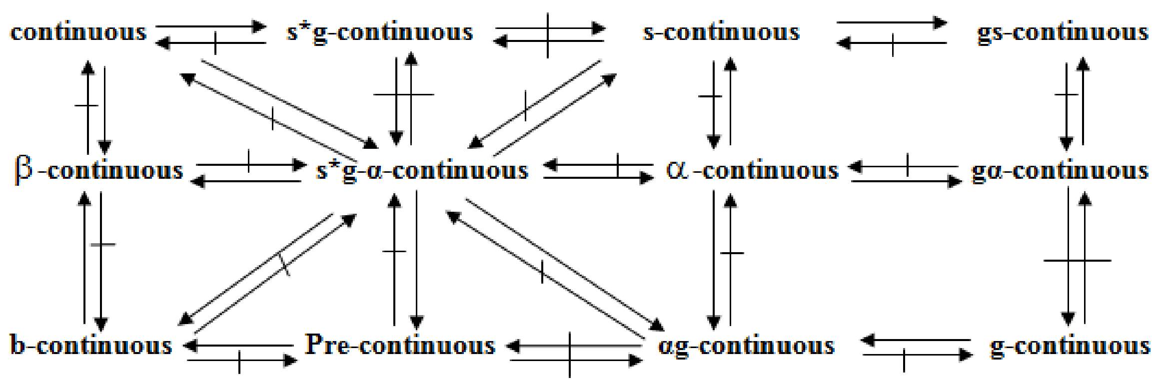
Example(3.15): Let $X = Y = \Re$, $\tau = \mu =$ usual topology & $\sigma = \{\Re,\phi,\{Q\}\}$. Define $f:(\Re,\mu)\to(\Re,\sigma)$ by : $f(x) = x$ for each $x \in \Re \Rightarrow$ f is not αg-continuous , since Q is open in Y, but $f^{-1}(\{Q\}) = Q$ is not αg-open set in X . But f is pre-continuous .

Remark(3.16): g-continuous functions and gα-continuous functions are in general independent . Consider the following examples:-

Example(3.17): Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\}\}$ & $\sigma = \{Y,\phi,\{b\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by : $f(a) = a$, $f(b) = c$ & $f(c) = b \Rightarrow$ f is g-continuous , but f is not gα-continuous, since $\{b\}$ is open set in Y, but $f^{-1}(\{b\}) = \{c\}$ is not gα-open set in X , since $\{c\}^c$ = $\{a,b\}$ is not gα-closed set in X .

Example(3.18): Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{a,c\}\}$ & $\sigma = \{Y,\phi,\{b\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by : $f(a) = b$, $f(b) = b$ & $f(c) = a$ $\Rightarrow$ f is gα-continuous , but f is not g-continuous, since $\{b\}$ is open set in Y, but $f^{-1}(\{b\}) = \{a,b\}$ is not g-open set in X , since $\{a,b\}^c = \{c\}$ is not g-closed in X .

The following diagram shows the relationships between s*g-α-continuous functions and some other continuous functions:



Proposition(3.19): If $f:(X,\tau)\to(Y,\sigma)$ is s*g-α-continuous, then $f(cl_{s^*g\alpha}(A))\subseteq cl(f(A))$ for every subset A of X.

Proof: Since $f(A)\subseteq cl(f(A))\Rightarrow A\subseteq f^{-1}(cl(f(A)))$. Since $cl(f(A))$ is a closed set in Y and f is s*g-α-continuous ,then by (3.2) $f^{-1}(cl(f(A)))$ is an s*g-α-closed set in X containing A. Hence $cl_{s^*g\alpha}(A)\subseteq f^{-1}(cl(f(A)))$. Therefore $f(cl_{s^*g\alpha}(A))\subseteq cl(f(A))$.

Theorem(3.20: Let $f:(X,\tau)\to(Y,\sigma)$ be a function . Then the following statements are equivalent:-

i) f is s*g-α-continuous .

ii) For each point x in X and each open set V in Y with $f(x)\in V$, there is an s*g-α-open set U in X such that $x\in U$ and $f(U)\subseteq V$.

iii) For each subset A of X, $f(cl_{s^*g\alpha}(A))\subseteq cl(f(A))$.

iv) For each subset B of Y, $cl_{s^*g\alpha}(f^{-1}(B))\subseteq f^{-1}(cl(B))$.

Proof: (i) $\Rightarrow$ (ii). Let $f:X\to Y$ be an s*g-α-continuous function and V be an open set in Y s.t $f(x)\in V$. To prove that , there is an s*g-α-open set U in X s.t $x\in U$ and $f(U)\subseteq V$. Since f is s*g-α-continuous , then $f^{-1}(V)$ is an s*g-α-open set in X s.t $x\in f^{-1}(V)$. Let $U=f^{-1}(V)\Rightarrow f(U)=f(f^{-1}(V))\subseteq V\Rightarrow f(U)\subseteq V$.

(ii) $\Rightarrow$ (i). To prove that $f:X\to Y$ is s*g-α-continuous . Let V be any open set in Y . To prove that $f^{-1}(V)$ is an s*g-α-open set in X . Let $x\in f^{-1}(V)\Rightarrow f(x)\in V$ .By hypothesis there is an s*g-α-open set U in X s.t $x\in U$ and $f(U)\subseteq V\Rightarrow x\in U\subseteq f^{-1}(V)$. Thus by Theorem ((2.26),vii) $f^{-1}(V)$ is an s*g-α-open set in X . Hence $f:X\to Y$ is an s*g-α-continuous function .

(ii) $\to$ (iii).

Suppose that (ii) holds and let $y\in f(cl_{s^*g\alpha}(A))$ and let V be any open neighborhood of y in Y. Since $y\in f(cl_{s^*g\alpha}(A))\Rightarrow\exists\ x\in cl_{s^*g\alpha}(A)$ s.t $f(x)=y$. Since $f(x)\in V$, then by (ii) $\exists$ an s*g-α-open set U in X s.t $x\in U$ and $f(U)\subseteq V$. Since $x\in cl_{s^*g\alpha}(A)$, then by Theorem ((2.26),viii) $U\bigcap A\neq\phi$ and hence $f(A)\bigcap V\neq\phi$. Therefore we have $y\in cl(f(A))$. Hence $f(cl_{s^*g\alpha}(A))\subseteq cl(f(A))$.

$(iii) \to (ii)$.

Let $x \in X$ and V be any open set in Y containing $f(x)$. Let $A = f^{-1}(V^c) \Rightarrow x \notin A$. Since $f(cl_{s*g\alpha}(A)) \subseteq cl(f(A)) \subseteq V^c \Rightarrow cl_{s*g\alpha}(A) \subseteq f^{-1}(V^c) = A$. Since $x \notin A \Rightarrow$ $x \notin cl_{s*g\alpha}(A)$ and

by Theorem ((2.26),viii) there exists an s*g-α-open set U containing x such that $U \bigcap A = \phi$ and hence $f(U) \subseteq f(A^c) \subseteq V$.

$(iii) \to (iv)$.

Suppose that (iii) holds and let B be any subset of Y. Replacing A by $f^{-1}(B)$ we get from (iii) $f(cl_{s*g\alpha}(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$. Hence $cl_{s*g\alpha}(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

$(iv) \to (iii)$.

Suppose that (iv) holds and let $B = f(A)$ where A is a subset of X. Then we get from (iv) $cl_{s*g\alpha}(A) \subseteq cl_{s*g\alpha}(f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. Therefore $f(cl_{s*g\alpha}(A)) \subseteq cl(f(A))$.

Definition(3.21): A function $f:(X,\tau) \to (Y,\sigma)$ is called s*g-α-irresolute if the inverse image of every s*g-α-open set in Y is an s*g-α-open set in X .

Proposition(3.22): Every s*g-α-irresolute function is s*g-α-continuous .
Proof: It is Obvious .
Remark(3.23): The converse of Proposition (3.22) may not be true in general as shown in the following example:
Example(3.24): Let $X = Y = \{a,b,c\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a,c\}\}$ & $\sigma = \{Y, \phi, \{a\}, \{a,c\}\}$ $\Rightarrow$
$\tau^{s*g-\alpha} = \tau$ and $\sigma^{s*g-\alpha} = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Define $f:(X,\tau) \to (Y,\sigma)$ by : $f(a) = a$, $f(b) = b$ & $f(c) = c \Rightarrow$ f is s*g-α-continuous, but f is not s*g-α-irresolute since $\{a,b\}$ is an s*g-α-open set in Y , but $f^{-1}(\{a,b\}) = \{a,b\}$ is not s*g-α-open set in X .

Remark(3.25): continuous functions and s*g-α-irresolute functions are in general independent

Consider the following examples:-

Example(3.26): Let $X = Y = \{a,b,c\}$, $\tau = \{X, \phi, \{a\}, \{b,c\}\}$ & $\sigma = \{Y, \phi, \{a\}\}$. Also ,
$\tau^{s*g-\alpha} = \{X, \phi, \{a\}, \{b,c\}\}$ & $\sigma^{s*g-\alpha} = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Define $f:(X,\tau) \to (Y,\sigma)$ by : $f(a) = a$,
$f(b) = b$ & $f(c) = c \Rightarrow$ f is continuous , but f is not s*g-α-irresolute , since $\{a,b\}$ is s*g-α-open
set in Y , but $f^{-1}(\{a,b\}) = \{a,b\}$ is not s*g-α-open set in X .
Example(3.27): Let $X = Y = \{a,b,c\}$, $\tau = \{X, \phi, \{a\}\}$ & $\sigma = \{Y, \phi, \{a,b\}\}$. Also , $\tau^{s*g-\alpha} = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ & $\sigma^{s*g-\alpha} = \{Y, \phi, \{a,b\}\}$. Define $f:(X,\tau) \to (Y,\sigma)$ by : $f(a) = a$, $f(b) = b$ & $f(c) = c \Rightarrow$ f is s*g-α-irresolute , but f is not continuous , Since $\{a,b\}$ is open in Y , but $f^{-1}(\{a,b\}) = \{a,b\}$ is not open in X .
Theorem(3.28): Let $f:(X,\tau) \to (Y,\sigma)$ be a function . Then the following statements are equivalent:-
(i) f is s*g-α-irresolute .
(ii) For each $x \in X$ and each s*g-α-neighborhood V of $f(x)$ in Y, there is an s*g-α-neighborhood

U of x in X such that $f(U) \subseteq V$.

(iii) The inverse image of every s*g-α-closed subset of Y is an s*g-α-closed subset of X .

Proof: (i) $\Rightarrow$ (ii) **.** Let $f : X \to Y$ be an s*g-α-irresolute function and V be an s*g-α-neighborhood of $f(x)$ in Y . To prove that , there is an s*g-α-neighborhood U of x in X such that $f(U) \subseteq V$. Since f is an s*g-α-irresolute then , $f^{-1}(V)$ is an s*g-α-neighborhood of x in X . Let $U = f^{-1}(V) \Rightarrow f(U) = f(f^{-1}(V)) \subseteq V \Rightarrow f(U) \subseteq V$.

(ii) $\Rightarrow$ (i) **.** To prove that $f : X \to Y$ is s*g-α-irresolute . Let V be an s*g-α-open set in Y . To prove that $f^{-1}(V)$ is an s*g-α-open set in X . Let $x \in f^{-1}(V) \Rightarrow f(x) \in V \Rightarrow$ V is an s*g-α-neighborhood of $f(x)$ .By hypothesis there is an s*g-α-neighborhood U_x of x such that $f(U_x) \subseteq V \Rightarrow U_x \subseteq f^{-1}(V), \forall x \in f^{-1}(V) \Rightarrow \exists$ an s*g-α-open set W_x of x such that $W_x \subseteq U_x \subseteq f^{-1}(V), \forall x \in f^{-1}(V) \Rightarrow \bigcup_{x \in f^{-1}(V)} W_x \subseteq f^{-1}(V)$. Since $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(V)} W_x$

$\Rightarrow f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} W_x \Rightarrow f^{-1}(V)$ is an s*g-α-open set in Y, since its a union of s*g-α-open sets . Thus $f : X \to Y$ is an s*g-α-irresolute function .

(i) $\Leftrightarrow$ (iii) . It is a obvious .

Corollary(3.29): Let (X_1, τ_1) and (X_2, τ_2) be topological spaces . Then the projection functions

$\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are s*g-α-irresolute functions .

Proof: Let U be an s*g-α-open set in X_1 , then $\pi_1^{-1}(U) = U \times X_2$. Since U is s*g-α-open in X_1 and X_2 is s*g-α-open in X_2 , then by Proposition (2.27) $U \times X_2$ is s*g-α-open in $X_1 \times X_2$. Thus

$\pi_1 : X_1 \times X_2 \to X_1$ is an s*g-α-irresolute function . Similaly we can prove that $\pi_2 : X_1 \times X_2 \to X_2$ is s*g-α-irresolute function .

However the following theorem holds . The proof is easy and hence omitted .

Theorem(3.30): If $f : (X, \tau) \to (Y, \sigma)$ and $f : (Y, \sigma) \to (Z, \eta)$ are functions, then:-

i) If f and g are both s*g-α-irresolute functions , then so is $g \circ f$.

ii) If f is s*g-α-irresolute and g is s*g-α-continuous , then $g \circ f$ is s*g-α-continuous .

iii) If f is s*g-α-continuous and g is continuous , then $g \circ f$ is s*g-α-continuous .

References

1 . Levine, N. (1963) Semi-open sets and semi-continuity in topological spaces , Amer. Math. Monthly, 70, 36-41 .

2 . Levine,N.(1970)Generalized closed sets in topology,Rend.Circ.Math.Palermo,_19 (2), 89-96 .

3 . Njasta, O. (1965) On some classes of nearly open sets , Pacific J. Math. 15, 961-970 .

4 . Mashhour, A.S. ; Abd El-Monsef,M.E. and El-Deeb,S.N. (1982) On precontinuous and weak precontinuous functions , Proc. Math. Phys. Soc. Egypt , 51, 47-53 .

5 . Andrijevic, D. (1996) On b-open sets, Mat. Vesnik , 48 (1-2) , 59-64 .

6 . Abd El-Monsef,M.E ; El-Deeb , S.N. and Mahmoud,R.A. (1983) β-open sets and β-continuous Mappings , Bull. Fac. Sci. Assuit Univ. 12, 77-90 .

7 . Arya,S.P. and Nour,T.M. (1990) Characterizations of s-normal spaces , Indian J. Pure Appl. Math. 21 (8) , 717-719 .

8 . Maki,H. ; Devi ,R. and Balachandran,K. (1993) Generalized α-closed sets in topology , Bull.Fukuoka Univ. Ed. Part III, 42_, 13-21.

9 . Maki,H. ; Devi,R. and Balachandran,K. (1994) Associated topologies of generalized α-closed sets and α-generalized closed sets , Mem. Fac. Sci. Kochi Univ. Ser.A.Math. 15_, 51-63.

10 . Khan,M. ; Noiri, T . and Hussain, M .(2008) On s*g-closed sets and s*-normal spaces , 48 , 31-41 .

11 . Mashhour,A.S.; Hasanein,I.A and El-Deeb,S.N. (1983) α -continuous and α -open mappings , Acta Math Hung. 41 (3-4) , 213-218 .

12 . Ekici, E. and Caldas,M. (2004) Slightly γ -continuous functions , Bol. Soc. Parana Mat. 22 (2), 63-74 .

13 . Balachandran,K.; Sundaram,P. and Maki,H. (1991) On generalized continuous maps in topological spaces ,Mem. Fac. Sci. Kochi Univ. Ser. A. Math. 12 , 5-13 .

14 . Devi,R. ; Balachandran,K. and Maki,H. (1995) Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces , Indian J. Pure Appl. Math. 26 (3), 271-284 .

15 . Gnanambal,Y. (1997) On generalized preregular closed sets in topological spaces , Indian J. Pure Appl .Math. 28(3), 351-360 .

16 . S. I . and Afrah M .(2010) S*-separation axioms , Iraqi Journal of Science, University of Baghdad, 51(1), 145-153 .

17 . S. I . (2014) On Weak D_{s*g} -Sets And Associative Separation Axioms , Ibn Al-Haitham Journal for Pure and Applied Science , (to appear) .

18 . S. I., (2014) Another Type Of Compactness In Bitopological Spaces , Journal of Al Rafidain University College , (to appear) .

حول المجموعات المفتوحة - α -S*g في الفضاءات التبولوجية

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الخلاصة

قدمنا في هذا البحث صنفا جديدا من المجموعات أسميناها بالمجموعات المفتوحة من النمط - α -s*g و من ثم اثبتنا ان عائلة كل المجموعات الجزئية المفتوحة من النمط - α -s*g من الفضاء التبولوجي (X,τ) تشكل تبولوجي على X الذي هو انعم من τ . كذلك درسنا المكافئات والخواص الأساسية للمجموعات المفتوحة من النمط- α -s*g والمجموعات المغلقة من النمط - α -s*g . فضلا عن ذلك استخدمنا هذه المجموعة في تعريف ودراسة صنف جديد من الدوال في الفضاءات التبولوجية أسميناه بالدوال المستمرة من النمط - α -s*g والدوال المحيرة من النمط- α -s*g وقد درست بعض خواص هذه الدوال.

الكلمات المفتاحية: المجموعات المفتوحة من النمط -- α s*g , المجموعات المغلقة من النمط - α s*g - , الدوال المستمرة من النمط-- α s*g , الدوال المحيرة من النمط-- α s*g .