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Novel Approximate Solutions for Nonlinear Blasius Equations

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Abstract
The method of operational matrices based on different types of polynomials such as Bernstein, shifted Legendre and Bernoulli polynomials is introduced and implemented to solve the nonlinear Blasius equations approximately. The nonlinear differential equation is converted into a system of nonlinear algebraic equations that can be solved using Mathematica®12. The efficiency of these methods has been studied by calculating the maximum error remainder (,MER-n.), and it was found that their efficiency increases with increasing polynomial degree (n) as the errors decrease. Moreover, the approximate solutions obtained by the proposed methods are compared with the solution of the fourth-order Runge-Kutta method (RK4), which gives a very good agreement. In addition, the convergence of the proposed approximation methods is given based on one of the results of the Banach fixed point theorem.

Keywords: Blasius equations; Bernstein polynomial; Legendre polynomial; Bernoulli polynomial; operational matrices.

1. Introduction

The nonlinear ordinary differential equations (NODE) have many practical applications in engineering and applied sciences, such as fluid flow, current flow in electric circuits, heat dissipation in solid bodies, seismic wave propagation, population increase or decrease, and many other topics [1,2]. In approximation theory and numerical analysis, polynomials are particularly useful tools [3]. Consequently, the polynomial series then the operational matrices (OM) are used to simplify the unknown function by transforming it into a system of algebraic equations that can be easily solved without integration and differentiation. Several studies have been conducted on this technique, which uses OM methods to solve many problems based on various polynomials.
Many researchers have solved various problems using OM based on the Bernstein polynomial (BOM), such as: [4] solved odd boundary value problems, [5] studied third order equations ODE, [6] solved fractional integral equations. Moreover, there are many researchers who have used operational matrices based on different polynomials, such as [7] used the Legendre operational

matrix (LOM) method to solve the fractional-order two-dimensional integral equations. In [8] Sharma et al. solved the Lane-Emden equations using the Chebyshev operational matrix (ChOM) method. Also, OM with Bernoulli polynomials (BrOM) was used by Bazm [9] to solve some types of integral equations. [10] studied the magnetohydrodynamic squeezing fluid, straight fin problem, Jeffery-Hamel flow, and Falkner-Skan equation using BOM and ChOM. They also studied the solution of nonlinear thin- film flow of 3rd-grade fluid problems with LOM [11]. In [12-20], other types of polynomials were used to solve different types of problems.

There is an application for third-order NODEs that occurs in fluid mechanics with the laminar viscous flow and the various aspects of the hydrodynamic boundary layer problem is the nonlinear Blasius equation [21,22]. Several researchers have worked on the solution of this equation: [23] obtained a numerical solution by the Runge-Kutta method. [21] solved the Blasius equation, the Duffing equation, the Van der Pol equation, and the Jerk equation using the numerical approach of the inverse Laplace transform based on the BrOM integration technique. Also, He [24] used the variational iteration method to obtain an analytic approximate solution for the Blasius equation. [25] solved it using Adomian's decomposition method. Moreover, [26] studied it with the new homotopy perturbation method.

The aim of this paper is to use the method of operational matrices based on different types of polynomials such as Bernstein, shifted Legendre and Bernoulli polynomials will be used to solve the nonlinear Blasius equations and novel approximate solutions will be obtained.

The structure of this paper is as follows: In Section two, the Blasius equation is introduced. Section three gives the orthogonal polynomials on which the operational matrices depend, namely the Bernstein polynomial, the shifted Legendre polynomial, and the Bernoulli polynomial. In Section four, the convergence of the proposed methods is explained. In section five, the problem will be solved using the proposed methods. Finally, Section six presents the conclusions.

2. The Blasius Equation

This equation is important because it appears in many hydrodynamic boundary layer problems as well as in the fluid mechanics of laminar viscous flows and its formula is given by [22]:

$$f'''(x) + \frac{1}{2} f(x) f''(x) = 0, \tag{1}$$

with boundary conditions: $f(0) = f'(0) = 0, f'(\infty) = 1.$ (2)

To solve this equation, the boundary conditions were converted to initial conditions by computing $f''(0)$ instead of $f'(\infty)$, as Liao [27] did and found $f''(0) = 0.3320573$, followed by Khataybeh et al. [5] used the value $f''(0) = 1$.

In general, the Blasius equation represents a model of the attitude of a two-dimensional stable laminar viscous flow on a semi-infinite flat plate in which the flowing fluid is incompressible, but the boundary layer assumption must be governed by the continuity and Navier-Stokes equations of motion, but it should be noted that the fluid flow velocity decreases sharply from U to 0 at $y = 0$ as x changes from -0 to $+0$.

3. The operational matrices of the orthogonal polynomials

Orthogonal polynomials play an important role in pure and applied mathematics as well as in numerical computation [28]. Three types of these polynomials will be used: Bernstein, shifted Legendre, and Bernoulli to solve the Blasius equation.

3.1 The Bernstein polynomials

The Bernstein Polynomials of n^{th} degree in [0,1] are defined by [4,10,29]:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0,1,2,\dots,n, \tag{3}$$

where $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$.

Or the recursive definition over [0,1] is used to generate these polynomials, resulting in Bernstein polynomial being represented as follows:

$$B_{i,n}(x) = (1-x)\, B_{i,n-1}(x) + x\, B_{i-1,n-1}(x). \tag{4}$$

Practically only the first $(n+1)$ terms of the Bernstein polynomials of degree n are satisfied, because $B_{i,n}(x) = 0$ if $i < 0$ or $n < i$. There are many properties make Bernstein polynomial important, some of them are:

i. Property of Positivity: $B_{i,n}(x) > 0$ for all $0 < i < n$ and all $x \epsilon [0,1]$.

ii. Unity partition property: $\sum_{i=0}^{n} B_{i,n}(x) = \sum_{i=0}^{n-1} B_{i,n-1}(x) = \ldots = \sum_{i=0}^{1} B_{i,1}(x) = 1$.

Moreover, the type of linear combination shown below can be used to approximate any polynomial of n^{th}degree in $L^2[0,1]$:

$$f(x) = \sum_{i=0}^{n} c_i\, B_{i,n}(x) = C^T \phi(x), \tag{5}$$

where $C^T = [c_0, c_1, c_2, \dots, c_n]$, and $\phi(x) = \left[B_{0,n}, B_{1,n}, B_{2,n}, \dots,, B_{n,n}\right]^T$.

In addition, $\phi(x)$ can be decomposed as the product of a $(n+1) \times (n+1)$ matrix A and a $(n+1) \times 1$ vector X, i.e.:

$$\phi(x) = A\, X, \tag{6}$$

where:

$$A = \begin{bmatrix} (-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0} \binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0 \binom{n}{i} & \cdots & (-1)^{n-i} \binom{n}{i}\binom{n-i}{n-i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0 \binom{n}{n} \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ x \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

(7)

The determinant of the matrix A is $|A| = \prod_{i=0}^{n} \binom{n}{i}$. As a result, A is an invertible matrix.

Let us introduce the BOM. If D_B is the OM of derivative of size $(n+1) \times (n+1)$, then the derivative of $\phi(x)$ is:

$$\frac{d\phi(x)}{dx} = D_B \phi(x); \; x \epsilon [0,1], \tag{8}$$

where D_B is given as $D_B = A\,V\,B^*$,where:

$$V = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}_{(n+1)\times n} ,B^* = \begin{bmatrix} A_1^{-1} \\ A_2^{-1} \\ A_3^{-1} \\ \vdots \\ A_n^{-1} \end{bmatrix}_{n\times(n+1)} , \quad (9)$$

where B^* represents the α^{th} rows of $A^{-1}$for α=1, 2, …, n.

Furthermore, the generalization of (8) can be written as follows:

$$\frac{d^n\phi(x)}{dx^n} = (D_B)^n\,\phi(x);\ n = 1, 2, \dots\ . \quad (10)$$

Thus, it will be able to approximate the derivatives of $f(x)$ in terms of OM as follows:

$$f'(x) = C^T D_B\,\phi(x),\ f''(x) = C^T (D_B)^2\,\phi(x),\ \dots,\ f^{(n)}(x) = C^T (D_B)^n\,\phi(x). \quad (11)$$

This approximation and Eq.(5) are also applied to all conditions of the equation.

To solve the equation, we replace $f(x)$ and its derivatives by Eqs.(5) and (11), then modify x by appropriate points from Chebyshev roots, called collocation nodes, as follows:

$$x_r = \frac{1}{2}\,(cos\frac{r\,\pi}{n} + 1),\ \ r = 1, \dots, n-1. \quad (12)$$

This results in a system of algebraic equations that can be solved using software such as Mathematica or MATLAB to obtain the value of the vector C^T in Eq.(5).

3.2 The shifted Legendre polynomials

The Legendre polynomials of n^{th} order are defined on [-1,1] as [11,13]:

$$L_0(t) = 1,\ L_1(t) = t,\ \dots,\ L_{n+1}(t) = \frac{2\,n+1}{n+1}\,t\,L_n(t) - \frac{n}{n+1}\,L_{n-1}(t),\ n = 1,2, \dots\ . \quad (13)$$

Legendre polynomials can be defined on [0,1] by changing the variable $t = 2x - 1$ and denoting it as a shifted Legendre polynomial denoted by $P_n(x)$, and then calculated as follows:

$$P_0(x) = 1,\ P_1(x) = 2x - 1,\ \dots,\ P_{n+1}(x) = \frac{(2n+1)(2x-1)}{n+1} P_n(x) - \frac{n}{n+1} P_{n-1}(x), n = 1,2, \dots\ . \quad (14)$$

Also, $P_n(x)$ can be written as shown:

$$P_n(x) = \sum_{i=0}^{n} \frac{(-1)^{n+i}\,(n+i)}{(n-i)!\,(i!)^2}\,x^i. \quad (15)$$

Any function $f(x) \in L^2[0,1]$, could be characterized by a shifted Legendre polynomial as follows:

$$f(x) = \sum_{i=0}^{\infty} c_i\,P_i(x), \quad (16)$$

where $c_i = (2\,i + 1)\ \int_0^1 f(x)\,P_i(x)\,dx;\ i = 0, 1, 2, \dots\ .$

In practice, only the first $(n + 1)$ terms of Eq.(16) will be considered:

$$f(x) = \sum_{i=0}^{n} c_i\,P_i(x) = C^T\phi(x), \quad (17)$$

where $C^T = [c_0, c_1, \ldots, c_n]$, and $\phi(x) = [P_0(x), P_1(x), \ldots, P_n(x)]^T$.

The derivatives of $\phi(x)$can be defined as:

$$\frac{d\phi(x)}{dx} = D_L\phi(x),\ \frac{d^2\phi(x)}{dx^2} = (D_L)^2\phi(x),\ \ldots,\ \frac{d^n\phi(x)}{dx^n} = (D_L)^n\phi(x \tag{18}$$

where D_L is the $(n + 1) \times (n + 1)$ OM of derivative, given as:

$$D_L = (d_{ij})\begin{cases} 2\,(2j+1), & j = i - k, \quad \begin{cases} k = 1,3,\ldots,n, & \text{if } n \text{ odd}, \\ k = 1,3,\ldots,n-1, & \text{if } n \text{ even}, \end{cases} \\ 0 & \text{otherwise}. \end{cases} \tag{19}$$

Then write the derivatives of $f(x)$ with respect to the OM as follows:

$$f'(x) = C^T D_L\phi(x),\ f''(x) = C^T(D_L)^2\,\phi(x), \ldots,\ f^{(n)}(x) = C^T(D_L)^n\,\phi(x). \tag{20}$$

This helps to solve the NODE by substituting $f(x)$ and its derivatives as in Eqs.(17) and (20). Also, compensated in the conditions with the NODE. Then the collocation nodes are inserted into these equations to produce a system of algebraic equations $(n + 1)$ that can be solved with software such as MATLAB or Mathematica to obtain the coefficients of the vector C^T.

3.3 The Bernoulli polynomials

The n^{th}Bernoulli polynomials on [0,1] are defined as follows [9,21]:

$$Br_n(x) = \sum_{i=0}^{n}\binom{n}{i}Br_i\, x^{n-i}, \tag{21}$$

where $Br_i = Br_i(0)$ is the Bernoulli number for all $i = 0,1,\ldots,n$. These numbers are calculated by:

$$Br_n = -\sum_{i=1}^{n+1}\frac{(-1)^i}{i}\binom{n+1}{i}\sum_{j=1}^{i} j^n, \tag{22}$$

For $n \geq 0, n \neq 1$. If $n = 1$, then $Br_1 = -\frac{1}{2}$.

Therefore, some Bernoulli polynomials can be represented as follows:

$Br_0(x) = 1$, $Br_1(x) = x - \frac{1}{2}$, $Br_2(x) = x^2 - x + \frac{1}{6}$, $Br_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$,

$Br_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$.

Bernoulli polynomials are used in many fields of mathematics, which led to the discovery of important properties for them, some of which we mention below:

i. $\frac{dBr_n(x)}{dx} = n\, Br_{n-1}(x), n \geq 1$.

ii. $\int_a^z Br_n(x)\, dx = \frac{Br_{n+1}(z) - Br_{n+1}(a)}{n+1}$.

iii. $\int_0^1 Br_n(x)\, dx = 0, n \geq 1$.

iv. $\int_0^1 Br_n(x)\, Br_m(x)\, dx = (-1)^{n-1}\frac{n!\, m!}{(n+m)!}\, Br_{n+m}$.

v. $Br_n(1 - x) = (-1)^n Br_n(x)$.

vi. $Br(x + 1) - Br_n(x) = n\, x^{n-1}$.

Therefore, it is now easy to express any $f(x) \in L^2[0,1]$ by the linear combination of the Bernoulli polynomial as:

$$f(x) = \sum_{i=0}^{n} c_i \, Br_i(x) = C^T \, Br(x), \tag{23}$$

where $C^T = [c_0, c_1, \dots, c_n]$, and $Br(x) = [\, Br_0(x), Br_1(x), \dots, Br_n(x)]^T$. For all $i = 0,1, \dots, n$, the matrix form of $Br_i(x)$ can be obtained as follows:

$$Br_i(x) = \sum_{j=0}^{i} \binom{i}{j} Br_j \, x^{i-j},$$

$$= \binom{i}{i} Br_i \, x^0 + \binom{i}{i-1} Br_{i-1} \, x + \cdots + \binom{i}{1} Br_1 \, x^{i-1} + \binom{i}{0} Br_0 \, x^i,$$

$$= \begin{bmatrix} \binom{i}{i} Br_i & \binom{i}{i-1} Br_{i-1} & \cdots & \binom{i}{1} Br_1 & \binom{i}{0} Br_0 & \overbrace{0 \;\; 0 \;\; \dots \;\; 0}^{n-i} \end{bmatrix} \begin{bmatrix} 1 \dots \\ \vdots \\ x^i \\ x^{i+1} \\ \vdots \\ x^n \end{bmatrix},$$

$$= M_i \, H(x). \tag{24}$$

where

$$H(x) = [1, x, \dots, x^n]^T \text{and } M_i = \begin{bmatrix} \binom{i}{i} Br_i & \binom{i}{i-1} Br_{i-1} & \cdots & \binom{i}{1} Br_1 & \binom{i}{0} Br_0 & \overbrace{0 \;\; 0 \;\; \dots \;\; 0}^{n-i} \end{bmatrix}.$$

As a result, $Br(x)$ in Eq.(23) can be written as follows:

$$Br(x) = [Br_0(x), Br_1(x), \dots, Br_n(x)]^T,$$

$$= [M_0 H(x), M_1 H(x), \dots, M_n H(x)]^T,$$

$$= [M_0, M_1, \dots, M_n]^T H(x),$$

$$= \begin{bmatrix} \binom{0}{0} Br_0 & 0 & 0 & \cdots & 0 \\ \binom{1}{1} Br_1 & \binom{1}{0} Br_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \binom{n}{n} Br_n & \binom{n}{n-1} Br_{n-1} & \binom{n}{n-2} Br_{n-2} & \cdots & \binom{n}{0} Br_0 \end{bmatrix} H(x),$$

$$= M \, H(x). \tag{25}$$

While the derivative of it will be:

$$\frac{dBr(x)}{dx} = D_{Br} \, Br(x), \tag{26}$$

where $D_{Br} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}$ is the $(n+1) \times (n+1)$ OM.

Moreover, the n^{th}derivative:

$$\frac{d^n Br(x)}{dx^n} = (D_{Br})^n \, Br(x). \tag{27}$$

To complete the solution, we must find the value of the vector C^T in Eq.(23), using the same procedures as the previous methods.

4. The convergence of the proposed methods

In this section, the convergence analysis for the proposed methods and fundamental theorem are discussed.

Theorem 4.1. Let a Banach space $A \subset R$ be given with a norm $\| \ \|$ defined on it. Taking $f_1(x)$ as an approximate solution obtain from the first iteration n, we construct the following sequence regarding the solution of Eq.(1):

$$v_1(x) = f_1(x), \qquad v_i(x) = f_i(x) - f_{i-1}(x), (i \geq 2).$$

Then, the assertions are:

- **(i)** Provided that for all i there exist $0 < \beta_i < 1$ such that $\|v_{i+1}(x)\| \leq \beta_i \|v_i(x)\|$, the series $\sum_{i=1}^{\infty} v_i(x)$ is than convergent and so $f(x) = \sum_{i=1}^{\infty} v_i(x)$ in the interval of interest $x \in \Gamma$.
- **(ii)** Otherwise, for all i there exist $\beta_i > 1$ leading to $\|v_{i+1}(x)\| \geq \beta_i \|v_i(x)\|$, the series $\sum_{i=1}^{\infty} v_i(x)$ and thus, the proposed method diverges in the interval of interest $x \in \Gamma$.

Proof: See [30].

Remark 1. Defining a ratio β_i via:

$$\beta_i = \frac{\|v_{i+1}(x)\|}{\|v_i(x)\|}, \tag{28}$$

if this ratio stays less than one for all i, the approximate solution converges to the exact solution.

5. Numerical results and discussions

In this section, the methods presented in section three: the BOM, the shifted LOM, and the BrOM are applied to solve the nonlinear Blasius equation. However, first, the boundary conditions must be converted to initial conditions, i.e., the value of the second derivative at zero must be found. This was done in [27] using the homotopy-Padé method for approximate and finding that the best result is $f''(0) = 0.3320573$.

To solve this equation using the BOM method, we apply the technique from Section 3.1 with $n = 3$. Let us first assume the approximate solution as:

$$f(x) = c_0\, B_{0,3}(x) + c_1\, B_{1,3}(x) + c_2\, B_{2,3}(x) + c_3\, B_{3,3}(x) = C^T \phi(x), \tag{29}$$

Where $B_{0,3}(x) = 1 - 3x + 3x^2 - x^3,\ B_{1,3}(x) = 3x - 6x^2 + 3x^3,\ B_{2,3}(x) = 3x^2 - 3x^3,$ $B_{3,3}(x) = x^3$.

From Eq.(8) we obtain the OM:

$$D_B = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \qquad (D_B)^2 = \begin{bmatrix} 6 & 4 & 2 & 0 \\ -12 & -6 & 0 & 6 \\ 6 & 0 & -6 & -12 \\ 0 & 2 & 4 & 6 \end{bmatrix},$$

$$(D_B)^3 = \begin{bmatrix} -6 & -6 & -6 & -6 \\ 18 & 18 & 18 & 18 \\ -18 & -18 & -18 & -18 \\ 6 & 6 & 6 & 6 \end{bmatrix}.$$

After converting each $f(x)$ and its derivatives in the nonlinear Blasius equation and its initial conditions into terms of operational matrices, we substitute the collocation nodes Eq.(12) instead of each x, which gives us the following system of nonlinear algebraic equations:

$$-6c_0 + 18c_1 - 18c_2 + 6c_3 + \tfrac{1}{2}(0.015625c_0 + 0.140625c_1 + 0.421875c_2 + 0.421875c_3)(1.5c_0 + 1.5c_1 - 7.5c_2 + 4.5c_3) = 0,$$

$$-3c_0 + 3c_1 = 0,$$

$$c_0 = 0,$$

$$6c_0 - 12c_1 + 6c_2 = 0.3320573. \tag{30}$$

Finally, by solving (30) we obtain:

$c_0 = 0,\ \ c_1 = 0,\ \ c_2 = 0.055342883,\ \ c_3 = 0.1635587513832345.$

So, the approximate solution will be as follows:

$$f(x) = [0, \quad 0, \quad 0.055342883, \quad 0.1635587513832345] \begin{bmatrix} 1 - 3x + 3x^2 - x^3 \\ 3x - 6x^2 + 3x^3 \\ 3x^2 - 3x^3 \\ x^3 \end{bmatrix},$$

$$= 0.16602865x^2 - 0.002469898616765498x^3.$$

Thus, until $n = 11$, the approximate solution will be:

$$f(x) = 0.16602865x^2 + 2.491375994395639 \times 10^{-10}x^3 - 1.878031063995422 \times 10^{-9}x^4 - 0.000459416599124296x^5 - 2.537478849262697 \times 10^{-8}x^6 + 4.931977670707965 \times 10^{-8}x^7 + 0.0000024338763893183 57x^8 + 5.216138276864513 \times 10^{-8}x^9 - 2.538929066986384 \times 10^{-8}x^{10} - 8.5669178417902 \times 10^{-9}x^{11}. \tag{31}$$

Also, this equation is solved by using the shifted LOM method, and the description in Section 3.2 is followed when $n = 3$, the approximate solution is assumed to be as follows:

$$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) = C^T\ \phi(x), \tag{32}$$

Where $P_0(x) = 1,\ \ P_1(x) = 2x - 1,\ \ P_2(x) = 6x^2 - 6x + 1,$

$P_3(x) = 20x^3 - 30x^2 + 12x - 1.$

From Eq.(19) we have the OM:

$$D_L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 10 & 0 \end{bmatrix},\ (D_L)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 \end{bmatrix},\ (D_L)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 120 & 0 & 0 & 0 \end{bmatrix}.$$

After writing $f(x)$ and its derivatives in NODE and its initial conditions in terms of operational matrices, the collocation nodes Eq.(12) are substituted in place of each x to obtain the system of nonlinear algebraic equations:

$6c_0c_2 + 3.c_1c_2 - 0.75c_2^2 + 120c_3 + 15.c_0c_3 + 7.5c_1c_3 - 4.5c_2c_3 - 6.5625c_3^2 = 0,$

$2c_1 - 6c_2 + 12c_3 = 0,$

$c_0 - c_1 + c_2 - c_3 = 0,$

$$12c_2 - 60c_3 = 0.3320573. \tag{33}$$

Solving this system and substituting the value of C^T into Eq.(32) yields the approximate solution:

$$f(x) = -3.469446951953614 \times 10^{-18} + 0.16602865x^2 - 0.002469898616765488x^3.$$

Consequently, until $n = 11$, we obtain the following approximate solution:

$$f(x) = -3.696547211931118 \times 10^{-12} + 3.253866343033706 \times 10^{-10}x + 0.16602864303933304x^2 + 6.283554605411232 \times 10^{-8}x^3 - 2.920702954934285 \times 10^{-7}x^4 - 0.00045866085596424073x^5 - 0.0000011485624698356698x^6 + 9.397408390670055 \times 10^{-7}x^7 + 0.0000021633505483296\,46x^8. \tag{34}$$

Moreover, if we use the BrOM method, then the explanation in Section 3.3 is followed and the approximate solution is taken as assumed:

$$f(x) = c_0Br_0(x) + c_1Br_1(x) + c_2Br_2(x) + c_3Br_3(x) = C^T\ Br(x), \tag{35}$$

Where $Br_0(x) = 1,\ \ Br_1(x) = x - \frac{1}{2},\ \ Br_2(x) = x^2 - x + \frac{1}{6},\ \ Br_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$

Here, from Eq.(26) we have the OM:

$$D_{Br} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix},\ (D_{Br})^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix},\ (D_{Br})^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}.$$

After writing the nonlinear Blasius equation and its initial conditions in the form of operational matrices and putting the collocation nodes instead of each x, we obtain the following system:

$c_0c_2 + 0.25c_1c_2 - 0.020833333333333337c_2^2 + 6c_3 + 0.75c_0c_3 + 0.1875c_1c_3 - 0.06250000000000003c_2c_3 - 0.03515625c_3^2 = 0,$

$c_1 - c_2 + \frac{c_3}{2} = 0,$

$c_0 - \frac{c_1}{2} + \frac{c_2}{6} = 0,$

$2c_2 - 3c_3 = 0.3320573.$ (36)

Once this system is solved, the roots are substituted into Eq.(35), and we obtain the approximate solution as follows:

$f(x) = -6.93889390391 \times 10^{-18} + 0.1660286500000003x^2 - 0.00246989861677x^3.$

Thus, up to $n = 11$, the approximate solution will be:

$f(x) = -3.529427472531059 \times 10^{-18} - 3.962758940434519 \times 10^{-17}x + 0.16602865000000003x^2 + 1.979266592913857 \times 10^{-10}x^3 - 1.499749921060817 \times 10^{-9}x^4 - 0.000459418278828476x^5 - 2.065013702012278 \times 10^{-8}x^6 + 4.070826065187125 \times 10^{-8}x^7 + 0.000002444007699522198x^8 + 4.474523682223838 \times 10^{-8}x^9 - 2.232041556339086 \times 10^{-8}x^{10} - 9.11508105682988 \times 10^{-9}x^{11}.$
(37)

The exact solution of the Blasius equation is unknown. Therefore, the maximum error remainder (MER_n) is calculated to determine the accuracy of the proposed methods. The MER_n is given by:

$$MER_n = \max_{0\le x\le 1} \left| f'''(x) + \frac{1}{2} f(x)\, f''(x) \right|. \quad (38)$$

Figure 1 shows the logarithmic plots for MER_n of the approximate solutions obtained by the three proposed methods at all n iterations ($n = 3$ to 11).

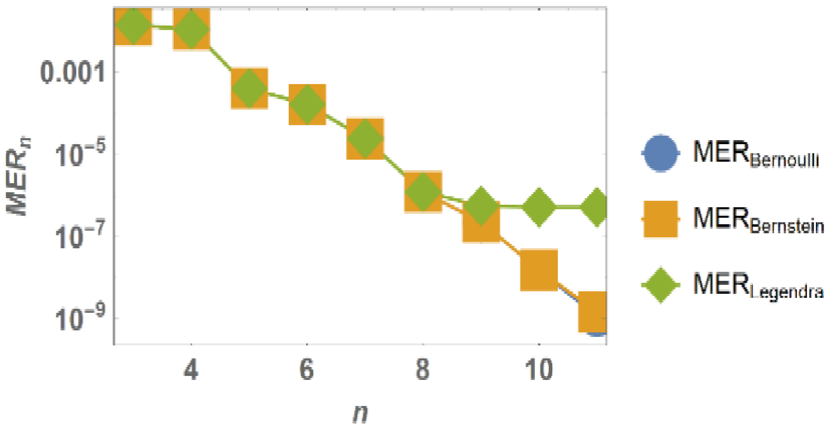

Figure 1.Logarithmic plots for the MER_n of BOM, LOM, and BrOM.

In addition, **Table 1** shows the MER_n value for all n studied in solving the nonlinear Blasius equation by using the proposed methods.

Table 1: The comparison of MER_n between the BOM, LOM, and BrOM methods for $n = 3$ to 11.

n	BOM	LOM	BrOM
3	0.014819391700592988	0.014819391700592925	0.014819391700593073
4	0.011427015738563973	0.011427015738558465	0.011427015738558968
5	0.0004197059747635956	0.00041970597546251153	0.0004197059754639646
6	0.00017127579193321196	0.00017127579194381904	0.00017127579189292255
7	0.000025137988745926876	0.00002513798831121161	0.00002513798889495492
8	0.0000012630050019168 94	0.0000012630033656508 16	0.0000012630035611465 65
9	$2.347967527072114 \times 10^{-7}$	$5.772589885717771 \times 10^{-7}$	$2.34800920268644 \times 10^{-7}$
10	$1.588884090963915 \times 10^{-8}$	$5.387294752107197 \times 10^{-7}$	$1.59061837140361 \times 10^{-8}$
11	$1.494825596637383 \times 10^{-9}$	$5.385043960887126 \times 10^{-7}$	$1.187559955162328 \times 10^{-9}$

From **Figure 1** and **Table 1**, we can conclude that the value of the error decreases as n increases. Thus, the BrOM method is better than the LOM method and slightly better than the BOM method.

Moreover, a comparison was made between the approximate solutions found by the proposed methods at $n = 11$, and the numerical solution obtained by the Range-Kutta method (*RK4*). This is shown in **Figure 2**, which shows a good agreement between them.

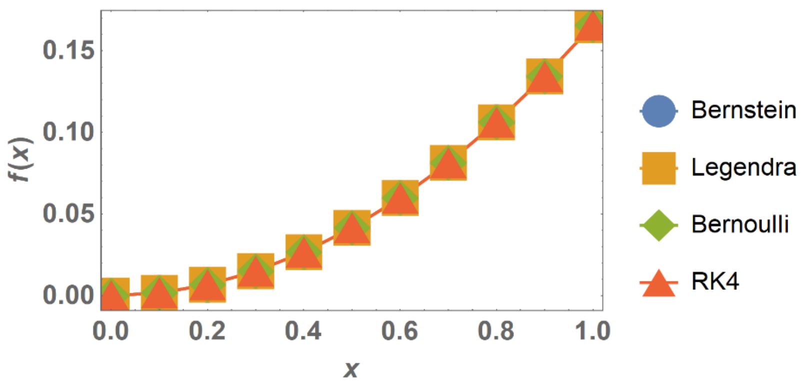

Figure 2. The comparison of the solutions of proposed methods and *RK4* at $n = 11$.

To investigate the convergence of the solutions of the nonlinear Blasius equation, the convergence condition described in Section 4 is applied to the solutions of the proposed methods for all iterations n (n=3 to 11) by calculating the values of $\beta_i = \frac{\|v_{i+1}(x)\|}{\|v_i(x)\|}$, as shown in

Table 2, the values of β_i for all $i \geq 3$ are less than 1, so these solutions converge to the exact solution.

Table 2. The value of β_i to the solutions of the proposed methods for $n = 3$ to 11 when $f''(0) = 0.3320573$.

β_i	BOM	LOM	BrOM
β_3	0.016073750074326664	0.01607375007432271	0.016073750074321977
β_4	0.29293061768966266	0.29293061766935835	0.2929306176693013
β_5	0.03639714771755241	0.03639714778922806	0.03639714778673125
β_6	0.24728635030717735	0.24728634929502963	0.24728634997162285
β_7	0.09393295721721101	0.09393296065173386	0.09393296261024817
β_8	0.0477950301697603	0.04779170199698017	0.047795085567504614
β_9	0.13563608954043813	0.1356410881800995	0.1356460394028394
β_{10}	0.047883452366514034	0.04874842641203115	0.04878982655985626

On the other hand, in [5] the Blasius equation for $f''(0) = 1$ was solved with BOM at $n = 11$, so **Figure 3** shows the difference between the approximate solution at $f''(0) = 0.3320573$ and $f''(0) = 1$.

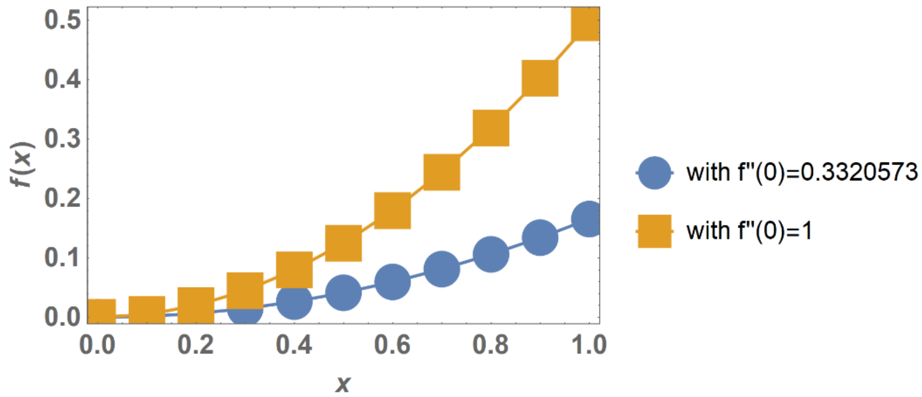

Figure 3. Comparing the solution to the Blasius equation by BOM for two different values of $f''(0)$ at $n = 11$.

From the above figure, it can be seen that the solution is better at $f''(0) = 0.3320573$. To illustrate this, the MER_n for both values of $f''(0)$ was calculated for all n ($n = 3$ to 11) as in **Table 3.**

Table 3. The comparison of MER_n of the solutions of the Blasius equation by BOM at *n*=3 to 11 at two different values of $f''(0)$

n	When $f''(0) = 0.3320573$	When $f''(0) = 1$
3	0.014819391700592988	0.12364105477870169
4	0.011427015738563973	0.09719389597650396
5	0.0004197059747635956	0.011027011129465247
6	0.00017127579193321196	0.004293535161145279
7	0.000025137988745926876	0.0004907797800370872
8	0.000001263005001916894	0.00009600784941188323
9	$2.347967527072114 \times 10^{-7}$	0.00001612130638850573
10	$1.588884090963915 \times 10^{-8}$	$4.033942939685175 \times 10^{-7}$
11	$1.494825596637383 \times 10^{-9}$	$2.636242850684311 \times 10^{-7}$

Thus, it can be clearly seen that MER_n converges to zero faster in the case of $f''(0) = 0.3320573$ than in the case of at $f''(0) = 1$, as n increases.

Moreover, we will solve this problem with the initial condition $f''(0) = 1$ using the shifted LOM and the BrOM. The solution with the shifted LOM when $n = 11$ is as follows:

$$f(x) = 2.717523012968703 \times 10^{-11} - 2.989454184199047 \times 10^{-9}x + 0.5000000807212571x^2 - 8.890725537863409 \times 10^{-7}x^3 + 0.0000053823541016 5247x^4 - 0.004185704926584 84x^5 + 0.000041144785309415706x^6 - 0.000054417856055302455x^7 + 0.00010990985811043829x^8 - 0.000015116787731315758x^9. \quad (39)$$

If the BrOM apply, the approximate solution is as follows:

$$f(x) = 4.28022688906543 \times 10^{-17} + 4.126066892665148 \times 10^{-17}x + 0.5x^2 + 4.379568419265968 \times 10^{-8}x^3 - 3.321493629909657 \times 10^{-7}x^4 - 0.0041651294231496725x^5 - 0.0000045887951487605 94x^6 + 0.000009070467696073757x^7 + 0.000005631161302828 6645x^8 + 0.000010056188911186578x^9 - 0.000005051467149564113x^{10} + 5.856448890320508 \times 10^{-9}x^{11}. \quad (40)$$

Table 4 shows a comparison between the MER_n of the three proposed methods for this problem for all iterations n (n=3 to 11). It can be seen that the accuracy increases as n increases. Also, the BrOM and the BOM methods are better than the LOM method, and the BrOM method is slightly better than the BOM method.

Table 4. The comparison of MER_n between the BOM, LOM, and BrOM methods for $n = 3$ to 11 at $f''(0) = 1$.

n	BOM	LOM	BrOM
3	0.12364105477870169	0.12364105477870149	0.12364105477870134
4	0.09719389597650396	0.09719389597647592	0.09719389597650352
5	0.011027011129465247	0.011027011129505574	0.011027011129506844
6	0.004293535161145279	0.004293535161116567	0.00429353516106645
7	0.0004907797800370872	0.0004907797800190347	0.0004907797804556991
8	0.00009600784941188323	0.00009600784484897892	0.00009600784504393617
9	0.000016121306388850573	0.000016121319056666043	0.000016121340651173055
10	$4.033942939685175 \times 10^{-7}$	0.0000005672419462032785	$4.034414582290297 \times 10^{-7}$
11	$2.636242850684311 \times 10^{-7}$	0.0000005624000721099476	$2.627741051773592 \times 10^{-7}$

Moreover, the convergence condition given in Section 4 is applied to the solutions of the proposed methods for all iterations n (n=3 to 11) when $f''(0) = 1$ by calculating the values of $\beta_i = \frac{\|v_{i+1}(x)\|}{\|v_i(x)\|}$, as shown in **Table 5**, the values of β_i for all $i \geq 3$ are less than 1. Consequently, the convergence condition is satisfied.

Table 5. The value of β_i to the solutions of the proposed methods for $n = 3$ to 11 when $f''(0) = 1$.

β_i	BOM	LOM	BrOM
β_3	0.04510659083967248	0.04510659083966797	0.04510659083967211
β_4	0.277633864539339	0.27763386453933614	0.04510659083967211
β_5	0.11985758340437944	0.11985758340397297	0.1198575834036301
β_6	0.23281227458321288	0.23281227457927967	0.23281227459976303
β_7	0.06500373117417597	0.06500373174199665	0.0650037318024709
β_8	0.21189027133781216	0.21189028400881857	0.21189031026366478
β_9	0.1197929121542225	0.11979856224759194	0.11979340082544765
β_{10}	0.008393018377721006	0.008431707509080347	0.008431055448947825

6. Conclusions

In this work, the operational matrices of differentiation were used based on: Bernstein, shifted Legendre, and Bernoulli polynomials to solve the nonlinear Blasius equation. We concluded that the Bernoulli operational matrix method is better than the Bernstein operational matrix method and the shifted Legendre operational matrix method. We also compared the results numerically with the Runge-Kutta method and found good agreement between them. The calculations in this study were performed using the Mathematica® 12 program. Also, the maximal error remainder value for the proposed approximation methods was calculated.

Moreover, after examining a certain value of the second derivative ($f''(0) = 0.3320573$) calculated in [27], the result was compared with the value obtained in [5] ($f''(0) = 1$). It was found that the solution is more accurate at $f''(0) = 0.3320573$.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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