IHJPAS. 36 (3) 2023

Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq

Analytical Solutions to Investigate Fractional Newell-Whitehead Nonlinear Equation using *Sumudu* Transform Decomposition Method

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Article history: Received 26 January 2023, Accepted 20 February 2023, Published in July 2023.

doi.org/10.30526/36.3.3241

Abstract

Some nonlinear differential equations with fractional order are evaluated using a novel approach *Sumudu and Adomian* Decomposition technique (STADM). To get the results of the given model, the Sumudu transformation and iterative technique are employed. The suggested method has an advantage over alternative strategies in that it does not require for additional resources or calculations. This approach works well, easy to use, and yield good results. Besides, the solution graphs are plotted using MATLAB software. Also, the true solution of the fractional Newell-Whitehead equation is shown together with the approximate solutions of STADM. The results showed our approach is a great way, fantastic, reliable and easy method to deal with specific problems in a variety of applied sciences and engineering fields.

Keywords: *Sumudu* Transformation, Caputo derivative, Fractional Calculus, Approximate solutions, Decomposition method.

1. Introduction

Partial differential equations with fractional order (FPDEs) are a modification of integer order differential equations. The study of FPDEs has attracted more attention recently. The fractional approach has developed into a powerful modeling technique that is frequently used in the fields of chaotic dynamics, wave propagation, turbulence, turbulent flow, diffusion processes, [1]-[4]. Because some fractional order models cannot be tested analytically and the results for FPDEs should be have. Many researchers have focused on developing effective and reliable techniques for FPDEs which include Laplace transform [7], Laplace Variational method (LVIM) [8], perturbation method [9], and differential transform method [5], Variational iteration method [6], and many others. Several analytical and approximation methods using SVIM solving nonlinear problems of fractional order [9,10], and others[11-17] have been proposed. In this Work, we applied a new mixture which is a graceful coupling of two strong approaches STADM for solving fractional-order Nonlinear PDES .

2. Idea of *Sumudu* Transform and Decomposition Approach (STADM):

The STADM is discussed in this section as it relates to solving FPDEs. The following general FPDEs as:

$$D_t^w \mathrm{y(x,t)} + \mathrm{Ly(x,t)} + \mathrm{Ny(x,t)} = \mathrm{q(x,t)}\,,\ \mathrm{x} \geq 0, \mathrm{t} > 0\,, \mathrm{n} - 1 < w \leq \mathrm{n} \qquad ....(1)$$

With the initial conditions

$$\mathrm{y(x,0)} = \mathrm{g(x)} \qquad \text{or} \quad \frac{\partial^{\mathrm{r}} \mathrm{y(x,0)}}{\partial\, \mathrm{t}^{\mathrm{r}}} = \mathrm{g}^{(\mathrm{r})}(\mathrm{x},0) = \mathrm{g_r(x)}\,, \ldots\ldots\ldots\ (2)$$

$$\mathrm{r} = 0,1, \ldots \ldots .. \mathrm{n} - 1$$

The *Caputo* fractional derivative $D_t^w y(x,t)$ of $y(x,t)$ denoted below:

$$\frac{\partial^w}{\partial t^w}\ y(x,t) = \begin{Bmatrix} \dfrac{1}{\Gamma(\mathrm{n-w})} \displaystyle\int_0^t (t-s)^{n-w-1}\ \dfrac{\partial^n y(x,s)}{\partial t^n} & n-1 < w < n \\ \dfrac{\partial^n y\,(x,t)}{\partial t^n} & w = n \in N \end{Bmatrix}$$

Taking *Sumudu* transform of the Eq. (1), we have:

$$S[D_t^w y(x,t)] + S\big[R[y(x,t)]\big] + S\big[N[y(x,t)]\big] = S[q(x,t)] \ldots\ldots\ldots.(3)$$

The property of *Sumudu* transform of function derivatives used, then

$$\frac{S[y(x,t)]}{u^w} = \sum_{k=0}^{n-1} \frac{y(x,0)^k}{u^{w-k}} + S[q(x,t)] - S[L(y(x,t)) + N(y(x,t))]$$

$$S[y(x,t)] = y(x,0) + u^w S[q(x,t)] - u^w S\big[L\big(y(x,t)\big) + N\big(y(x,t)\big)\big] \qquad (4)$$

Application of *Sumudu* inverse transform on Eq. (4) yielded:

$$y(x,t) = f(x) + S^{-1}(u^w S\,[q(x,t)]) - S^{-1}(u^w S[L(y(x,t) + N(y(x,t)]) \qquad (5)$$

The representation of the solution for Eq. (5) as an infinite series is given below:

$$y(x,t) = \sum_{i=0}^{\infty} y_i(x,t) \qquad (6)$$

And the nonlinear term is being decomposed as:

$$N[y(x,t)] = \sum_{i=0}^{\infty} A_i(y_0, y_1, \ldots., y_i) \qquad (7)$$

Where, A_i are the *Adomian polynomials* of functions $y_0, y_1, \ldots., y_i$ can be calculated by formula given as:

$$A_i = \frac{1}{i!}\frac{\partial^i}{\partial \lambda^i}\left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}$$

Substituting Eqs. (6) and (7) in Eq. (5):

$$\sum_{i=0}^{\infty} y_i(x,t) = \sum_{k=0}^{\infty} \frac{y(x,0)^{(k)}}{u^{w-k}} + S^{-1}(u^w S[q(x,t)]) - S^{-1}\left[u^w S\left[L\left(\sum_{i=0}^{\infty} y_i(x,t)\right) + \sum_{i=0}^{\infty} A_i\right]\right] \quad (8)$$

Simplification of Eq. (7) as many times as possible resulted into series solution, we get:

$$y_0(x,t) = \sum_{k=0}^{\infty} \frac{y(x,0)^{(k)}}{u^{w-k}} + S^{-1}(s^w S[g(x,t)])$$
$$y_1(x,t) = -S^{-1}\left[u^w S[L(y_0(x,t)) + A_0]\right]$$
$$y_2(x,t) = -S^{-1}\left[u^w S\left[L\left(y_1(x,t)\right) + A_1\right]\right]$$
$$\vdots$$
$$y_i(x,t) = -S^{-1}\left[u^w S[L(y_{i-1}(x,t)) + A_i]\right]$$

Finally, the iteration $y_0, y_1, \dots, y_i$ were obtained and we approximate the analytical solution $y(x,t)$ by truncated series $y(x,t) = \sum_{i=0}^{\infty} y_i(x,t)$.

3. Test Example:

The test example shows the reliable and efficient of STADM. All results are calculated using the software MATLAB R2021b.

Example 3.1: Consider Newell-Whitehead PDES as follows:

$$\frac{\partial^w y(x,t)}{\partial t^w} - \frac{\partial^2 y(x,t)}{\partial x^2} = y(x,t) - y^3(x,t) \quad \dots\dots\dots\dots\dots\dots(9)$$

With initial condition

$$y(x,0) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{2\sqrt{2}}\right)\right] \quad \dots\dots\dots\dots\dots\dots(10)$$

And the true solution $y(x,t) = \frac{1}{2}\left[1 + \tanh\left(\frac{\sqrt{2}x + 3t}{4}\right)\right]$

Solution :

The Adomain polynomials for the nonlinear terms $-y^3$ Can be computed as follows:

$$A_0 = -y_0^3$$
$$A_1 = -3\, y_0^2\, y_1$$
$$A_2 = -3(y_0^2\, y_2 + y_0\, y_1^2)$$
$$A_3 = -(3(y_0^2\, y_3 + 6y_0\, y_1\, y_2 + y_1^3)$$

By using Eq.(10), we have

$$y_0(x,t) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{2\sqrt{2}}\right)\right]$$

$$y_1(x,t) = \int_t^{-1} u\, S_t\left(\frac{\partial^2 y_0(x,t)}{\partial x^2} + y_0(x,t) + A_0\right)$$
$$= \frac{3}{8}\operatorname{sech}^2\left(\frac{x}{2\sqrt{2}}\right).\left(\frac{t^w}{w}\right)$$

We determine the terms below using the same pattern:

$$y_2(x,t) = -\frac{9}{4}\sin h^4\left(\frac{x}{2\sqrt{2}}\right)\csc h^3\left(\frac{x}{\sqrt{2}}\right).\left(\frac{t^w}{w}\right)^2$$
$$y_3(x,t) = \frac{9}{128}.\left(\frac{t^w}{w}\right)^3.\left(\cosh\left(\frac{x}{\sqrt{2}}\right) - 2\right)\operatorname{sech}^4\left(\frac{x}{2\sqrt{2}}\right)$$

The analytical solution is provided by:

$$y_n(x,t) = y_0(x,t) + y_1(x,t) + y_2(x,t) + y_3(x,t)$$
$$y_n(x,t) = \frac{1}{2}\left[1 + \tan h\left(\frac{x}{2\sqrt{2}}\right)\right] + \frac{3}{8}\operatorname{sech}^2\left(\frac{x}{2\sqrt{2}}\right).\left(\frac{t^\alpha}{\alpha}\right) - \frac{9}{4}\sin h^4\left(\frac{x}{2\sqrt{2}}\right)\csc h^3\left(\frac{x}{\sqrt{2}}\right).\left(\frac{t^\alpha}{\alpha}\right)^2$$
$$+ \frac{9}{128}.\left(\frac{t^\alpha}{\alpha}\right)^3.\left(\cosh\left(\frac{x}{\sqrt{2}}\right) - 2\right)\operatorname{sech}^4\left(\frac{x}{2\sqrt{2}}\right)$$

Where n = 1 , 2 , 3 and 4 for the set of α values that applied in this example, which are 0.4, 0.6, 0.8, and 1.

4. Results

The following Figures present the absolute error at t=0.002 with various value of x . We employ a few terms to approximate the solution, and the suggested approach, FSTADM, has a high convergence order and higher accuracy. Similarly, Figure4.1–Figure4.6 show the 3D exact and achieved results are plotted at α =0.4, 0.6, 0.8, and 1. All the accurate and approximate results on the graphs have shown are much closed and indicates the validity of the present technique.

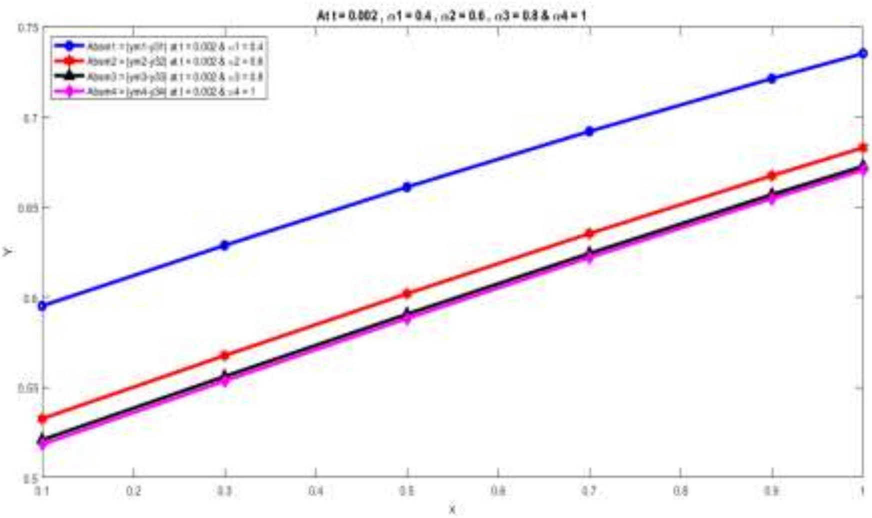

Figure 4.1: ABS error of the solutions at α=0.4,0.6,0.8,1.

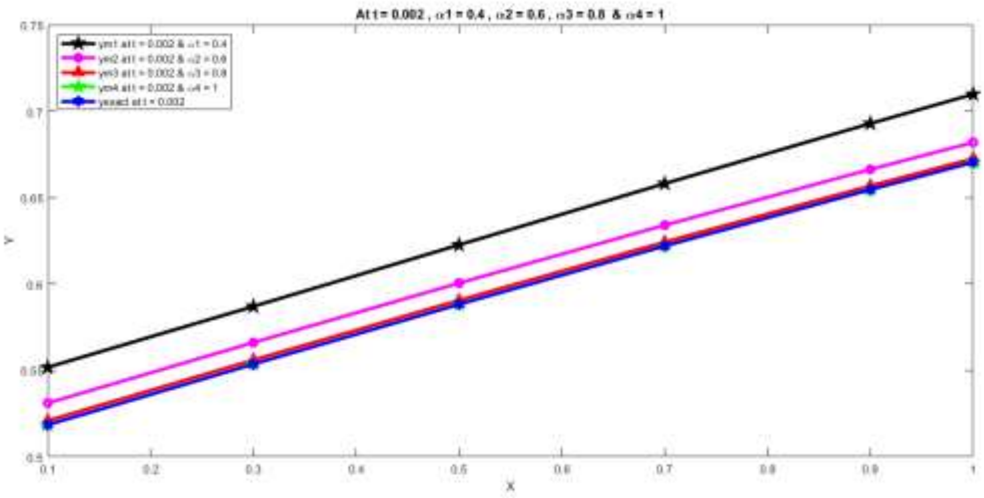

Figure4.2: Approximate solutions ym at α=0.4,0.6,0.8 and 1 and exact solution .

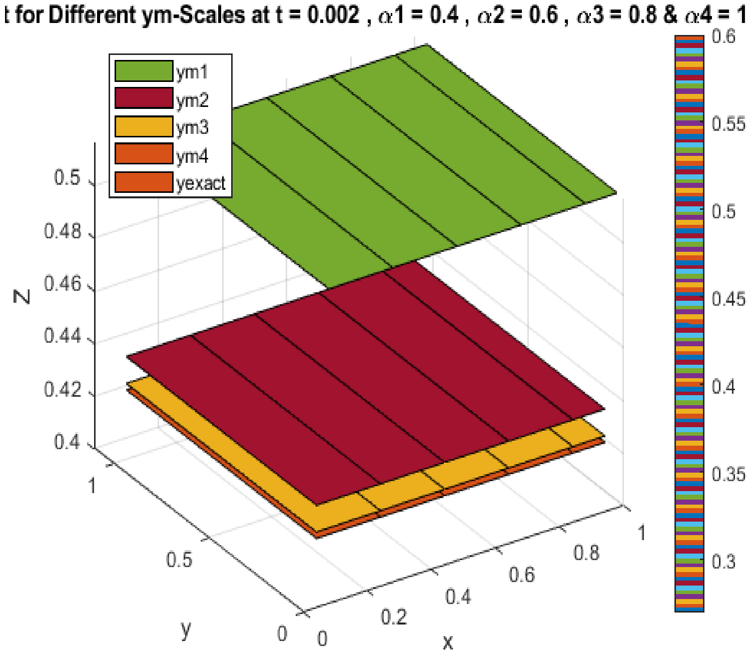

Figure 4.3: The 3D approximate solutions plots at α = 0.4 , 0.6 , 0.8 and 1.

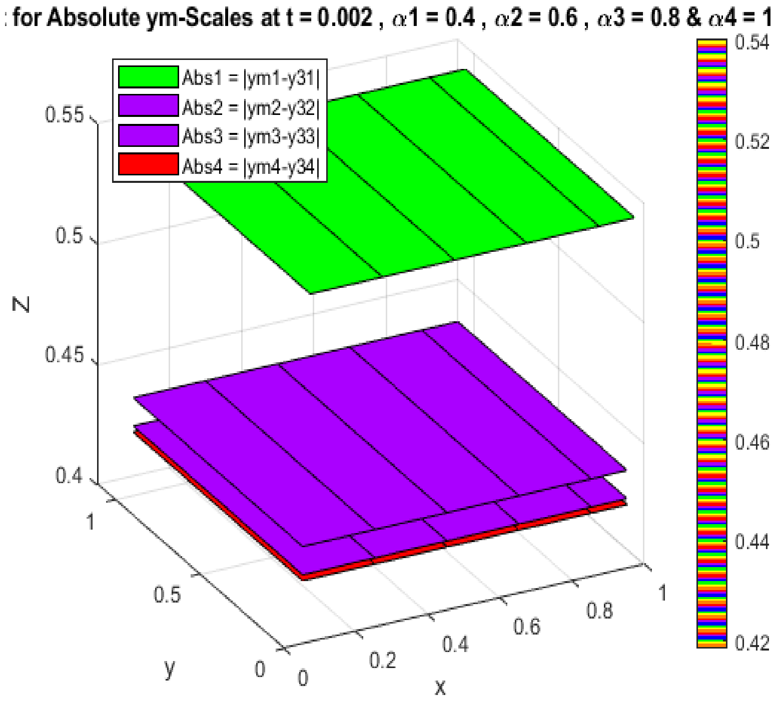

Figure 4.4: The 3D absolute solution plots.

5. Conclusion

It is difficult to find analytical solutions to some FPDEs with initial and boundary conditions, so the solution here rarely exists. Here, the solutions of the time-fractional Newell-Whitehead equation are made successfully. The results are convergent and much closed to the true solutions. The results are shown through 2D and 3D at various fractional-orders. So, the presented method has an excellent convergent rate and can be used to solve non-linear applications.

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