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Extension of Size and Degree of (k,r)-Caps in $PG(3,13)$

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Abstract

The aim of this work is an extension of the (k,r)-caps (k is the order, r is the degree), where $0 < r < 14$, in the three projective space of dimension over the Galois field of order thirteen, $PG(3,13)$. The extensions have been done on the caps founded by action subgroups of projective general linear of order four over the finite field of order thirteen on $PG(3,13)$. The main condition for the completion of the expansion process on the degree of caps is 14 (number of points on the line in $PG(3,13)$) and the size of the cap is points with a zero index zero, as it becomes complete when it is equal to zero. In this paper, we present fifteen caps (completes and incompletes) in $PG(3,13)$ of degrees 2,3,4,7 are extended in size until they reach 14, which are then complete caps, and then the c_i-distribution are computed for each new cap.

Keywords: Cap, Complete cap, Finite projective space, Group action**.**

1. Introduction

Let F_q be the Galois field of q elements and $PG(3,q)$ be the projective space of dimension three. The points $[x_0, x_1, x_2, x_3]$ of $PG(3,q)$ are the 1-dimensional subspaces of the vector space F_q^4 over F_q. Subspaces of dimension two are called lines and dimension three planes. The number of points and the number of planes in $PG(3,q)$ is $q^3 + q^2 + q + 1$. The number of lines is $(q^{n+1} - 1)(q^n - 1)/(q^2 - 1)(q - 1)$. There are $q + 1$ points on every line, $q + 1$ lines through every point, and $q^2 + q + 1$ points on every plane; see [1],[2] and [3].

Many articles have been published about finding (k,r)-caps in the projective space especially caps of degree 2. The classification of caps and determine their completeness are hard tasks because of the number of points and lines in the projective space. So, the number of researchers focused on the theoretical research method to find a specific case gives complete caps. But no one has been able to give a full classification of the caps on specific projective spaces. In [4], Segre constructed complete $((3q + 2), 2)$-caps in $PG(3, 2^h)$. In [5], Alexander gave a generalization of Segre's construction of the complete caps in $PG(3, 2^h)$. In [6], Anbar *et. al.* used a geometrical object arc in the projective plane to produce complete caps of size $kq^{(N-2)/2}$ in affine spaces of dimension $N \equiv 0 \ (mod\ 4)$. A computer search has been used by a number of authors to find new caps and complete caps such as Alexander *et. al.* [7] used a computer search in the finite projective spaces $PG(n,q)$ for the spectrum of possible sizes k of complete k-caps. Also, in [8-10], a group action on the projective space of dimension three have been used to construct special types of caps for $q = 8,11,13,23$. As in [9], in this paper, we did an extension of the caps that have been constructed in [10] with respect to their size and degree when $q = 13$. All the results obtained from this research could be of interest to many researchers because of their relationship to linear coding, see[11-18], as well as their relationship to cryptography, see [19-21] and also the network system, see [22-24]. Regarding finite projective spaces with a dimension higher than three, some studies have appeared recently discussing some projective concepts, as in [25-31]. The algorithms in this paper have been implemented by the GAP program [32].

2. Preliminary Concepts

Definition 2.1 [1]: A (k,r)-cap in $PG(n \geq 3, q)$ is a set of k points such that no $r+1$ points are collinear, but at most r points of which lie in any line. Here, r is called a degree of the (k,r)-cap. The (k,r)-cap is called a complete cap if it is not contained in $(k+1,r)$-cap.

Definition 2.2 [2]: Let K be a cap of degree r, an i-secant of a K in $PG(n,q)$ is a line such that $|k \cap \pi| = i$. The number of i-secants of K denoted by τ_i.

Definition 2.3 [1]: Let Q be a point not on the (k,r)-cap, K. The number of i-secant of K passing through Q denoted by $\sigma_i(Q)$. The number $\sigma_r(Q)$ of r-secants is called the index $of\, Q$ with respect to K.

Definition 2.4 [2]: The set of all points of index i will be denoted by C_i and the cardinality of C_i is denoted by c_i. The sequence $(t_0, \dots, t_r)$ will represent the secant distribution and the sequences $(c_0, \dots, c_d)$ refer to the index distribution.

Definition 2.5 [2,3]: The group of projectivities of $PG(n,q)$ is called the projective general linear group, and is denoted by $PGL\,(n+1,q)$.

3. (k,r)-Caps in $PG(3,13)$

Let $F_{13} = \langle \tau \rangle = \{0,1,\tau,\tau^2,\tau^3,\tau^4,\tau^5,\tau^6,\tau^7,\tau^8,\tau^9,\tau^{10},\tau^{11}\}$ be the 13th-order Galois field, where τ is the primitive element of F_{13}. Let S be the non-singular 4×4 companion matrix over F_{13}

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tau^5 & \tau^3 & 0 & 1 \end{bmatrix}.$$

The cyclic subgroup $\langle S \rangle$ of $PGL(4,13)$ is of order $\theta_3(13) = 2380$. The projective space $PG(3,13)$ has $\theta_3(13) = 2380$ points and planes, 31110 lines, 14 points on each line and 183 lines passing through each point.

The number 2380 has 22 non-trivial divisors P which are: 2,4,5,7,10,14,17,20,28,34,35,68, 70,85,119,140,170,238,340,476,595,1190. The cyclic subgroups $\langle S^P \rangle$ of the cyclic group $\langle S \rangle$ are constructed in [10] and used to formed caps. The following is the summary of these caps results.

From the action of the subgroups $\langle S^P \rangle$ on $PG(3,13)$ deduced 22 equivalence classes (orbits) in $PG(3,13)$ formed caps are summarized below: Let χ_P be the first orbit from the action of $\langle S^P \rangle$ on $PG(3,13)$.

1. $\chi_2 = \{1+2k \mid k = 0,\dots,1189\}$ is incomplete (1190,14)-cap.
2. $\chi_4 = \{1+4k \mid k = 0,\dots,594\}$ is complete (595,7)-cap.
3. $\chi_5 = \{1+5k \mid k = 0,\dots,475\}$ is incompletes (476,14)-cap.
4. $\chi_7 = \{1+7k \mid k = 0,\dots,339\}$ is completes (340,4)-cap.
5. $\chi_{10} = \{1+10k \mid k = 0,\dots,237\}$ is incompletes (238,14)-cap.
6. $\chi_{14} = \{1+14k \mid k = 0,\dots,169\}$ is completes (170,2)-cap.

7. $\chi_{17} = \{1 + 17k|k = 0, \dots, 139\}$ is incompletes (140,14)-cap.
8. $\chi_{20} = \{1 + 20k|k = 0, \dots, 118\}$ is incompletes (119,7)-cap.
9. $\chi_{28} = \{1 + 28k|k = 0, \dots, 84\}$ is incompletes (85,2)-cap.
10. $\chi_{34} = \{1 + 34k|k = 0, \dots, 69\}$ is incompletes (70,14)-cap.
11. $\chi_{35} = \{1 + 35k|k = 0, \dots, 67\}$ is incompletes (68,3)-cap.
12. $\chi_{68} = \{1 + 68k|k = 0, \dots, 34\}$ is incompletes (35,7)-cap.
13. $\chi_{70} = \{1 + 70k|k = 0, \dots, 33\}$ is incompletes (34,2)-cap.
14. $\chi_{85} = \{1 + 85k|k = 0, \dots, 27\}$ is incompletes (28,14)-cap.
15. $\chi_{119} = \{1 + 119k|k = 0, \dots, 19\}$ is incompletes (20,2)-cap.
16. $\chi_{140} = \{1 + 140k|k = 0, \dots, 16\}$ is incompletes (17,2)-cap.
17. $\chi_{170} = \{1 + 170k|k = 0, \dots, 13\}$ is incompletes (14,14)-cap.
18. $\chi_{238} = \{1 + 238k|k = 0, \dots, 9\}$ is incompletes (10,2)-cap.
19. $\chi_{340} = \{1 + 340k|k = 0, \dots, 6\}$ is incompletes (7,7)-cap.
20. $\chi_{476} = \{1 + 476k|k = 0, \dots, 4\}$ is incompletes (5,2)-cap.
21. $\chi_{595} = \{1 + 595k|k = 0, \dots, 3\}$ is incompletes (4,2)-cap.
22. $\chi_{1190} = \{1,1191\}$ is incompletes (2,2)-cap.

2. Extension of Caps

An exhaustive computer search proves the following theorem:

Theorem: The caps $\chi_i, i = 4,7,14,20,28,35,68,70,119,140,238,340,476,595,1190$ can be extended by their sizes and degrees to complete caps.

Proof:

Let $\mathcal{D}_P^{p_i} = \chi_P \cup p_i$, where p_i is the set of addition points, $1 \le i \le 14$.

1. The line $\mathbf{I}(0,0,1,0,0,0)$ meets the cap χ_4 in 7 points, so, seven extension points have been added to χ_P in the following orders: $4, 144, 358, 470, 754, 755, 923$.

Let $p_1 = \{4\}, \mathcal{D}_4^{p_1} = \chi_4 \cup p_1$. The maximum number of the intersection points between $\chi_4^{p_1}$ and the lines of $PG(3,13)$ are eight, so $\chi_4^{p_1}$ is (596,8)-cap and c_i values are $c_0 = 1736$, $c_1 = 48$. Since $c_0 \neq 0$, then $\chi_4^{p_1}$ is incomplete. The (596,8)-cap will be complete when you add 196 points to it.

Let $p_2 = \{4, 144\}, \mathcal{D}_4^{p_2} = \chi_4 \cup p_2$. The maximum number of the intersection points between $\chi_4^{p_2}$ and the lines of $PG(3,13)$ are nine, so $\chi_4^{p_2}$ is (597,9)-cap and c_i values are c_0 =1778, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_4^{p_2}$ is incomplete. The (597,9)-cap will be complete when you add 413 points to it.

Let $p_3 = \{4, 144, 358\}, \mathcal{D}_4^{p_3} = \chi_4 \cup p_3$. The maximum number of the intersection points between $\chi_4^{p_3}$ and the lines of $PG(3,13)$ are ten, so $\chi_4^{p_3}$ is (598,10)-cap and c_i values are $c_0 = 1778$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_4^{p_3}$ is incomplete. The (598,10)-cap will be complete when you add 600 points to it.

Let $p_4 = \{4, 144, 358, 470\}, \mathcal{D}_4^{p_4} = \chi_4 \cup p_4$. The maximum number of the intersection points between $\chi_4^{p_4}$ and lines of $PG(3,13)$ are eleven, so $\chi_4^{p_4}$ is (599,11)-cap and c_i values are $c_0 = 1778$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_4^{p_4}$ is incomplete. The (599,11)-cap will be complete when you add 702 points to it.

Let $p_5 = \{4, 144, 358, 470, 754\}, \mathcal{D}_4^{p_5} = \chi_4 \cup p_5$. The maximum number of intersection points between $\chi_4^{p_5}$ and lines of $PG(3,13)$ are twelve, so $\chi_4^{p_5}$ is (600,12)-cap and c_i values are $c_0 =$

1778, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_4^{p_5}$ is incomplete. The (600,12)-cap will be complete when you add 1016 points to it.

Let $p_6 = \{4, 144,358,470,754,755\}, \mathcal{D}_4^{p_6} = \chi_4 \cup p_6$. The maximum number of the intersection points between $\chi_4^{p_6}$ and lines of $PG(3,13)$ are thirteen, so $\chi_4^{p_6}$ is (601,13)-cap and c_i values are $c_0 = 1778$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_4^{p_6}$ is incomplete. The (601,13)-cap will be complete when you add 1148 points to it.

Let $p_7 = \{4, 144,358,470,754,755,923\}, \mathcal{D}_4^{p_7} = \chi_4 \cup p_7$. The maximum number of the intersection points between $\chi_4^{p_7}$ and lines of $PG(3,13)$ are fourteen, so $\chi_4^{p_7}$ is (602,14)-cap and c_i values are $c_0 = 1778$. Since $c_0 \neq 0$, then $\chi_4^{p_7}$ is incomplete. The (602,14)-cap will be complete when you add 1778 points to it.

2. The line **I**(0,1,0,0,0,0) meets the cap χ_7 in 4 points, so we have ten extension points to be added to χ_P in the following orders: 3, 249, 488, 602, 1006, 1075, 1232, 1251, 1433, 2341.

Let $p_1 = \{3\}, \mathcal{D}_7^{p_1} = \chi_7 \cup p_1$. The maximum number of the intersection points between $\chi_7^{p_1}$ and the lines of $PG(3,13)$ are five, so $\chi_7^{p_1}$ is (341,5)-cap and c_i values are $c_0 = 1751$, $c_1 = 288$. Since $c_0 \neq 0$, then $\chi_7^{p_1}$ is incomplete. The (341,5)-cap will be complete when you add 66 points to it.

Let $p_2 = \{3,\ 249\}$, $\mathcal{D}_7^{p_2} = \chi_7 \cup p_2$. The maximum number of the intersection points between $\chi_7^{p_2}$ and the lines of $PG(3,13)$ are six, so $\chi_7^{p_2}$ is (342,6)-cap and c_i values are c_0 =1011, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_7^{p_2}$ is incomplete. The (342,6)-cap will be complete when you add 196 points to it.

Let $p_3 = \{3,\ 249,\ 488\}$, $\mathcal{D}_7^{p_3} = \chi_7 \cup p_3$. The maximum number of the intersection points between $\chi_7^{p_3}$ and the lines of $PG(3,13)$ are seven, so $\chi_7^{p_3}$ is (343,7)-cap and c_i values are $c_0 = 2030$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_7^{p_3}$ is incomplete. The (343,7)-cap will be complete when you add 335 points to it.

Let $p_4 = \{3,\ 249,\ 488,\ 602\}$, $\mathcal{D}_7^{p_4} = \chi_7 \cup p_4$. The maximum number of the intersection points between $\chi_7^{p_4}$ and the lines of $PG(3,13)$ are eight, so $\chi_7^{p_4}$ is (344,8)-cap and c_i values are $c_0 = 2030$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_7^{p_4}$ is incomplete. The (344,8)-cap will be complete when you add 463 points to it.

Let $p_5 = \{3,\ 249,\ 488,\ 602,\ 1006\}$, $\mathcal{D}_7^{p_5} = \chi_7 \cup p_5$. The maximum number of the intersection points between $\chi_7^{p_5}$ and the lines of $PG(3,13)$ are nine, so $\chi_7^{p_5}$ is (345,9)-cap and c_i values are $c_0 = 2030$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_7^{p_5}$ is incomplete. The (345,9)-cap will be complete when you add 685 points to it.

Let $p_6 = \{3,\ 249,\ 488,\ 602, 1006, 1075\}$, $\mathcal{D}_7^{p_6} = \chi_7 \cup p_6$. The maximum number of the intersection points between $\chi_7^{p_6}$ and the lines of $PG(3,13)$ are ten, so $\chi_7^{p_6}$ is (346,10)-cap and c_i values are $c_0 = 2030$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_7^{p_6}$ is incomplete. The (346,10)-cap will be complete when you add 816 points to it.

Let $p_7 = \{3,\ 249,\ 488,\ 602, 1006,\ 1075,\ 1232\}$, $\mathcal{D}_7^{p_7} = \chi_7 \cup p_7$. The maximum number of the intersection points between $\chi_7^{p_7}$ and the lines of $PG(3,13)$ are eleven, so $\chi_7^{p_7}$ is (347,11)-cap and c_i values are $c_0 = 2030$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_7^{p_7}$ is incomplete. The (347,11)-cap will be complete when you add 991 points to it.

Let $p_8 = \{3,\ 249,\ 488, 602, 1006, 1075, 1232, 1251\}$, $\mathcal{D}_7^{p_8} = \chi_7 \cup p_8$. The maximum number of the intersection points between $\chi_7^{p_8}$ and the lines of $PG(3,13)$ are twelve, so $\chi_7^{p_8}$ is (348,12)-cap and c_i values are $c_0 = 2030,\ c_1 = 2$. Since $c_0 \neq 0$, then $\chi_7^{p_8}$ is incomplete. The (348,12)-cap will be complete when you add 1180 points to it.

Let $p_9 = \{3,\ 249,\ 488,\ 602, 1006, 1075, 1232, 1251, 1433\}$, $\mathcal{D}_7^{p_9} = \chi_7 \cup p_9$. The maximum number of intersection points between $\chi_7^{p_9}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_7^{p_9}$ is (349,13)-cap and c_i values are $c_0 = 2030,\ c_1 = 1$. Since $c_0 \neq 0$, then $\chi_7^{p_9}$ is incomplete. The (349,13)-cap will be complete when you add 1395 points to it.

Let $p_{10} = \{3, 249,\ 488, 602, 1006,\ 1075,\ 1232,\ 1251, 1433,\ 2341\}$, $\mathcal{D}_7^{p_{10}} = \chi_7 \cup p_{10}$. The maximum number of the intersection points between $\chi_7^{p_{10}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_7^{p_{10}}$ is (350,14)-cap and c_i values are $c_0 = 2030$. Since $c_0 \neq 0$, then $\chi_7^{p_{10}}$ is incomplete. The (350,14)-cap will be complete when adding 2030 points to it.

3. The line $\mathbf{I}(0,1,0,0,0,0)$ meets the cap χ_{14} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order:

3,176,249,488,602,638,1006,1075,1232,1251,1433, 2341.

Let $p_1 = \{3\}$, $\mathcal{D}_{14}^{p_1} = \chi_{14} \cup p_1$. The maximum number of the intersection points between $\chi_{14}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{14}^{p_1}$ is (171,3)-cap and c_i values are $c_0 = 1351$, $c_1 = 858$. Since $c_0 \neq 0$, then $\chi_{14}^{p_1}$ is incomplete. The (171,3)-cap will be complete when you add 20 points to it.

Let $p_2 = \{3, 176\}$, $\mathcal{D}_{14}^{p_2} = \chi_{14} \cup p_2$. The maximum number of the intersection points between $\chi_{14}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{14}^{p_2}$ is (172,4)-cap and c_i values are c_0 =2198, c_1=10. Since $c_0 \neq 0$, then $\chi_{14}^{p_2}$ is incomplete. The (172,4)-cap will be complete when adding 119 points to it.

Let $p_3 = \{3, 176, 249\}$, $\mathcal{D}_{14}^{p_3} = \chi_{14} \cup p_3$. The maximum number of the intersection points between $\chi_{14}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{14}^{p_3}$ is (173,5)-cap and c_i values are $c_0 = 2198,\ c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{14}^{p_3}$ is incomplete. The (173,5)-cap will be complete when you add 254 points to it.

Let $p_4 = \{3, 176, 249, 488\}$, $\mathcal{D}_{14}^{p_4} = \chi_{14} \cup p_4$. The maximum number of the intersection points between $\chi_{14}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{14}^{p_4}$ is (174,6)-cap and c_i values are $c_0 = 2198,\ c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{14}^{p_4}$ is incomplete. The (174,6)-cap will be complete when you add 352 points to it.

Let $p_5 = \{3, 176, 249, 488, 602\}$, $\mathcal{D}_{14}^{p_5} = \chi_{14} \cup p_5$. The maximum number of the intersection points between $\chi_{14}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{14}^{p_5}$ is (175,7)-cap and c_i values are $c_0 = 2198,\ c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{14}^{p_5}$ is incomplete. The (175,7)-cap will be complete when you add 537 points to it.

Let $p_6 = \{3, 176, 249, 488, 602, 638\}$, $\mathcal{D}_{14}^{p_6} = \chi_{14} \cup p_6$. The maximum number of the intersection points between $\chi_{14}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{14}^{p_6}$ is (176,8)-cap and c_i values are $c_0 = 2198,\ c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{14}^{p_6}$ is incomplete. The (176,8)-cap will be complete when you add 629 points to it.

Let $p_7 = \{3,\ 176,\ 249,\ 488,\ 602,\ 638,\ 1006\}$, $\mathcal{D}_{14}^{p_7} = \chi_{14} \cup p_7$. The maximum number of the intersection points between $\chi_{14}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{14}^{p_7}$ is (177,9)-cap and c_i values are $c_0 = 2198$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{14}^{p_7}$ is incomplete. The (177,9)-cap will be complete when you add 818 points to it.

Let $p_8 = \{3, 176, 249, 488, 602, 638, 1006, 1075\}$, $\mathcal{D}_{14}^{p_8} = \chi_{14} \cup p_8$. The maximum number of the intersection points between $\chi_{14}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{14}^{p_8}$ is (178,10)-cap and c_i values are $c_0 = 2198$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{14}^{p_8}$ is incomplete. The (178,10)-cap will be complete when you add 970 points to it.

Let $p_9 = \{3, 176, 249, 488, 602, 638, 1006, 1075, 1232\}$, $\mathcal{D}_{14}^{p_9} = \chi_{14} \cup p_9$. The maximum number of the intersection points between $\chi_{14}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{14}^{p_9}$ is (179,11)-cap and c_i values are $c_0 = 2198$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{14}^{p_9}$ is incomplete. The (179,11)-cap will be complete when you add 1200 points to it.

Let $p_{10} = \{3, 176, 249, 488, 602, 638, 1006, 1075, 1232, 1251\}$, $\mathcal{D}_{14}^{p_{10}} = \chi_{14} \cup p_{10}$. The maximum number of the intersection points between $\chi_{14}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{14}^{p_{10}}$ is (180,12)-cap and c_i values are $c_0 = 2198$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{14}^{p_{10}}$ is incomplete. The (180,12)-cap will be complete when you add 1325 points to it.

Let $p_{11} = \{3, 176, 249, 488, 602, 638, 1006, 1075, 1232, 1251, 1433\}$, $\mathcal{D}_{14}^{p_{11}} = \chi_{14} \cup p_{11}$. The maximum number of the intersection points between $\chi_{14}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{14}^{p_{11}}$ is (181,13)-cap and c_i values are $c_0 = 2198$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{14}^{p_{11}}$ is incomplete. The (181,13)-cap will be complete when you add 1516 points to it.

Let $p_{12} = \{3, 176, 249, 488, 602, 638, 1006, 1075, 1232, 1251, 1433, 2341\}$.

Put $\mathcal{D}_{14}^{p_{12}} = \chi_{14} \cup p_{12}$. The maximum number of the intersection points between $\chi_{14}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{14}^{p_{12}}$ is (182,14)-cap and c_i values are $c_0 = 2198$. Since $c_0 \neq 0$, then $\chi_{14}^{p_{12}}$ is incomplete. The (182,14)-cap will be complete when you add 2198 points to it.

4. The line $\mathbf{I}(\tau^{11}, 1,0,0,0,0)$ meets the cap χ_{20} in 7 points, so we have seven extension points that can be added to $\chi_{P,}$ as in the following orders: $171, 511, 851, 1191, 1531, 1871, 2211$.

Let $p_1 = \{171\}$, $\mathcal{D}_{20}^{p_1} = \chi_{20} \cup p_1$. The maximum number of the intersection points between $\chi_{20}^{p_1}$ and the lines of $PG(3\ 13)$ are eight, so $\chi_{20}^{p_1}$ is (120,8)-cap and c_i values are $c_0 = 2254$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{20}^{p_1}$ is incomplete. The (120,8)-cap will be complete when you add 658 points to it.

Let $p_2 = \{171, 511\}$, $\mathcal{D}_{20}^{p_2} = \chi_{20} \cup p_2$. The maximum number of the intersection points between $\chi_{20}^{p_2}$ and the lines of $PG(3,13)$ are nine, so $\chi_{20}^{p_2}$ is (121,9)-cap and c_i values are $c_0 = 2254$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{20}^{p_2}$ is incomplete. The (121,9)-cap will be complete when you add 827 points to it.

Let $p_3 = \{171, 511, 851\}$, $\mathcal{D}_{20}^{p_3} = \chi_{20} \cup p_3$. The maximum number of the intersection points between $\chi_{20}^{p_3}$ and the lines of $PG(3,13)$ are ten, so $\chi_{20}^{p_3}$ is (122,10)-cap and c_i values are $c_0 = 2254$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{20}^{p_3}$ is incomplete. The (122,10)-cap will be complete when you add 1033 points to it.

Let $p_4 = \{171, 511, 851, 1191\}$, $\mathcal{D}_{20}^{p_4} = \chi_{20} \cup p_4$. The maximum number of intersection points between $\chi_{20}^{p_4}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{20}^{p_4}$ is (123,11)-cap and c_i values are $c_0 = 2254$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{20}^{p_4}$ is incomplete. The (123,11)-cap will be complete when you add 1250 points to it.

Let $p_5 = \{171, 511, 851, 1191, 1531\}$, $\mathcal{D}_{20}^{p_5} = \chi_{20} \cup p_5$. The maximum number of the intersection points between $\chi_{20}^{p_5}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{20}^{p_5}$ is (124,12)-cap and c_i values are $c_0 = 2254$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{20}^{p_5}$ is incomplete. The (124,12)-cap will be complete when you add 1383 points to it.

Let $p_6 = \{171, 511, 851, 1191, 1531, 1871\}$, $\mathcal{D}_{20}^{p_6} = \chi_{20} \cup p_6$. The maximum number of the intersection points between $\chi_{20}^{p_6}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{20}^{p_6}$ is (125,13)-cap and c_i values are $c_0 = 2254$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{20}^{p_6}$ is incomplete. The (125,13)-cap will be complete when you add 1555 points to it.

Let $p_7 = \{171, 511, 851, 1191, 1531, 1871, 2211\}$, $\mathcal{D}_{20}^{p_7} = \chi_{20} \cup p_7$. The maximum number of the intersection points between $\chi_{20}^{p_7}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{20}^{p_7}$ is (126,14)-cap and c_i values are $c_0 = 2254$. Since $c_0 \neq 0$, then $\chi_{20}^{p_7}$ is incomplete. The (126,14)-cap will be complete when you add 2254 points to it.

5. The line $\mathbf{I}(0,0,1,0,0,0)$ meets the cap χ_{28} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order:

4,144,358,470,593,754,755,923,1213,2105,2165, 2369.

Let $p_1 = \{4\}$, $\mathcal{D}_{28}^{p_1} = \chi_{28} \cup p_1$. The maximum number of the intersection points between $\chi_{28}^{p_1}$ and the lines of $PG(3,13)$ are three so $\chi_{28}^{p_1}$ is (86,3)-cap and c_i values are $c_0 = 2096$, $c_1 = 198$. Since $c_0 \neq 0$, then $\chi_{28}^{p_1}$ is incomplete. The (86,3)-cap will be complete when you add 74 points to it.

Let $p_2 = \{4, 144\}$, $\mathcal{D}_{28}^{p_2} = \chi_{28} \cup p_2$. The maximum number of the intersection points between $\chi_{28}^{p_2}$ and the lines of $PG(3,13)$ are four so $\chi_{28}^{p_2}$ is (87,4)-cap and c_i values are c_0 =2283, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{28}^{p_2}$ is incomplete. The (87,4)-cap will be complete when you add 176 points to it.

Let $p_3 = \{4, 144, 358\}$, $\mathcal{D}_{28}^{p_3} = \chi_{28} \cup p_3$. The maximum number of the intersection points between $\chi_{28}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{28}^{p_3}$ is (88,5)-cap and c_i values are $c_0 = 2283$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{28}^{p_3}$ is incomplete. The (88,5)-cap will be complete when you add 319 points to it.

Let $p_4 = \{4, 144, 358, 470\}$. Put $\mathcal{D}_{28}^{p_4} = \chi_{28} \cup p_4$. The maximum number of the intersection points between $\chi_{28}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{28}^{p_4}$ is (89,6)-cap and c_i values are $c_0 = 2283$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{28}^{p_4}$ is incomplete. The (89,6)-cap will be complete when adding 433 points to it. Let $p_5 = \{4, 144, 358, 470, 593\}$, $\mathcal{D}_{28}^{p_5} = \chi_{28} \cup p_5$. The maximum number of the intersection points between $\chi_{28}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{28}^{p_5}$ is (90,7)-cap and c_i values are $c_0 = 2283$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{28}^{p_5}$ is incomplete. The (90,7)-cap will be complete when you add 623 points to it.

Let $p_6 = \{4, 144, 358, 470, 593, 754\}$, $\mathcal{D}_{28}^{p_6} = \chi_{28} \cup p_6$. The maximum number of the intersection points between $\chi_{28}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{28}^{p_6}$ is (91,8)-cap and c_i values are $c_0 = 2283$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{28}^{p_6}$ is incomplete. The (91,8)-cap will be complete when you add 671 points to it.

Let $p_7 = \{4, 144, 358, 470, 593, 754, 755\}$, $\mathcal{D}_{28}^{p_7} = \chi_{28} \cup p_7$. The maximum number of the intersection points between $\chi_{28}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{28}^{p_7}$ is (92,9)-cap and c_i values are $c_0 = 2283$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{28}^{p_7}$ is incomplete. The (92,9)-cap will be complete when you add 892 points to it.

Let $p_8 = \{4, 144, 358, 470, 593, 754, 755, 923\}$, $\mathcal{D}_{28}^{p_8} = \chi_{28} \cup p_8$. The maximum number of the intersection points between $\chi_{28}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{28}^{p_8}$ is (93,10)-cap and

c_i values are $c_0 = 2283,\ c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{28}^{p_8}$ is incomplete. The (93,10)-cap will be complete when you add 1047 points to it.

Let $p_9 = \{4, 144, 358, 470, 593, 754, 755, 923, 1213, \}$. Put $\mathcal{D}_{28}^{p_9} = \chi_{28} \cup p_9$. The maximum number of the intersection points between $\chi_{28}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{28}^{p_9}$ is $(94,11)$-cap and c_i values are $c_0 = 2283,\ c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{28}^{p_9}$ is incomplete. The $(94,11)$-cap will be complete when you add 1299 points to it.

Let $p_{10} = \{ 4, 144,\ 358, 470, 593, 754, 755, 923,\ 1213, 2105\}$. Put $\mathcal{D}_{28}^{p_{10}} = \chi_{28} \cup p_{10}$. The maximum number of the intersection points between $\chi_{28}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{28}^{p_{10}}$ is $(95,12)$-cap and c_i values are $c_0 = 2283,\ c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{28}^{p_{10}}$ is incomplete. The $(95,12)$-cap will be complete when you add 1367 points to it.

Let $p_{11} = \{4, 144, 358, 470, 593, 754, 755, 923, 1213, 2105, 2165\}$, $\mathcal{D}_{28}^{p_{11}} = \chi_{28} \cup p_{11}$. The maximum number of the intersection points between $\chi_{28}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{28}^{p_{11}}$ is $(96,13)$-cap and c_i values are $c_0 = 2283,\ c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{28}^{p_{11}}$ is incomplete. The $(96,13)$-cap will be complete when you add 1577 points to it.

Let $p_{12} = \{4, 144, 358, 470, 593, 754, 755, 923, 1213, 2105, 2165, 2369\}$, $\mathcal{D}_{28}^{p_{12}} = \chi_{28} \cup p_{12}$. The maximum number of the intersection points between $\chi_{28}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{28}^{p_{12}}$ is $(97,14)$-cap and c_i values are $c_0 = 2283$. Since $c_0 \neq 0$, then $\chi_{28}^{p_{12}}$ is incomplete. The $(97,14)$-cap will be complete when you add 2283 points to it.

6. The line $\mathbf{I}(\tau^9, \tau^{10}, 1,0,0,0)$ meets the cap χ_{35} in 3 points, so we have eleven extension points that can be added to χ_P, as in the following order:

356,565,718,1049,1054,1314,1377,1756, 1818,2253,2331.

Let $p_1 = \{356\}$, $\mathcal{D}_{35}^{p_1} = \chi_{35} \cup p_1$. The maximum number of the intersection points between $\chi_{35}^{p_1}$ and the lines of $PG(3,13)$ are four, so $\chi_{35}^{p_1}$ is $(69,4)$-cap and c_i values are $c_0 = 2291,\ c_1 = 20$. Since $c_0 \neq 0$, then $\chi_{35}^{p_1}$ is incomplete. The $(69,4)$-cap will be complete when you add 189 points to it.

Let $p_2 = \{356, 565\}$, $\mathcal{D}_{35}^{p_2} = \chi_{35} \cup p_2$. The maximum number of the intersection points between $\chi_{35}^{p_2}$ and the lines of $PG(3,13)$ are five, so $\chi_{35}^{p_2}$ is $(70,5)$-cap and c_i values are $c_0 = 2301$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{35}^{p_2}$ is incomplete. The $(70,5)$-cap will be complete when you add 334 points to it.

Let $p_3 = \{356, 565, 718\}$, $\mathcal{D}_{35}^{p_3} = \chi_{35} \cup p_3$. The maximum number of the intersection points between $\chi_{35}^{p_3}$ and the lines of $PG(3,13)$ are six, so $\chi_{35}^{p_3}$ is $(71,6)$-cap and c_i values are $c_0 = 2301,\ c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{35}^{p_3}$ is incomplete. The $(71,6)$-cap will be complete when you add 486 points to it.

Let $p_4 = \{356, 565, 718, 1049\}$, $\mathcal{D}_{35}^{p_4} = \chi_{35} \cup p_4$. The maximum number of the intersection points between $\chi_{35}^{p_4}$ and the lines of $PG(3,13)$ are seven, so $\chi_{35}^{p_4}$ is $(72,7)$-cap and c_i values are $c_0 = 2301,\ c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{35}^{p_4}$ is incomplete. The $(72,7)$-cap will be complete when you add 627 points to it.

Let $p_5 = \{356, 565, 718, 1049, 1054\}$, $\mathcal{D}_{35}^{p_5} = \chi_{35} \cup p_5$. The maximum number of the intersection points between $\chi_{35}^{p_5}$ and the lines of $PG(3,13)$ are eight, so $\chi_{35}^{p_5}$ is $(73,8)$-cap and c_i values are $c_0 = 2301,\ c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{35}^{p_5}$ is incomplete. The $(73,8)$-cap will be complete when you add 692 points to it.

Let $p_6 = \{356, 565, 718, 1049, 1054, 1314\}$, $\mathcal{D}_{35}^{p_6} = \chi_{35} \cup p_6$. The maximum number of the intersection points between $\chi_{35}^{p_6}$ and the lines of $PG(3,13)$ are nine, so $\chi_{35}^{p_6}$ is (74,9)-cap and c_i values are $c_0 = 2301$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{35}^{p_6}$ is incomplete. The (74,9)-cap will be complete when you add 876 points to it.

Let $p_7 = \{356, 565, 718, 1049, 1054, 1314, 1377\}$, $\mathcal{D}_{35}^{p_7} = \chi_{35} \cup p_7$. The maximum number of the intersection points between $\chi_{35}^{p_7}$ and the lines of $PG(3,13)$ are ten, so $\chi_{35}^{p_7}$ is (75,10)-cap and c_i values are $c_0 = 2301$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{35}^{p_7}$ is incomplete. The (75,10)-cap will be complete when you add 1037 points to it.

Let $p_8 = \{356, 565, 718, 1049, 1054, 1314, 1377, 1756\}$, $\mathcal{D}_{35}^{p_8} = \chi_{35} \cup p_8$. The maximum number of the intersection points between $\chi_{35}^{p_8}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{35}^{p_8}$ is (76,11)-cap and c_i values are $c_0 = 2301$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{35}^{p_8}$ is incomplete. The (76,11)-cap will be complete when you add 1312 points to it.

Let $p_9 = \{356, 565, 718, 1049, 1054, 1314, 1377, 1756, 1818\}$. Put $\mathcal{D}_{35}^{p_9} = \chi_{35} \cup p_9$. The maximum number of intersection points between $\chi_{35}^{p_9}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{35}^{p_9}$ is (77,12)-cap and c_i values are $c_0 = 2301$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{35}^{p_9}$ is incomplete. The (77,12)-cap will be complete when you add 1401 points to it.

Let $p_{10} = \{356, 565, 718, 1049, 1054, 1314, 1377, 1756, 1818, 2253\}$, $\mathcal{D}_{35}^{p_{10}} = \chi_{35} \cup p_{10}$. The maximum number of the intersection points between $\chi_{35}^{p_{10}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{35}^{p_{10}}$ is (78,13)-cap and c_i values are $c_0 = 2301$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{35}^{p_{10}}$ is incomplete. The (78,13)-cap will be complete when you add 1591 points to it.

Let $p_{11} = \{356, 565, 718, 1049, 1054, 1314, 1377, 1756, 1818, 2253, 2331\}$.
Put $\mathcal{D}_{35}^{p_{11}} = \chi_{35} \cup p_{11}$. The maximum number of the intersection points between $\chi_{35}^{p_{11}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{35}^{p_{11}}$ is (79,14)-cap and c_i values are $c_0 = 2301$. Since $c_0 \neq 0$, then $\chi_{35}^{p_{11}}$ is incomplete. The (79,14)-cap will be complete when you add 2301 points to it.

7. The line $\mathbf{l}(\tau^{11}, 1,0,0,0,0)$ meets the cap χ_{68} in 7 points, so we have seven extension points that can be added to $\chi_{P,}$ as in the following order: $171, 511, 851, 1191, 1531, 1871, 2211$.

Let $p_1 = \{171\}$, $\mathcal{D}_{68}^{p_1} = \chi_{68} \cup p_1$. The maximum number of the intersection points between $\chi_{68}^{p_1}$ and the lines of $PG(3,13)$ are eight, so $\chi_{68}^{p_1}$ is (36,8)-cap and c_i values are $c_0 = 2338$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{68}^{p_1}$ is incomplete. The (36,8)-cap will be complete when you add 709 points to it.

Let $p_2 = \{171, 511\}$, $\mathcal{D}_{68}^{p_2} = \chi_{68} \cup p_2$. The maximum number of the intersection points between $\chi_{68}^{p_2}$ and the lines of $PG(3,13)$ are nine, so $\chi_{68}^{p_2}$ is (37,9)-cap and c_i values are $c_0 = 2338$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{68}^{p_2}$ is incomplete. The (37,9)-cap will be complete when you add 901 points to it.

Let $p_3 = \{171, 511, 851\}$, $\mathcal{D}_{68}^{p_3} = \chi_{68} \cup p_3$. The maximum number of the intersection points between $\chi_{68}^{p_3}$ and the lines of $PG(3,13)$ are ten, so $\chi_{68}^{p_3}$ is (38,10)-cap and c_i values are $c_0 = 2338$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{68}^{p_3}$ is incomplete. The (38,10)-cap will be complete when you add 1065 points to it.

Let $p_4 = \{171, 511, 851, 1191\}$, $\mathcal{D}_{68}^{p_4} = \chi_{68} \cup p_4$. The maximum number of the intersection points between $\chi_{68}^{p_4}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{68}^{p_4}$ is (39,11)-cap and c_i values are $c_0 = 2338$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{68}^{p_4}$ is incomplete. The (39,11)-cap will be complete when you add 1395 points to it.

Let $p_5 = \{171, 511, 851, 1191, 1531\}$, $\mathcal{D}_{68}^{p_5} = \chi_{68} \cup p_5$. The maximum number of the intersection points between $\chi_{68}^{p_5}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{68}^{p_5}$ is (40,12)-cap and c_i values are $c_0 = 2338$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{68}^{p_5}$ is incomplete. The (40,12)-cap will be complete when you add 1427 points to it.

Let $p_6 = \{171, 511, 851, 1191, 1531, 1871\}$, $\mathcal{D}_{68}^{p_6} = \chi_{68} \cup p_6$. The maximum number of the intersection points between $\chi_{68}^{p_6}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{68}^{p_6}$ is (41,13)-cap and c_i values are $c_0 = 2338$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{68}^{p_6}$ is incomplete. The (41,13)-cap will be complete when you add 1615 points to it.

Let $p_7 = \{171, 511, 851, 1191, 1531, 1871, 2211\}$, $\mathcal{D}_{68}^{p_7} = \chi_{68} \cup p_7$. The maximum number of the intersection points between $\chi_{68}^{p_7}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{68}^{p_7}$ is (42,14)-cap and c_i values are $c_0 = 2338$. Since $c_0 \neq 0$, then $\chi_{68}^{p_7}$ is incomplete. The (42,14)-cap will be complete when you add 2338 points to it.

8. The line $\mathbf{I}(\tau^5, \tau^5, 1,0,0,0)$ meets the cap χ_{70} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order: 20,202,772,1110,1150,1152,1325,1398, 1637,1787,2155,2224.

Let $p_1 = \{20\}$, $\mathcal{D}_{70}^{p_1} = \chi_{70} \cup p_1$. The maximum number of the intersection points between $\chi_{70}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{70}^{p_1}$ is (35,3)-cap and c_i values are $c_0 = 2323$, $c_1 = 22$. Since $c_0 \neq 0$, then $\chi_{70}^{p_1}$ is incomplete. The (35,3)-cap will be complete when you add 109 points to it.

Let $p_2 = \{20, 202,\}$, $\mathcal{D}_{70}^{p_2} = \chi_{70} \cup p_2$. The maximum number of the intersection points between $\chi_{70}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{70}^{p_2}$ is (36,4)-cap and c_i values are $c_0 = 2334$, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{70}^{p_2}$ is incomplete. The (36,4)-cap will be complete when you add 226 points to it.

Let $p_3 = \{20, 202, 772\}$, $\mathcal{D}_{70}^{p_3} = \chi_{70} \cup p_3$. The maximum number of the intersection points between $\chi_{70}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{70}^{p_3}$ is (37,5)-cap and c_i values are $c_0 = 2334$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{70}^{p_3}$ is incomplete. The (37,5)-cap will be complete when you add 368 points to it.

Let $p_4 = \{20, 202, 772, 1110\}$, $\mathcal{D}_{70}^{p_4} = \chi_{70} \cup p_4$. The maximum number of the intersection points between $\chi_{70}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{70}^{p_4}$ is (38,6)-cap and c_i values are $c_0 = 2334$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{70}^{p_4}$ is incomplete. The (38,6)-cap will be complete when you add 468 points to it.

Let $p_5 = \{20, 202, 772, 1110, 1150\}$, $\mathcal{D}_{70}^{p_5} = \chi_{70} \cup p_5$. The maximum number of the intersection points between $\chi_{70}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{70}^{p_5}$ is (39,7)-cap and c_i values are $c_0 = 2334$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{70}^{p_5}$ is incomplete. The (39,7)-cap will be complete when you add 639 points to it.

Let $p_6 = \{20, 202, 772, 1110, 1150, 1152\}$, $\mathcal{D}_{70}^{p_6} = \chi_{70} \cup p_6$. The maximum number of the intersection points between $\chi_{70}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{70}^{p_6}$ is (40,8)-cap and c_i values are $c_0 = 2334$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{70}^{p_6}$ is incomplete. The (40,8)-cap will be complete when you add 697 points to it.

Let $p_7 = \{20, 202, 772, 1110, 1150, 1152, 1325\}$, $\mathcal{D}_{70}^{p_7} = \chi_{70} \cup p_7$. The maximum number of the intersection points between $\chi_{70}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{70}^{p_7}$ is (41,9)-cap and c_i values are $c_0 = 2334$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{70}^{p_7}$ is incomplete. The (41,9)-cap will be complete when you add 989 points to it.

Let $p_8 = \{20, 202, 772, 1110, 1150, 1152, 1325, 1398\}$, $\mathcal{D}_{70}^{p_8} = \chi_{70} \cup p_8$. The maximum number of the intersection points between $\chi_{70}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{70}^{p_8}$ is (42,10)-cap and c_i values are $c_0 = 2334$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{70}^{p_8}$ is incomplete. The (42,10)-cap will be complete when you add 1056 points to it.

Let $p_9 = \{20, 202, 772, 1110, 1150, 1152, 1325, 1398, 1637\}$, $\mathcal{D}_{70}^{p_9} = \chi_{70} \cup p_9$. The maximum number of the intersection points between $\chi_{70}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{70}^{p_9}$ is (43,11)-cap and c_i values are $c_0 = 2334$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{70}^{p_9}$ is incomplete. The (43,11)-cap will be complete when you add 1330 points to it.

Let $p_{10} = \{20, 202, 772, 1110, 1150, 1152, 1325, 1398, 1637, 1787\}$, $\mathcal{D}_{70}^{p_{10}} = \chi_{70} \cup p_{10}$. The maximum number of the intersection points between $\chi_{70}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{70}^{p_{10}}$ is (44,12)-cap and c_i values are $c_0 = 2334$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{70}^{p_{10}}$ is incomplete. The (44,12)-cap will be complete when you add 1421 points to it.

Let $p_{11} = \{20, 202, 772, 1110, 1150, 1152, 1325, 1398, 1637, 1787, 2155\}$. Put $\mathcal{D}_{70}^{p_{11}} = \chi_{70} \cup p_{11}$. The maximum number of the intersection points between $\chi_{70}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{70}^{p_{11}}$ is (45,13)-cap and c_i values are $c_0 = 2334$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{70}^{p_{11}}$ is incomplete. The (45,13)-cap will be complete when you add 1612 points to it.

Let $p_{12} = \{20, 202, 772, 1110, 1150, 1152, 1325, 1398, 1637, 1787, 2155, 2224\}$. Put $\mathcal{D}_{70}^{p_{12}} = \chi_{70} \cup p_{12}$. The maximum number of the intersection points between $\chi_{70}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{70}^{p_{12}}$ is (46,14)-cap and c_i values are $c_0 = 2334$. Since $c_0 \neq 0$, then $\chi_{70}^{p_{12}}$ is incomplete. The (46,14)-cap will be complete when you add 2334 points to it.

9. The line $\mathbf{I}(0,0,1,0,0,0)$ meets the cap χ_{119} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order: $4, 144, 470, 593, 754, 755, 923, 1213, 2017, 2105, 2165, 2369$.

Let $p_1 = \{4\}$, $\mathcal{D}_{119}^{p_1} = \chi_{119} \cup p_1$. The maximum number of the intersection points between $\chi_{119}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{119}^{p_1}$ is (21,3)-cap and c_i values are $c_0 = 2337$, $c_1 = 22$. Since $c_0 \neq 0$, then $\chi_{119}^{p_1}$ is incomplete. The (21,3)-cap will be complete when you add 117 points to it.

Let $p_2 = \{4, 144\}$, $\mathcal{D}_{119}^{p_2} = \chi_{119} \cup p_2$. The maximum number of the intersection points between $\chi_{119}^{p_2}$ and the lines of $PG(3,13)$ four, so $\chi_{119}^{p_2}$ is (22,4)-cap and c_i values are $c_0 = 2348$, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{119}^{p_2}$ is incomplete. The (22,4)-cap will be complete when you add 241 points to it.

Let $p_3 = \{4, 144, 470\}$, $\mathcal{D}_{119}^{p_3} = \chi_{119} \cup p_3$. The maximum number of the intersection points between $\chi_{119}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{119}^{p_3}$ is (23,5)-cap and c_i values are $c_0 = 2348$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{119}^{p_3}$ is incomplete. The (23,5)-cap will be complete when you add 392 points to it.

Let $p_4 = \{4, 144, 470, 593\}$, $\mathcal{D}_{119}^{p_4} = \chi_{119} \cup p_4$. The maximum number of the intersection points between $\chi_{119}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{119}^{p_4}$ is (24,6)-cap and c_i values are $c_0 = 2348$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{119}^{p_4}$ is incomplete. The (24,6)-cap will be complete when you add 507 points to it.

Let $p_5 = \{4, 144, 470, 593, 754\}$, $\mathcal{D}_{119}^{p_5} = \chi_{119} \cup p_5$. The maximum number of the intersection points between $\chi_{119}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{119}^{p_5}$ is (25,7)-cap and c_i values are $c_0 = 2348$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{119}^{p_5}$ is incomplete. The (25,7)-cap will be complete when you add 639 points to it.

Let $p_6 = \{4, 144, 470, 593, 754, 755\}$, $\mathcal{D}_{119}^{p_6} = \chi_{119} \cup p_6$. The maximum number of the intersection points between $\chi_{119}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{119}^{p_6}$ is (26,8)-cap and c_i values are $c_0 = 2348$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{119}^{p_6}$ is incomplete. The (26,8)-cap will be complete when you add 734 points to it.

Let $p_7 = \{4, 144, 470, 593, 754, 755, 923\}$, $\mathcal{D}_{119}^{p_7} = \chi_{119} \cup p_7$. The maximum number of the intersection points between $\chi_{119}^{4,144,470,593,754,755,923}$ and the lines of $PG(3,13)$ are nine, so $\chi_{119}^{p_7}$ is (27,9)-cap and c_i values are $c_0 = 2348$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{119}^{p_7}$ is incomplete. The (27,9)-cap will be complete when you add 891 points to it.

Let $p_8 = \{4, 144, 470, 593, 754, 755, 923, 1213\}$, $\mathcal{D}_{119}^{p_8} = \chi_{119} \cup p_8$. The maximum number of the intersection points between $\chi_{119}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{119}^{p_8}$ is (28,10)-cap and c_i values are $c_0 = 2348$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{119}^{p_8}$ is incomplete. The (28,10)-cap will be complete when you add 1066 points to it.

Let $p_9 = \{4, 144, 470, 593, 754, 755, 923, 1213, 2017\}$, $\mathcal{D}_{119}^{p_9} = \chi_{119} \cup p_9$. The maximum number of the intersection points between $\chi_{119}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{119}^{p_9}$ is (29,11)-cap and c_i values are $c_0 = 2348$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{119}^{p_9}$ is incomplete. The (29,11)-cap will be complete when you add 1335 points to it.

Let $p_{10} = \{4, 144, 470, 593, 754, 755, 923, 1213, 2017, 2105\}$, $\mathcal{D}_{119}^{p_{10}} = \chi_{119} \cup p_{10}$. The maximum number of the intersection points between $\chi_{119}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{119}^{p_{10}}$ is (30,12)-cap and c_i values are $c_0 = 2348$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{119}^{p_{10}}$ is incomplete. The (30,12)-cap will be complete when you add 1444 points to it.

Let $p_{11} = \{4, 144, 470, 593, 754, 755, 923, 1213, 2017, 2105, 2165\}$, $\mathcal{D}_{119}^{p_{11}} = \chi_{119} \cup p_{11}$. The maximum number of the intersection points between $\chi_{119}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{119}^{p_{11}}$ is (31,13)-cap and c_i values are $c_0 = 2348$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{119}^{p_{11}}$ is incomplete. The (31,13)-cap will be complete when you add 1623 points to it.

Let $p_{12} = \{4, 144, 470, 593, 754, 755, 923, 1213, 2017, 2105, 2165, 2369\}$.

Put $\mathcal{D}_{119}^{p_{12}} = \chi_{119} \cup p_{12}$. The maximum number of the intersection points between $\chi_{119}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{119}^{p_{12}}$ is (32,14)-cap and c_i values are $c_0 = 2348$. Since $c_0 \neq 0$, then $\chi_{119}^{p_{12}}$ is incomplete. The (32,14)-cap will be complete when you add 2348 points to it.

10. The line $\mathbf{I}(\tau^9, \tau^{10}, 1,0,0,0)$ meets the cap χ_{140} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order:

36,356,565,718,1049,1054,1314,1377, 1756,1818,2253,2331.

Let $p_1 = \{36\}$, $\mathcal{D}_{140}^{p_1} = \chi_{140} \cup p_1$. The maximum number of the intersection points between $\chi_{140}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{140}^{p_1}$ is (18,3)-cap and c_i values are $c_0 = 2340$, $c_1 =$

22. Since $c_0 \neq 0$, then $\chi_{140}^{p_1}$ is incomplete. The (18,3)-cap will be complete when you add 128 points to it.

Let $p_2 = \{36, 356\}$, $\mathcal{D}_{140}^{p_2} = \chi_{140} \cup p_2$. The maximum number of the intersection points between $\chi_{140}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{140}^{p_2}$ is (19,4)-cap and c_i values are $c_0 = 2351, c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{140}^{p_2}$ is incomplete. The (19,4)-cap will be complete when you add 248 points to it.

Let $p_3 = \{36, 356, 565\}$, $\mathcal{D}_{140}^{p_3} = \chi_{140} \cup p_3$. The maximum number of the intersection points between $\chi_{140}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{140}^{p_3}$ is (20,5)-cap and c_i values are $c_0 = 2351, c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{140}^{p_3}$ is incomplete. The (20,5)-cap will be complete when you add 388 points to it.

Let $p_4 = \{36, 356, 565, 718\}$, $\mathcal{D}_{140}^{p_4} = \chi_{140} \cup p_4$. The maximum number of the intersection points between $\chi_{140}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{140}^{p_4}$ is (21,6)-cap and c_i values are $c_0 = 2351, c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{140}^{p_4}$ is incomplete. The (21,6)-cap will be complete when you add 489 points to it.

Let $p_5 = \{36, 356, 565, 718, 1049\}$, $\mathcal{D}_{140}^{p_5} = \chi_{140} \cup p_5$. The maximum number of the intersection points between $\chi_{140}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{140}^{p_5}$ is (22,7)-cap and c_i values are $c_0 = 2351, c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{140}^{p_5}$ is incomplete. The (22,7)-cap will be complete when you add 647 points to it.

Let $p_6 = \{36, 356, 565, 718, 1049, 1054\}$, $\mathcal{D}_{140}^{p_6} = \chi_{140} \cup p_6$. The maximum number of the intersection points between $\chi_{140}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{140}^{p_6}$ is (23,8)-cap and c_i values are $c_0 = 2351, c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{140}^{p_6}$ is incomplete. The (23,8)-cap will be complete when you add 707 points to it.

Let $p_7 = \{36, 356, 565, 718, 1049, 1054, 1314\}$, $\mathcal{D}_{140}^{p_7} = \chi_{140} \cup p_7$. The maximum number of the intersection points between $\chi_{140}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{140}^{p_7}$ is (24,9)-cap and c_i values are $c_0 = 2351, c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{140}^{p_7}$ is incomplete. The (24,9)-cap will be complete when you add 904 points to it.

Let $p_8 = \{36, 356, 565, 718, 1049, 1054, 1314, 1377\}$, $\mathcal{D}_{140}^{p_8} = \chi_{140} \cup p_8$. The maximum number of the intersection points between $\chi_{140}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{140}^{p_8}$ is (25,10)-cap and c_i values are $c_0 = 2351, c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{140}^{p_8}$ is incomplete. The (25,10)-cap will be complete when you add 1062 points to it.

Let $p_9 = \{36, 356, 565, 718, 1049, 1054, 1314, 1377, 1756\}$. Put $\mathcal{D}_{140}^{p_9} = \chi_{140} \cup p_9$. The maximum number of the intersection points between $\chi_{140}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{140}^{p_9}$ is (26,11)-cap and c_i values are $c_0 = 2351, c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{140}^{p_9}$ is incomplete. The (26,11)-cap will be complete when you add 1340 points to it.

Let $p_{10} = \{36, 356, 565, 718, 1049, 1054, 1314, 1377, 1756, 1818\}$, $\mathcal{D}_{140}^{p_{10}} = \chi_{140} \cup p_{10}$. The maximum number of the intersection points between $\chi_{140}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{140}^{p_{10}}$ is (27,12)-cap and c_i values are $c_0 = 2351, c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{140}^{p_{10}}$ is incomplete. The (27,12)-cap will be complete when you add 1383 points to it.

Let $p_{11} = \{36, 356, 565, 718, 1049, 1054, 1314, 1377, 1756, 1818, 2253\}$.

Put $\mathcal{D}_{140}^{p_{11}} = \chi_{140} \cup p_{11}$. The maximum number of the intersection points between $\chi_{140}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{140}^{p_{11}}$ is (28,13)-cap and c_i values are $c_0 = 2351, c_1 = 1$.

Since $c_0 \neq 0$, then $\chi_{140}^{p_{11}}$ is incomplete. The (28,13)-cap will be complete when you add 1625 points to it.

Let $p_{12} = \{36, 356, 565, 718, 1049, 1054, 1314, 1377, 1756, 1818, 2253, 2331\}$.

Put $\mathcal{D}_{140}^{p_{12}} = \chi_{140} \cup p_{12}$. The maximum number of the intersection points between $\chi_{140}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{140}^{p_{12}}$ is (29,14)-cap and c_i values are $c_0 = 2351$. Since $c_0 \neq 0$, then $\chi_{140}^{p_{12}}$ is incomplete. The (29,14)-cap will be complete when you add 2351 points to it.

11. The line $\mathbf{I}(\tau^7, \tau^9, 1,0,0,0)$ meets the cap χ_{238} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order: $24, 315, 453, 794, 860, 870, 968, 1324, 1534, 1658, 1895, 2373$.

Let $p_1 = \{24\}$, $\mathcal{D}_{238}^{p_1} = \chi_{238} \cup p_1$. The maximum number of the intersection points between $\chi_{238}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{238}^{p_1}$ is (11,3)-cap and c_i values are $c_0 = 2358$, $c_1 = 11$. Since $c_0 \neq 0$, then $\chi_{238}^{p_1}$ is incomplete. The (11,3)-cap will be complete when you add 119 points to it.

Let $p_2 = \{24, 315\}$, $\mathcal{D}_{238}^{p_2} = \chi_{238} \cup p_2$. The maximum number of the intersection points between $\chi_{238}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{238}^{p_2}$ is (12,4)-cap and c_i values are $c_0 = 2358$, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{238}^{p_2}$ is incomplete. The (12,4)-cap will be complete when you add 243 points to it.

Let $p_3 = \{24, 315, 453\}$, $\mathcal{D}_{238}^{p_3} = \chi_{238} \cup p_3$. The maximum number of the intersection points between $\chi_{238}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{238}^{p_3}$ is (13,5)-cap and c_i values are $c_0 = 2358$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{238}^{p_3}$ is incomplete. The (13,5)-cap will be complete when you add 393 points to it.

Let $p_4 = \{24, 315, 453, 794\}$, $\mathcal{D}_{238}^{p_4} = \chi_{238} \cup p_4$. The maximum number of the intersection points between $\chi_{238}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{238}^{p_4}$ is (14,6)-cap and c_i values are $c_0 = 2358$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{238}^{p_4}$ is incomplete. The (14,6)-cap will be complete when you add 509 points to it.

Let $p_5 = \{24, 315, 453, 794, 860\}$, $\mathcal{D}_{238}^{p_5} = \chi_{238} \cup p_5$. The maximum number of the intersection points between $\chi_{238}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{238}^{p_5}$ is (15,7)-cap and c_i values are $c_0 = 2358$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{238}^{p_5}$ is incomplete. The (15,7)-cap will be complete when you add 643 points to it.

Let $p_6 = \{24,\ 315,\ 453,\ 794,\ 860,\ 870\}$, $\mathcal{D}_{238}^{p_6} = \chi_{238} \cup p_6$. The maximum number of the intersection points between $\chi_{238}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{238}^{p_6}$ is (16,8)-cap and c_i values are $c_0 = 2358$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{238}^{p_6}$ is incomplete. The (16,8)-cap will be complete when you add 707 points to it.

Let $p_7 = \{24, 315, 453, 794, 860, 870, 968\}$, $\mathcal{D}_{238}^{p_7} = \chi_{238} \cup p_7$. The maximum number of the intersection points between $\chi_{238}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{238}^{p_7}$ is (17,9)-cap and c_i values are $c_0 = 2358$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{238}^{p_7}$ is incomplete. The (17,9)-cap will be complete when you add 896 points to it.

Let $p_8 = \{24, 315, 453, 794, 860, 870, 968, 1324\}$, $\mathcal{D}_{238}^{p_8} = \chi_{238} \cup p_8$. The maximum number of the intersection points between $\chi_{238}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{238}^{p_8}$ is (18,10)-cap and c_i values are $c_0 = 2358$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{238}^{p_8}$ is incomplete. The (18,10)-cap will be complete when you add 1065 points to it.

Let $p_9 = \{24,\ 315,\ 453,\ 794,\ 860, 870, 968, 1324, 1534\}$, $\mathcal{D}_{238}^{p_9} = \chi_{238} \cup p_9$. The maximum number of the intersection points between $\chi_{238}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{238}^{p_9}$ is (19,11)-cap and c_i values are $c_0 = 2358$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{238}^{p_9}$ is incomplete. The (19,11)-cap will be complete when you add 1406 points to it.

Let $p_{10} = \{24, 315, 453, 794, 860, 870, 968, 1324, 1534, 1658\}$, $\mathcal{D}_{238}^{p_{10}} = \chi_{238} \cup p_{10}$. The maximum number of the intersection points between $\chi_{238}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{238}^{p_{10}}$ is $(20,12)$-cap and c_i values are $c_0 = 2358$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{238}^{p_{10}}$ is incomplete. The (20,12)-cap will be complete when you add 1450 points to it.

Let $p_{11} = \{24, 315, 453, 794, 860, 870, 968, 1324, 1534, 1658, 1895\}$, $\mathcal{D}_{238}^{p_{11}} = \chi_{238} \cup p_{11}$. The maximum number of the intersection points between $\chi_{238}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{238}^{p_{11}}$ is $(21,13)$-cap and c_i values are $c_0 = 2358$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{238}^{p_{11}}$ is incomplete. The (21,13)-cap will be complete when you add 1629 points to it.

Let $p_{12} = \{24, 315, 453, 794, 860, 870, 968, 1324, 1534, 1658, 1895, 2373\}$.

Put $\mathcal{D}_{238}^{p_{12}} = \chi_{238} \cup p_{12}$. The maximum number of the intersection points between $\chi_{238}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{238}^{p_{12}}$ is (22,14)-cap and c_i values are $c_0 = 2358$. Since $c_0 \neq 0$, then $\chi_{238}^{p_{12}}$ is incomplete. The (22,14)-cap will be complete when you add 2358 points to it.

12. The line $\mathbf{I}(\tau^{11}, 1,0,0,0,0)$ meets the cap χ_{340} in 7 points, so we have seven extension points that can be added to $\chi_{P,}$ as in the following order:

$171, 511, 851, 1191, 1531, 1871, 2211.$

Let $p_1 = \{171\}$, $\mathcal{D}_{340}^{p_1} = \chi_{340} \cup p_1$. The maximum number of intersection points between $\chi_{340}^{p_1}$ and the lines of $PG(3,13)$ are eight, so $\chi_{340}^{p_1}$ is (8,8)-cap and c_i values are $c_0 = 2366$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{340}^{p_1}$ is incomplete. The (8,8)-cap will be complete when you add 702 points to it.

Let $p_2 = \{171, 511\}$, $\mathcal{D}_{340}^{p_2} = \chi_{340} \cup p_2$. The maximum number of the intersection points between $\chi_{340}^{p_2}$ and the lines of $PG(3,13)$ are nine, so $\chi_{340}^{p_2}$ is (9,9)-cap and c_i values are $c_0 = 2366$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{340}^{p_2}$ is incomplete. The (9,9)-cap will be complete when you add 912 points to it.

Let $p_3 = \{171, 511, 851\}$, $\mathcal{D}_{340}^{p_3} = \chi_{340} \cup p_3$. The maximum number of the intersection points between $\chi_{340}^{p_3}$ and the lines of $PG(3,13)$ are ten, so $\chi_{340}^{p_3}$ is (10,10)-cap and c_i values are $c_0 = 2366$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{340}^{p_3}$ is incomplete. The (10,10)-cap will be complete when you add 1077 points to it.

Let $p_4 = \{171, 511, 851, 1191\}$, $\mathcal{D}_{340}^{p_4} = \chi_{340} \cup p_4$. The maximum number of the intersection points between $\chi_{340}^{p_4}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{340}^{p_4}$ is (11,11)-cap and c_i values are $c_0 = 2366$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{340}^{p_4}$ is incomplete. The (11,11)-cap will be complete when you add 1408 points to it.

Let $p_5 = \{171, 511, 851, 1191, 1531\}$, $\mathcal{D}_{340}^{p_5} = \chi_{340} \cup p_5$. The maximum number of the intersection points between $\chi_{340}^{p_5}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{340}^{p_5}$ is (12,12)-cap and c_i values are $c_0 = 2366$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{340}^{p_5}$ is incomplete. The (12,12)-cap will be complete when you add 1451 points to it.

Let $p_6 = \{171, 511, 851, 1191, 1531, 1871\}$, $\mathcal{D}_{340}^{p_6} = \chi_{340} \cup p_6$. The maximum number of the intersection points between $\chi_{340}^{p_6}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{340}^{p_6}$ is (13,13)-

cap and c_i values are $c_0 = 2366$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{340}^{p_6}$ is incomplete. The (13,13)-cap will be complete when you add 1635 points to it.

Let $p_7 = \{171, 511, 851, 1191, 1531, 1871, 2211\}$, $\mathcal{D}_{340}^{p_7} = \chi_{340} \cup p_7$. The maximum number of the intersection points between $\chi_{340}^{p_7}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{340}^{p_7}$ is (14,14)-cap and c_i values are $c_0 = 2366$. Since $c_0 \neq 0$, then $\chi_{340}^{p_7}$ is incomplete. The (14,14)-cap will be complete when you add 2366 points to it.

13. The line $\mathbf{I}(\tau, \tau^6, 1,0,0,0)$ meets the cap χ_{476} in 2 points, so we have twelve extension points that can be added to χ_P, as in the following order: $205, 217, 220, 360, 574, 686, 809, 970, 971, 1139, 2233, 2321$.

Let $p_1 = \{205\}$, $\mathcal{D}_{476}^{p_1} = \chi_{476} \cup p_1$. The maximum number of the intersection points between $\chi_{476}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{476}^{p_1}$ is (6,3)-cap and c_i values are $c_0 = 2363$, $c_1 = 11$. Since $c_0 \neq 0$, then $\chi_{476}^{p_1}$ is incomplete. The (6,3)-cap will be complete when you add 137 points to it.

Let $p_2 = \{205, 217\}$, $\mathcal{D}_{476}^{p_2} = \chi_{476} \cup p_2$. The maximum number of the intersection points between $\chi_{476}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{476}^{p_2}$ is (7,4)-cap and c_i values are $c_0 = 2363$, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{476}^{p_2}$ is incomplete. The (7,4)-cap will be complete when you add 243 points to it.

Let $p_3 = \{205, 217, 220\}$, $\mathcal{D}_{476}^{p_3} = \chi_{476} \cup p_3$. The maximum number of the intersection points between $\chi_{476}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{476}^{p_3}$ is (8,5)-cap and c_i values are $c_0 = 2363$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{476}^{p_3}$ is incomplete. The (8,5)-cap will be complete when you add 399 points to it.

Let $p_4 = \{205, 217, 220, 360\}$, $\mathcal{D}_{476}^{p_4} = \chi_{476} \cup p_4$. The maximum number of the intersection points between $\chi_{476}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{476}^{p_4}$ is (9,6)-cap and c_i values are $c_0 = 2363$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{476}^{p_4}$ is incomplete. The (9,6)-cap will be complete when you add 509 points to it.

Let $p_5 = \{205, 217, 220, 360, 574\}$, $\mathcal{D}_{476}^{p_5} = \chi_{476} \cup p_5$. The maximum number of the intersection points between $\chi_{476}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{476}^{p_5}$ is (10,7)-cap and c_i values are $c_0 = 2363$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{476}^{p_5}$ is incomplete. The (10,7)-cap will be complete when you add 632 points to it.

Let $p_6 = \{205, 217, 220, 360, 574, 686\}$, $\mathcal{D}_{476}^{p_6} = \chi_{476} \cup p_6$. The maximum number of the intersection points between $\chi_{476}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{476}^{p_6}$ is (11,8)-cap and c_i values are $c_0 = 2363$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{476}^{p_6}$ is incomplete. The (11,8)-cap will be complete when you add 677 points to it.

Let $p_7 = \{205, 217, 220, 360, 574, 686, 809\}$, $\mathcal{D}_{476}^{p_7} = \chi_{476} \cup p_7$. The maximum number of the intersection points between $\chi_{476}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{476}^{p_7}$ is (12,9)-cap and c_i values are $c_0 = 2363$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{476}^{p_7}$ is incomplete. The (12,9)-cap will be complete when you add 896 points to it.

Let $p_8 = \{205, 217, 220, 360, 574, 686, 809, 970\}$, $\mathcal{D}_{476}^{p_8} = \chi_{476} \cup p_8$. The maximum number of the intersection points between $\chi_{476}^{p_8}$ and the lines of $PG(3,13)$ ten, so $\chi_{476}^{p_8}$ is (13,10)-cap and c_i values are $c_0 = 2363$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{476}^{p_8}$ is incomplete. The (13,10)-cap will be complete when you add 1069 points to it.

Let $p_9 = \{205,\ 217, 220,\ 360, 574,\ 686,\ 809,\ 970,\ 971\}$, $\mathcal{D}_{476}^{p_9} = \chi_{476} \cup p_9$. The maximum number of the intersection points between $\chi_{476}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{476}^{p_9}$ is (14,11)-cap and c_i values are $c_0 = 2363$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{476}^{p_9}$ is incomplete. The (14,11)-cap will be complete when you add 1407 points to it.

Let $p_{10} = \{205, 217, 220,\ 360, 574, 686,\ 809, 970, 971,\ 1139\}$, $\mathcal{D}_{476}^{p_{10}} = \chi_{476} \cup p_{10}$. The maximum number of the intersection points between $\chi_{476}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{476}^{p_{10}}$ is (15,12)-cap and c_i values are $c_0 = 2363$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{476}^{p_{10}}$ is incomplete. The (15,12)-cap will be complete when you add 1389 points to it.

Let $p_{11} = \{205, 217, 220,\ 360, 574, 686,\ 809, 970, 971,\ 1139,\ 2233\}$, $\mathcal{D}_{476}^{p_{11}} = \chi_{476} \cup p_{11}$. The maximum number of the intersection points between $\chi_{476}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{476}^{p_{11}}$ is (16,13)-cap and c_i values are $c_0 = 2363$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{476}^{p_{11}}$ is incomplete. The (16,13)-cap will be complete when you add 1630 points to it.

Let $p_{12} = \{205, 217, 220,\ 360, 574, 686,\ 809, 970, 971,\ 1139, 2233, 2321\}$.

Put $\mathcal{D}_{476}^{p_{12}} = \chi_{476} \cup p_{12}$. The maximum number of the intersection points between $\chi_{476}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{476}^{p_{12}}$ is (17,14)-cap and c_i values are $c_0 = 2363$. Since $c_0 \neq 0$, then $\chi_{476}^{p_{12}}$ is incomplete. The (17,14)-cap will be complete when you add 2363 points to it.

14. The line $\mathbf{I}(\tau^8, \tau^2, 1,0,0,0)$ meets the cap χ_{595} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order: $108, 483, 501,\ 539, 1201, 1321, 1392, 1424,\ 1507, 1741, 1840, 2235$.

Let $p_1 = \{108\}$, $\mathcal{D}_{595}^{p_1} = \chi_{595} \cup p_1$. The maximum number of the intersection points between $\chi_{595}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{595}^{p_1}$ is (5,3)-cap and c_i values are $c_0 = 2364$, $c_1 = 11$. Since $c_0 \neq 0$, then $\chi_{595}^{p_1}$ is incomplete. The (5,3)-cap will be complete when you add 118 points to it.

Let $p_2 = \{108, 483\}$, $\mathcal{D}_{595}^{p_2} = \chi_{595} \cup p_2$. The maximum number of the intersection points between $\chi_{595}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{595}^{p_2}$ is (6,4)-cap and c_i values are $c_0 = 2364$, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{595}^{p_2}$ is incomplete. The (6,4)-cap will be complete when you add 221 points to it.

Let $p_3 = \{108, 483, 501\}$, $\mathcal{D}_{595}^{p_3} = \chi_{595} \cup p_3$. The maximum number of the intersection points between $\chi_{595}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{595}^{p_3}$ is (7,5)-cap and c_i values are $c_0 = 2364$, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{595}^{p_3}$ is incomplete. The (7,5)-cap will be complete when you add 404 points to it.

Let $p_4 = \{108, 483, 501,\ 539\}$, $\mathcal{D}_{595}^{p_4} = \chi_{595} \cup p_4$. The maximum number of the intersection points between $\chi_{595}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{595}^{p_4}$ is (8,6)-cap and c_i values are $c_0 = 2364$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{595}^{p_4}$ is incomplete. The (8,6)-cap will be complete when you add 491 points to it.

Let $p_5 = \{108, 483, 501,\ 539, 1201\}$, $\mathcal{D}_{595}^{p_5} = \chi_{595} \cup p_5$. The maximum number of the intersection points between $\chi_{595}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{595}^{p_5}$ is (9,7)-cap and c_i values are $c_0 = 2364$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{595}^{108,483,501,539,1201}$ is incomplete. The (9,7)-cap will be complete when you add 633 points to it.

Let $p_6 = \{108, 483, 501,\ 539, 1201, 1321\}$, $\mathcal{D}_{595}^{p_6} = \chi_{595} \cup p_6$. The maximum number of the intersection points between $\chi_{595}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{595}^{p_6}$ is (10,8)-cap and

c_i values are $c_0 = 2364$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{595}^{p_6}$ is incomplete. The (10,8)-cap will be complete when you add 695 points to it.

Let $p_7 = \{108, 483, 501, 539, 1201, 1321, 1392\}$, $\mathcal{D}_{595}^{p_7} = \chi_{595} \cup p_7$. The maximum number of the intersection points between $\chi_{595}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{595}^{p_7}$ is (11,9)-cap and c_i values are $c_0 = 2364$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{595}^{p_7}$ is incomplete. The (11,9)-cap will be complete when you add 902 points to it.

Let $p_8 = \{108, 483, 501, 539, 1201, 1321, 1392, 1424\}$, $\mathcal{D}_{595}^{p_8} = \chi_{595} \cup p_8$. The maximum number of the intersection points between $\chi_{595}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{595}^{p_8}$ is $(12,10)$-cap and c_i values are $c_0 = 2364$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{595}^{p_8}$ is incomplete. The $(12,10)$-cap will be complete when you add 1073 points to it.

Let $p_9 = \{108, 483, 501, 539, 1201, 1321, 1392, 1424, 1507\}$, $\mathcal{D}_{595}^{p_9} = \chi_{595} \cup p_9$. The maximum number of the intersection points between $\chi_{595}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{595}^{p_9}$ is (13,11)-cap and c_i values are $c_0 = 2364$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{595}^{p_9}$ is incomplete. The $(13,11)$-cap will be complete when you add 1346 points to it.

Let $p_{10} = \{108, 483, 501, 539, 1201, 1321, 1392, 1424, 1507, 1741\}$.

Put $\mathcal{D}_{595}^{p_{10}} = \chi_{595} \cup p_{10}$. The maximum number of the intersection points between $\chi_{595}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{595}^{p_{10}}$ is (14,12)-cap and c_i values are $c_0 = 2364$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{595}^{p_{10}}$ is incomplete. The (14,12)-cap will be complete when you add 1411 points to it.

Let $p_{11} = \{108, 483, 501, 539, 1201, 1321, 1392, 1424, 1507, 1741, 1840\}$.

Put $\mathcal{D}_{595}^{p_{11}} = \chi_{595} \cup p_{11}$. The maximum number of the intersection points between $\chi_{595}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{595}^{p_{11}}$ is (15,13)-cap and c_i values are $c_0 = 2364$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{595}^{p_{11}}$ is incomplete. The (15,13)-cap will be complete when you add 1632 points to it.

Let $p_{12} = \{108, 483, 501, 539, 1201, 1321, 1392, 1424, 1507, 1741, 1840, 2235\}$. Put $\mathcal{D}_{595}^{p_{12}} = \chi_{595} \cup p_{12}$. The maximum number of the intersection points between $\chi_{595}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{595}^{p_{12}}$ is (16,14)-cap and c_i values are $c_0 = 2334$. Since $c_0 \neq 0$, then $\chi_{595}^{p_{12}}$ is incomplete. The (16,14)-cap will be complete when you add 2364 points to it.

15. The line $\mathbf{I}(\tau^{11}, 1,0,0,0,0)$ meets the cap χ_{1190} in 2 points, so we have twelve extension points that can be added to $\chi_{P,}$ as in the following order: $171, 341, 511, 681, 851, 1021, 1361, 1531, 1701, 1871, 2041, 2211$.

Let $p_1 = \{171\}$, $\mathcal{D}_{1190}^{p_1} = \chi_{1190} \cup p_1$. The maximum number of the intersection points between $\chi_{1190}^{p_1}$ and the lines of $PG(3,13)$ are three, so $\chi_{1190}^{p_1}$ is (3,3)-cap and c_i values are $c_0 = 2366$, $c_1 = 11$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_1}$ is incomplete. The (3,3)-cap will be complete when you add 119 points to it.

Let $p_2 = \{171, 341\}$, $\mathcal{D}_{1190}^{p_2} = \chi_{1190} \cup p_2$. The maximum number of the intersection points between $\chi_{1190}^{p_2}$ and the lines of $PG(3,13)$ are four, so $\chi_{1190}^{p_2}$ is (4,4)-cap and c_i values are $c_0 = 2366$, $c_1 = 10$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_2}$ is incomplete. The (4,4)-cap will be complete when you add 222 points to it.

Let $p_3 = \{171, 341, 511\}$, $\mathcal{D}_{1190}^{p_3} = \chi_{1190} \cup p_3$. The maximum number of the intersection points between $\chi_{1190}^{p_3}$ and the lines of $PG(3,13)$ are five, so $\chi_{1190}^{p_3}$ is (5,5)-cap and c_i values are $c_0 =$

2366, $c_1 = 9$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_3}$ is incomplete. The (5,5)-cap will be complete when you add 401 points to it.

Let $p_4 = \{171, 341, 511, 681\}$, $\mathcal{D}_{1190}^{p_4} = \chi_{1190} \cup p_4$. The maximum number of the intersection points between $\chi_{1190}^{p_4}$ and the lines of $PG(3,13)$ are six, so $\chi_{1190}^{p_4}$ is (6,6)-cap and c_i values are $c_0 = 2366$, $c_1 = 8$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_4}$ is incomplete. The (6,6)-cap will be complete when you add 491 points to it.

Let $p_5 = \{171, 341, 511, 681, 851\}$, $\mathcal{D}_{1190}^{p_5} = \chi_{1190} \cup p_5$. The maximum number of the intersection points between $\chi_{1190}^{p_5}$ and the lines of $PG(3,13)$ are seven, so $\chi_{1190}^{p_5}$ is (7,7)-cap and c_i values are $c_0 = 2366$, $c_1 = 7$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_5}$ is incomplete. The (7,7)-cap will be complete when you add 603 points to it.

Let $p_6 = \{171, 341, 511, 681, 851, 1021\}$, $\mathcal{D}_{1190}^{p_6} = \chi_{1190} \cup p_6$. The maximum number of the intersection points between $\chi_{1190}^{p_6}$ and the lines of $PG(3,13)$ are eight, so $\chi_{1190}^{p_6}$ is (8,8)-cap and c_i values are $c_0 = 2366$, $c_1 = 6$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_6}$ is incomplete. The (8,8)-cap will be complete when you add 681 points to it.

Let $p_7 = \{171, 341, 511, 681, 851, 1021, 1361\}$, $\mathcal{D}_{1190}^{p_7} = \chi_{1190} \cup p_7$. The maximum number of the intersection points between $\chi_{1190}^{p_7}$ and the lines of $PG(3,13)$ are nine, so $\chi_{1190}^{p_7}$ is (9,9)-cap and c_i values are $c_0 = 2366$, $c_1 = 5$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_7}$ is incomplete. The (9,9)-cap will be complete when you add 899 points to it.

Let $p_8 = \{171, 341, 511, 681, 851, 1021, 1361, 1531\}$, $\mathcal{D}_{1190}^{p_8} = \chi_{1190} \cup p_8$. The maximum number of the intersection points between $\chi_{1190}^{p_8}$ and the lines of $PG(3,13)$ are ten, so $\chi_{1190}^{p_8}$ is (10,10)-cap and c_i values are $c_0 = 2366$, $c_1 = 4$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_8}$ is incomplete. The (10,10)-cap will be complete when you add 1069 points to it.

Let $p_9 = \{171, 341, 511, 681, 851, 1021, 1361, 1531, 1701\}$, $\mathcal{D}_{1190}^{p_9} = \chi_{1190} \cup p_9$. The maximum number of the intersection points between $\chi_{1190}^{p_9}$ and the lines of $PG(3,13)$ are eleven, so $\chi_{1190}^{p_9}$ is (11,11)-cap and c_i values are $c_0 = 2366$, $c_1 = 3$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_9}$ is incomplete. The (11,11)-cap will be complete when you add 1347 points to it.

Let $p_{10} = \{171, 341, 511, 681, 851, 1021, 1361, 1531, 1701, 1871\}$, $\mathcal{D}_{1190}^{p_{10}} = \chi_{1190} \cup p_{10}$. The maximum number of the intersection points between $\chi_{1190}^{p_{10}}$ and the lines of $PG(3,13)$ are twelve, so $\chi_{1190}^{p_{10}}$ is (12,12)-cap and c_i values are $c_0 = 2366$, $c_1 = 2$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_{10}}$ is incomplete. The (12,12)-cap will be complete when you add 1434 points to it.

Let $p_{11} = \{171, 341, 511, 681, 851, 1021, 1361, 1531, 1701, 1871, 2041\}$, $\mathcal{D}_{1190}^{p_{11}} = \chi_{1190} \cup p_{11}$. The maximum number of the intersection points between $\chi_{1190}^{p_{11}}$ and the lines of $PG(3,13)$ are thirteen, so $\chi_{1190}^{p_{11}}$ is (13,13)-cap and c_i values are $c_0 = 2366$, $c_1 = 1$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_{11}}$ is incomplete. The (13,13)-cap will be complete when you add 1635 points to it.

Let $p_{12} = \{171, 341, 511, 681, 851, 1021, 1361, 1531, 1701, 1871, 2041, 2211\}$. Put $\mathcal{D}_{1190}^{p_{12}} = \chi_{1190} \cup p_{12}$. The maximum number of the intersection points between $\chi_{1190}^{p_{12}}$ and the lines of $PG(3,13)$ are fourteen, so $\chi_{1190}^{p_{12}}$ is (14,14)-cap and c_i values are $c_0 = 2366$. Since $c_0 \neq 0$, then $\chi_{1190}^{p_{12}}$ is incomplete. The (14,14)-cap will be complete when you add 2366 points to it.

4. Results and discussions

The order pairs $(s+i, d+i)$ refer to the size and degree of the new extension cap from (s, d)-cap, $\mathcal{D}_P^{p_i}$. Let # denote the number of added points to the orbit to be complete. Table 1 contains a summary of the getting results.

Table 1. Details about the extension complete caps.

	$\mathcal{D}_P^{p_i}$	(size, degree) of the cap	#
1	$\mathcal{D}_4^{p_i}$	$(595,7) \rightarrow (595+i, 7+i),\ i=1,\dots,7$	196, 413, 600, 702, 1016, 1148, 1778
2	$\mathcal{D}_7^{p_i}$	$(340,4) \rightarrow (340+i, 4+i),\ i=1,\dots,10$	66, 196, 335, 463, 685, 816, 991, 1180, 1395, 2030
3	$\mathcal{D}_{14}^{p_i}$	$(170,2) \rightarrow (170+i, 2+i),\ i=1,\dots,12$	20, 119, 254, 352, 537, 629, 818, 970, 1200, 1325, 1516, 2198
4	$\mathcal{D}_{20}^{p_i}$	$(119,7) \rightarrow (119+i, 7+i),\ i=1,\dots,7$	658, 827, 1033, 1250, 1383, 1555, 2254
5	$\mathcal{D}_{28}^{p_i}$	$(85,2) \rightarrow (85+i, 2+i),\ i=1,\dots,12$	74, 176, 319, 433, 623, 671, 892, 1047, 1299, 1367, 1577, 2283
6	$\mathcal{D}_{35}^{p_i}$	$(68,3) \rightarrow (68+i, 3+i),\ i=1,\dots,11$	189, 334, 486, 627, 692, 876, 1037, 1312, 1401, 1591, 2301
7	$\mathcal{D}_{68}^{p_i}$	$(35,7) \rightarrow (35+i, 7+i),\ i=1,\dots,7$	709, 901, 1065, 1395, 1427, 1615, 2338
8	$\mathcal{D}_{70}^{p_i}$	$(34,2) \rightarrow (34+i, 2+i),\ i=1,\dots,12$	109, 226, 368, 468, 639, 697, 989, 1056, 1330, 1421, 1612, 2334
9	$\mathcal{D}_{119}^{p_i}$	$(20,2) \rightarrow (20+i, 2+i),\ i=1,\dots,12$	117, 241, 392, 507, 639, 734, 891, 1066, 1335, 1444, 1623, 2348
10	$\mathcal{D}_{140}^{p_i}$	$(19,2) \rightarrow (19+i, 2+i),\ i=1,\dots,12$	128, 248, 388, 489, 647, 707, 904, 1062, 1340, 1383, 1625, 2351
11	$\mathcal{D}_{238}^{p_i}$	$(10,2) \rightarrow (10+i, 2+i),\ i=1,\dots,12$	119, 243, 393, 509, 643, 707, 896, 1065, 1406, 1450, 1629, 2358
12	$\mathcal{D}_{340}^{p_i}$	$(7,7) \rightarrow (7+i, 7+i),\ i=1,\dots,7$	702, 912, 1077, 1408, 1451, 1635, 2366
13	$\mathcal{D}_{476}^{p_i}$	$(5,2) \rightarrow (5+i, 2+i),\ i=1,\dots,12$	137, 243, 399, 509, 632, 677, 896, 1069, 1407, 1389, 1630, 2363
14	$\mathcal{D}_{595}^{p_i}$	$(4,2) \rightarrow (4+i, 2+i),\ i=1,\dots,12$	118, 221, 404, 491, 633, 695, 902, 1073, 1346, 1411, 1632, 2364
15	$\mathcal{D}_{1190}^{p_i}$	$(2,2) \rightarrow (2+i, 2+i),\ i=1,\dots,12$	119, 222, 401, 491, 603, 681, 899, 1069, 1347, 1434, 1635, 2366

5. Conclusion

Theory of Group action is very helpful to introduce new caps in the finite projective space of dimension higher than two. All caps that are founded in this paper can be used to construct linear codes. Also, all these caps can be used to construct more caps which will deal with it in the next paper.

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Conflicts of Interest

The authors declare no conflict of interest.

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Non

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