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Bayesian Estimation of The unknown parameter of Inverse Rayleigh Distribution (IRD) based on GELF

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Abstract

In this paper, we investigated the Bayesian estimation of the unknown parameter of the Inverse Rayleigh Distribution (IRD) under different priors, represented by the inverse gamma distribution, the inverse chi-squared distribution, and the standard Levy distribution as priors. We obtained the posterior distributions for the unknown parameter of IRD under the different priors based on the general entropy loss function (GELF). We assumed different values for the shape parameter of GELF. Also, the maximum likelihood estimator is used to estimate the scale parameter of IRD. Then, a study is conducted to obtain the results, based on the different parameters of Inverse Rayleigh distribution and sample sizes. We found that Bayes estimators perform better than MLE according to the Least Mean Square Error (MSE) Criterion.

Keywords: Inverse Rayleigh distribution, Maximum Likelihood Estimator, Bayes estimation, Entropy loss function, Mean Square Error.

1. Introduction

The Inverse Rayleigh distribution (IRD) is one of the continuous distributions. It is a very useful lifetime distribution. Many authors have discussed its applications in survival analysis and reliability theory, as well as in life test study and industrial reliability problems. Therefore, many authors have studied different methods for estimating the unknown parameters of the Inverse Rayleigh Distribution (IRD). We list some of these studies below: [1] derived a maximum likelihood (MLE) and a Bayes estimator for the unknown parameters of the inverse Rayleigh distribution (IRD) based on a set of lower record values. They also derived the r^{th} moment around the origin and Bayes point and interval estimators for the parameter under informative priors based on quadratic error loss functions. They used a simulation study to illustrate the theoretical results of the prediction interval. [2] derived Bayes estimators for the unknown parameter and the reliability function of an inverse Rayleigh distribution (IRD) under a non-informative prior distribution based on the squared error loss function (SELF) and the LINEX loss function which are an asymmetric function. He used a simulation study to compare the different estimators of the parameters and the reliability function. He concludes that the Bayes estimators for the parameter under the LINEX loss function are better than the Bayes estimators based on the SE loss function. He also finds that the Bayes estimators for the reliability function under the LINEX loss function are better than the Bayes.

Estimators based on the SE loss function, corresponding to the smallest value of the root-mean-square error measure [3] maximum likelihood (ML) based on the lower record values and Bayes estimation to estimate the unknown parameter of the inverse Rayleigh distribution (IRD), as well as the reliability and cumulative failure rate function for the inverse Rayleigh distribution. They derived Bayes estimators under the Gamma distribution as a prior distribution based on quadratic error losses and LINEX loss functions. They also obtained Bayesian prediction intervals for the future record values. They used a simulation study to calculate the estimated mean and mean squared errors for each estimator. They find that the Bayesian estimates are superior to the MLE estimates, as indicated by the smallest value of the mean squared error. In [4], Bayesian estimation was used to estimate the unknown parameter of the inverse Rayleigh distribution (IRD). They derived posterior distributions and posterior risks under informative and non-informative priors in the presence of left censoring. They obtained the Bayes estimators of the unknown parameter of the IRD under different loss functions such as squared error loss, prudence loss, weighted squared error loss, quasi-squared loss, squared-logarithmic error loss) and entropy loss functions, they assumed different informative priors (exponential distribution and gamma distribution and inverse Levy distribution) and non-informative priors (the uniform prior and the Jeffrey' prior).

In [5], introduces the modified Inverse Rayleigh (MIR) distribution, He studied the statistical properties of the MIR distribution, represented by quantile and median, and gives closed form to generate the random number of the MIR distribution. He also presents the moment and moment generating. He derives the mean deviation about mean and about the median M. He gives the close form to the distribution having first order statistic and n^{th} order probability density function. He also discussed some of its properties to illustrating the usefulness of the MIR distribution to real data using MLE. [6] maximum likelihood estimators and Bayes estimators were derived for the unknown parameters of the inverse Rayleigh distribution (IRD). He obtains Bayes estimators assuming quasi-density as a non- informative prior based on squared error losses, LINEX losses and entropy loss functions. He simulations to obtain the results about the estimators. He concludes that these estimators have the same values for a large (n>50). In [7] proposed a new three parameter of inverse Rayleigh Distribution (IRD) using Alpha Power Transformation (APT) known as the Alpha Power inverse Rayleigh distribution (APIRD). They derived the mathematical forms of the APIRD such as moments, mgf., entropies, mean waiting time, mean residual time. They also estimated the parameters of the APIRD by using maximum likelihood estimation method. [8] derived minimum expected estimators using the maximum likelihood (MLE) of the unknown parameter of the inverse Rayleigh distribution (IRD). They consider four different loss functions represented by the linear exponential loss function and the precautionary loss function, with the other two proposed loss functions. They depend on the relative efficiency of the estimators to compare the different estimators of the scale parameter of the IRD. [9] obtain the maximum likelihood estimate of the scale parameter of the weighted Rayleigh distribution (WRD). They used the Bayesian method to estimate the scale parameter of the WRD. They derived posterior distributions by using different priors, namely informative priors such as the Gumbel type II distribution and the Levy distribution and non-informative priors such as the quasi-prior. They obtained Bayes estimators based on different loss functions represented by squared error losses and quadratic losses and precautionary loss functions. They conduct a simulation study to compare classical and Bayesian estimates of scale parameters according to the mean squared error (MSE) criterion for different sample sizes and for different values of the parameters.

In this study will discuss different estimation methods for the unknown scale parameter φ for the Inverse Rayleigh distribution (IRD) using the maximum likelihood estimator (MLE) and Bayes' estimators based on the general entropy loss function (GELF), under different prior distributions (informative priors) which are the inverse Gamma distribution, the inverse chi-square distribution, and the standard Levy distribution . In order to compared the accuracy for Bayes' estimators with the corresponding maximum likelihood estimator (MLE) of the scale parameter of IRD using the mean square error criterion (MSE).

2.The Inverse Rayleigh Distribution (IRD)

Suppose that $(t_1, t_2,..., t_n)$ be a random variable from the inverse Rayleigh distribution (IRD) with the scale parameter $\varphi > 0$ with the following probability density function (pdf) [2]:

$$f(t;\varphi) = \frac{2}{\varphi\, t^3}\exp(-\frac{1}{\varphi\, t^2}) \quad , \; t > 0 \;,\; \varphi > 0 \tag{1}$$

and the cumulative distribution function is given by

$$F(t;\varphi) = \exp(-\frac{1}{\varphi\, t^2}) \quad , \; t > 0 \;,\; \varphi > 0 \tag{2}$$

With The reliability function is given by

$$R(t) = \bar{F}(t) = 1 - \exp(-\frac{1}{\varphi\, t^2}) \;,\; t > 0 \;,\; \varphi > 0 \tag{3}$$

3. Maximum likelihood Estimation (MLE)

Then the likelihood function for the $(t_1, t_2,..., t_n)$ observations is defined by equation (1) [10,11]:

$$\ell(\varphi \backslash \underline{t}) = \prod_{i=1}^{n} f(t_i;\varphi) = 2^n \varphi^{-n} \prod_{i=1}^{n}\frac{1}{t_i^3}\exp(-\frac{1}{\varphi}\sum_{i=1}^{n}\frac{1}{t_i^2}) \tag{4}$$

The log-likelihood function $L = \ln(\ell)$ is given by

$$L = \log(2^n) - n\log(\varphi) + \log(\prod_{i=1}^{n}\frac{1}{t_i^3}) - \frac{1}{\varphi}\sum_{i=1}^{n}\frac{1}{t_i^2} \tag{5}$$

The first partial derivative of log of the likelihood L with respect to φ is obtained as follows:

$$\frac{\partial L}{\partial \varphi} = \frac{-n}{\varphi} + \frac{1}{\varphi^2}\sum_{i=1}^{n}\frac{1}{t_i^2} = 0 \tag{6}$$

Then the maximum likelihood estimator(MLE) of φ is given by

$$\hat{\varphi}_{MLE} = \frac{(\sum_{i=1}^{n}\frac{1}{t_i^2})}{n} \tag{7}$$

4. Bayesian Estimation

Let us consider the Bayes estimators for scale parameter of an IRD under the general entropy loss function (GELF), based on different priors. The respective expressions have been presented by informative priors for unknown scale parameter φ which are the inverse Gamma distribution, the inverse chi-square distribution, and the standard Levy distribution as prior distributions. And we derived the posterior density using the different informative priors as shown in the next section.

4.1 Posterior distribution

We derive posterior distribution for the unknown parameter φ of the inverse Rayleigh distribution (IRD) as the following.

a. Posterior distribution using the inverse Gamma as prior distribution

We consider the random variables for φ to be from the inverse Gamma as prior distribution with hyper parameters $(\lambda \ , \eta)$ [12, 13, 14] as

$$g_1(\varphi;\lambda,\eta)\ \alpha\ \frac{\lambda^{\eta}}{\Gamma(\eta)}\varphi^{-(\eta+1)}\exp(-\frac{\lambda}{\varphi}) \qquad \text{with} \qquad \varphi>0 \ , \ \lambda,\eta>0 \tag{8}$$

We can define the posterior distribution of φ as follow[15, 16, 17,18] :

$$\pi_1(\varphi \setminus t)=\frac{\ell(\varphi \setminus t_1,t_2,...,t_n)\ g_1(\varphi)}{\int_{\varphi} \ell(\varphi \setminus t_1,t_2,...,t_n)\ g_1(\varphi)d\varphi} \tag{9}$$

Substituting equation (4) and equation (7) in equation (9), yields the posterior probability density function of the scale parameter φ as the following:

$$\pi_1(\varphi \setminus t)=\frac{[2^n\varphi^{-n}\prod_{i=1}^{n}\frac{1}{t_i^3}\exp(-\frac{1}{\varphi}\sum_{i=1}^{n}\frac{1}{t_i^2})][\frac{\lambda^{\eta}}{\Gamma(\eta)}\varphi^{-(\eta+1)}\exp(-\frac{\lambda}{\varphi})]}{\int_{\varphi=0}^{\infty}[2^n\varphi^{-n}\prod_{i=1}^{n}\frac{1}{t_i^3}\exp(-\frac{1}{\varphi}\sum_{i=1}^{n}\frac{1}{t_i^2})][\frac{\lambda^{\eta}}{\Gamma(\eta)}\varphi^{-(\eta+1)}\exp(-\frac{\lambda}{\varphi})]d\varphi} \tag{10}$$

$$\pi_1(\varphi \setminus t)=\frac{\varphi^{-((n+\eta)+1)}\ \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda))}{\int_{\varphi=0}^{\infty}\varphi^{-((n+\eta)+1)}\ \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda))\,d\varphi} \tag{11}$$

By multiplying the integral in equation (11) by the quantity which is equal

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda)^{(n+\eta)}}{\Gamma(n+\eta)}\right)\left(\frac{\Gamma(n+\eta)}{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda)^{(n+\eta)}}\right)$$

,where $\Gamma(.)$ is a gamma function , we obtain

$$\pi_1(\varphi \setminus t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda)^{(n+\eta)}}{\Gamma(n+\eta)\quad A1(t;\varphi)}\varphi^{-((n+\eta)+1)}\ \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda)) \tag{12}$$

where

$$A1(t;\varphi)=\int_{0}^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda)^{(n+\eta)}}{\Gamma(n+\eta)}\varphi^{-((n+\eta)+1)}\ \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda))d\varphi=1 \tag{13}$$

The equation (13) is the integral of the pdf of the inverse gamma distribution [12]. The posterior distribution of φ is the inverse gamma distribution as

$$\pi_1(\varphi \backslash t) = \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)^{(n+\eta)}}{\Gamma(n+\eta)} \varphi^{-((n+\eta)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)) \tag{14}$$

i.e. $(\varphi \backslash t)$ be the inverse gamma distribution with the shape parameter $(n+\eta)$ and the scale parameter $(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)$ with posterior mean is $E(\varphi \backslash t) = \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)}{((n+\eta)-1)}$ and posterior variance is $var(\varphi \backslash t) = \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)^2}{((n+\eta)-1)^2 (n+\eta-2)}$.

b. Posterior distribution using the inverse chi-square as prior distribution

We consider the random variables for φ to be from the inverse chi-square as prior distribution with hyper parameters (k) [12, 19 , 20]as

$$g_2(\varphi; k) \,\alpha\, \frac{(k\ /2)^{\frac{k}{2}}}{\Gamma(\ k/2)} \varphi^{-1-(\frac{k}{2})} \exp(-\frac{k}{2\varphi}) \quad ,\varphi > 0\ ,\ k > 0 \tag{15}$$

Substituting equation (4) and equation (15) in equation (9), yields the posterior probability density function of the scale parameter φ as the following[15, 16, 17,18] :

$$\pi_2(\varphi \backslash t) = \frac{[2^n \varphi^{-n} \prod_{i=1}^{n} \frac{1}{t_i^3} \exp(-\frac{1}{\varphi} \sum_{i=1}^{n} \frac{1}{t_i^2})][\frac{(k\ /2)^{\frac{k}{2}}}{\Gamma(\ k/2)} \varphi^{-1-(\frac{k}{2})} \exp(-\frac{k}{2\varphi})]}{\int_{\varphi=0}^{\infty} [2^n \varphi^{-n} \prod_{i=1}^{n} \frac{1}{t_i^3} \exp(-\frac{1}{\varphi} \sum_{i=1}^{n} \frac{1}{t_i^2})][\frac{(k\ /2)^{\frac{k}{2}}}{\Gamma(\ k/2)} \varphi^{-1-(\frac{k}{2})} \exp(-\frac{k}{2\varphi})]\, d\varphi} \tag{16}$$

$$\pi_2(\varphi \backslash t) = \frac{\varphi^{-((n+0.5k)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5k))}{\int_{\varphi=0}^{\infty} \varphi^{-((n+0.5k)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5k))\, d\varphi} \tag{17}$$

By multiplying the integral in equation (17) by the quantity which is equal $\left(\frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5k)^{(n+0.5k)}}{\Gamma(n+0.5k)}\right) \left(\frac{\Gamma(n+0.5k)}{(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5k)^{(n+0.5k)}}\right)$,where $\Gamma(.)$ is a gamma function , we obtain

$$\pi_2(\varphi \backslash t) = \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5k)^{(n+0.5k)}}{\Gamma(n+0.5k)\, A2(t;\varphi)} \varphi^{-((n+0.5k)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5k)) \tag{18}$$

where

$$A2(t;\varphi)=\int_{0}^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^{(n+0.5k)}}{\Gamma(n+0.5k)}\varphi^{-((n+0.5k)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k))d\varphi=1 \quad (19)$$

The equation (19) is the integral of the pdf of the inverse gamma distribution [12]. The posterior distribution of φ is the inverse gamma distribution as

$$\pi_2(\varphi\setminus t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^{(n+0.5k)}}{\Gamma(n+0.5k)}\varphi^{-((n+0.5k)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)) \quad (20)$$

i.e. $(\varphi\setminus t)$ be the inverse gamma distribution with the shape parameter $(n+0.5k)$ and the scale parameter $(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)$ with posterior mean is $E(\varphi\setminus t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)}{((n+0.5k)-1)}$ and posterior variance is $var(\varphi\setminus t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^2}{((n+0.5k)-1)^2(n+0.5k-2)}$.

c. Posterior distribution using the standard Levy as prior distribution

We consider the random variables for φ to be from the standard levy as prior distribution with hyper parameters (ω) [21, 22, 23,24]as

$$g_3(\varphi\ ;\ \omega)\ \alpha\ (\frac{\omega}{2\pi})^{\frac{1}{2}}\varphi^{-(\frac{3}{2})}\exp(-\frac{\omega}{2\varphi})\quad ,\varphi>0\ ,\ \omega>0 \quad (21)$$

Substituting equation (4) and equation (21) in equation (9), yields the posterior probability density function of the scale parameter φ as the following[15-18] :

$$\pi_3(\varphi\setminus t)=\frac{[2^n\varphi^{-n}\prod_{i=1}^{n}\frac{1}{t_i^3}\exp(-\frac{1}{\varphi}\sum_{i=1}^{n}\frac{1}{t_i^2})][(\frac{\omega}{2\pi})^{\frac{1}{2}}\varphi^{-(\frac{3}{2})}\exp(-\frac{\omega}{2\varphi})]}{\int_{\varphi=0}^{\infty}[2^n\varphi^{-n}\prod_{i=1}^{n}\frac{1}{t_i^3}\exp(-\frac{1}{\varphi}\sum_{i=1}^{n}\frac{1}{t_i^2})][(\frac{\omega}{2\pi})^{\frac{1}{2}}\varphi^{-(\frac{3}{2})}\exp(-\frac{\omega}{2\varphi})]\,d\varphi} \quad (22)$$

$$\pi_3(\varphi\setminus t)=\frac{\varphi^{-((n+0.5)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega))}{\int_{\varphi=0}^{\infty}\varphi^{-((n+0.5)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega))\,d\varphi} \quad (23)$$

By multiplying the integral in equation (23) by the quantity which is equal $\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^{(n+0.5)}}{\Gamma(n+0.5)}\right)\left(\frac{\Gamma(n+0.5)}{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^{(n+0.5)}}\right)$,where $\Gamma(.)$ is a gamma function , we obtain

$$\pi_3(\varphi \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^{(n+0.5)}}{\Gamma(n+0.5)\ A3(t;\varphi)}\ \varphi^{-((n+0.5)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)) \quad (24)$$

Where

$$A3(t;\varphi)=\int_0^\infty \frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^{(n+0.5)}}{\Gamma(n+0.5)}\ \varphi^{-((n+0.5)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega))d\varphi=1 \quad (25)$$

The equation (25) is the integral of the pdf of the inverse gamma distribution [12]. The posterior distribution of φ is the inverse gamma distribution as

$$\pi_3(\varphi \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^{(n+0.5)}}{\Gamma(n+0.5)}\ \varphi^{-((n+0.5)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)) \quad (26)$$

i.e. $(\varphi \backslash t)$ be the inverse gamma distribution with the shape parameter $(n+0.5)$ and the scale parameter $(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)$ with posterior mean is $E(\varphi \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)}{((n+0.5)-1)}$ and posterior variance is $var(\varphi \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^2}{((n+0.5)-1)^2(n+0.5-2)}$.

4.2 Bayes estimation under general entropy loss function

The general entropy loss function (GELF) presented by Calabria and Pulcini [25] (1994), then many authors used the GELF, such as [26] used the GELF by setting the shape parameter equal to (± 3). [27] also used GELF, by setting the shape parameter to be one and two. [28] used GELF, by setting the shape parameter to be one. Here we used the general entropy loss function (GELF) to obtain Bayes estimators, which is defined as follows [29, 30]:

$$L(\hat{\varphi},\varphi)=(\frac{\hat{\varphi}}{\varphi})^a - a\log_e(\frac{\hat{\varphi}}{\varphi})-1 \qquad , \quad a>0 \quad (27)$$

With the shape parameter (a) of general entropy loss function [25]. Then we can define the risk function as follows

$$R(\hat{\varphi},\varphi)=E[L(\hat{\varphi},\varphi)]=E((\frac{\hat{\varphi}}{\varphi})^a - a\log_e(\frac{\hat{\varphi}}{\varphi})-1)=\int_\varphi ((\frac{\hat{\varphi}}{\varphi})^a - a\log_e(\frac{\hat{\varphi}}{\varphi})-1)\ \pi(\varphi \backslash t)d\varphi \quad (28)$$

$$R(\hat{\varphi},\varphi)=\int_\varphi (\frac{\hat{\varphi}}{\varphi})^a \pi(\varphi \backslash t)d\varphi - a\int_\varphi \log_e(\frac{\hat{\varphi}}{\varphi})\pi(\varphi \backslash t)d\varphi - \int_\varphi 1\,\pi(\varphi \backslash t)d\varphi)\,d\varphi \quad (29)$$

$$\frac{\partial R(\hat{\varphi}, \varphi)}{\partial \hat{\varphi}} = a\,(\hat{\varphi})^{a-1} E(\varphi^{-a} \setminus t) - \frac{a}{\hat{\varphi}} \tag{30}$$

The equation (30) satisfies the following condition $\frac{\partial K(\hat{\varphi}, \varphi)}{\partial \hat{\varphi}} = 0$ and we have

$$a\,(\hat{\varphi})^{a-1} E(\varphi^{-a} \setminus t) = a\,(\hat{\varphi})^{-1} \tag{31}$$

So , we obtain Bayes estimator based on general entropy loss function (GELF) of φ denoted by $\hat{\varphi}$ under different priors as follows

$$\hat{\varphi}_{GELF} = [E(\varphi^{-a} \setminus t)]^{-\frac{1}{a}} \Rightarrow \hat{\varphi}_{GELF} = [\int_{\varphi} \varphi^{-a} \pi(\varphi \setminus t) d\varphi]^{-\frac{1}{a}} \tag{32}$$

It is easy to verify that if $a = 1$, it gives Bayes estimator of φ based on entropy loss function. And if $a = 1$, it gives Bayes estimator of φ based on weighted square error loss function .Also if $a = -1$, it gives Bayes estimator of φ based on square error loss function. So we can derive Bayes estimator for unknown scale parameter φ based on general entropy loss function (GELF) as follows.

a. Bayes estimator using the inverse gamma as prior distribution

Here , we can derive Bayes estimator for φ based on general entropy loss function (GELF), By substituting equation (14) in equation (32) , yields

$$\hat{\varphi}_{GELF(1)} = [\int_{\varphi} \varphi^{-a} \pi_1(\varphi \setminus \underline{t}) d\varphi]^{-\frac{1}{a}} \tag{32}$$

$$\hat{\varphi}_{GELF(1)} = [\int_{\varphi=0}^{\infty} \varphi^{-a} \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)^{(n+\eta)}}{\Gamma(n+\eta)} \varphi^{-((n+\eta)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda))\, d\varphi]^{-\frac{1}{a}} \tag{33}$$

$$\hat{\varphi}_{GELF(1)} = [\int_{\varphi=0}^{\infty} \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)^{(n+\eta)}}{\Gamma(n+\eta)} \varphi^{-((n+\eta+a)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda))\, d\varphi]^{-\frac{1}{a}} \tag{34}$$

By multiplying the integral in equation (34) by the quantity which equal $(\frac{\Gamma(n+\eta+a)}{\Gamma(n+\eta+a)})$, where $\Gamma(.)$ is a gamma function .After some simplification, it yields

$$\hat{\varphi}_{GELF(1)} = [\frac{\Gamma(n+\eta+a)}{\Gamma(n+\eta)\,(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)^{a}} B1(t,\varphi)]^{-\frac{1}{a}} \tag{35}$$

Where

$$B1(t,\varphi) = \int_{\varphi=0}^{\infty} \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda)^{(n+\eta+a)}}{\Gamma(n+\eta+a)} \varphi^{-((n+\eta+a)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + \lambda))\, d\varphi = 1 \tag{36}$$

The equation (36) is the integral of the pdf of inverse gamma distribution [12] .So the Bayes estimator for φ will be

$$\hat{\varphi}_{GELF(1)}=[\frac{\Gamma(n+\eta+a)}{\Gamma(n+\eta)\ \ (\sum_{i=1}^{n}\frac{1}{t_i^2}+\lambda)^a}]^{-\frac{1}{a}} \tag{37}$$

b. Bayes estimator using the inverse chi-square as prior distribution

And, we can derive Bayes estimator for φ based on general entropy loss function (GELF) ,By substituting equation (20) in equation (32) , yields

$$\hat{\varphi}_{GELF(2)}=[\int_{\varphi}\varphi^{-a}\pi_2(\varphi\setminus \underline{t})d\varphi]^{-\frac{1}{a}} \tag{32}$$

$$\hat{\varphi}_{GELF(2)}=[\int_{\varphi=0}^{\infty}\varphi^{-a}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^{(n+0.5k)}}{\Gamma(n+0.5k)}\ \varphi^{-((n+0.5k)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k))\ d\varphi]^{-\frac{1}{a}} \tag{38}$$

$$\hat{\varphi}_{GELF(2)}=[\int_{\varphi=0}^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^{(n+0.5k)}}{\Gamma(n+0.5k)}\ \varphi^{-((n+0.5k+a)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k))\,d\varphi]^{-\frac{1}{a}} \tag{39}$$

By multiplying the integral in equation (39) by the quantity which equal $(\frac{\Gamma(n+0.5k+a)}{\Gamma(n+0.5k+a)})$, where $\Gamma(.)$ is a gamma function .After some simplification, it yields

$$\hat{\varphi}_{GELF(2)}=[\frac{\Gamma(n+0.5k+a)}{\Gamma(n+0.5k)\ \ (\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^a}B2(t,\varphi)]^{-\frac{1}{a}} \tag{40}$$

Where

$$B2(t,\varphi)=\int_{\varphi=0}^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^{(n+0.5k+a)}}{\Gamma(n+0.5k+a)}\ \varphi^{-((n+0.5k+a)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k))\ d\varphi=1 \tag{41}$$

The equation (41) is the integral of the pdf of inverse gamma distribution [12] .So the Bayes estimator for φ will be

$$\hat{\varphi}_{GELF(2)}=[\frac{\Gamma(n+0.5k+a)}{\Gamma(n+0.5k)\ \ (\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5k)^a}]^{-\frac{1}{a}} \tag{42}$$

c. Bayes estimator using the standard Levy as prior distribution

And, we can derive Bayes estimator for φ based on general entropy loss function (GELF) ,By substituting equation (26) in equation (32), yields

$$\hat{\varphi}_{GELF(3)}=[\int_{\varphi}\varphi^{-a}\pi_3(\varphi\setminus \underline{t})d\varphi]^{-\frac{1}{a}} \tag{32}$$

$$\hat{\varphi}_{GELF(3)}=[\int_{\varphi=0}^{\infty}\varphi^{-a}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega)^{(n+0.5)}}{\Gamma(n+0.5)}\ \varphi^{-((n+0.5)+1)}\exp(-\frac{1}{\varphi}(\sum_{i=1}^{n}\frac{1}{t_i^2}+0.5\omega))\,d\varphi]^{-\frac{1}{a}} \tag{43}$$

$$\hat{\varphi}_{GELF(3)} = [\int_{\varphi=0}^{\infty} \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5\omega)^{(n+0.5)}}{\Gamma(n+0.5)} \varphi^{-((n+0.5+a)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5\omega))\, d\varphi\,]^{-\frac{1}{a}} \quad (44)$$

By multiplying the integral in equation (44) by the quantity which equals to $(\frac{\Gamma(n+0.5+a)}{\Gamma(n+0.5+a)})$, where $\Gamma(.)$ is a gamma function .After some simplification, it yields

$$\hat{\varphi}_{GELF(3)} = [\frac{\Gamma(n+0.5+a)}{\Gamma(n+0.5)\ (\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5\omega)^{a}} B3(t,\varphi)]^{-\frac{1}{a}} \quad (45)$$

Where

$$B3(t,\varphi) = \int_{\varphi=0}^{\infty} \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5\omega)^{(n+0.5+a)}}{\Gamma(n+0.5+a)} \varphi^{-((n+0.5+a)+1)} \exp(-\frac{1}{\varphi}(\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5\omega))\, d\varphi = 1 \quad (46)$$

The equation (46) is the integral of the pdf of inverse gamma distribution [12] .So the Bayes estimator for φ will be

$$\hat{\varphi}_{GELF(3)} = [\frac{\Gamma(n+0.5+a)}{\Gamma(n+0.5)\ (\sum_{i=1}^{n} \frac{1}{t_i^2} + 0.5\omega)^{a}}]^{-\frac{1}{a}} \quad (47)$$

5. Simulation and Discussion

We used simulation study to compare between the maximum likelihood estimator(MLE) and the Bayes estimators for unknown scale parameter φ of an IRD. The simulation program was written in matlabR2018b.The data is generated for different samples sizes n= (15,25,50,75,100) from IRD using the quantile function from equation(2) as $t_i = (\frac{-1}{\varphi\ \ln(F_i)})^{1/2}$, where $F_i = U_i$ is a uniform distribution with (0,1) for several values of the true value parameter $\varphi = 2, 3, 5$ and the values for the hyper parameters $(\lambda, \eta, k, \omega)$ of the prior distributions can be chosen arbitrarily to compare the accuracy of the different estimates for φ as follows :

- The values for the parameters of the inverse Gamma prior have been selected arbitrarily as $(\lambda = 4, \eta = 3)$.
- The value for a parameter of the inverse chi-square prior have been selected arbitrarily as $(k = 4)$.
- The value for a parameter of the standard Levy prior have been selected arbitrarily as $(\omega = 0.6)$.

Based on the general entropy loss function (GELF) with the shape parameter can be chosen arbitrarily as (a=1, -1,2), with replications number of the experiments (r=5000) .In order to compare the accuracy of the different estimates for φ , we depend on the mean square error criterion, i.e. the estimates with the smallest MSE's will be the best estimates.

$$MSE = \frac{1}{5000} \sum_{r=1}^{5000} (\hat{\varphi}(r) - \varphi)^{2} \quad (28)$$

The results of simulation study listed in tables for each estimator and for all sample sizes.

Table 1. The estimated values of the maximum likelihood estimator ($\hat{\varphi}_{\text{MLE}}$) and MSE's for the estimators of the inverse Rayleigh distribution with the true values ($\varphi = 2,3,5$) and r =5000.

	true value ($\varphi = 2$)		true value ($\varphi = 3$)		true value ($\varphi = 5$)	
n	$\hat{\varphi}_{\text{MLE}}$	MSE	$\hat{\varphi}_{\text{MLE}}$	MSE	$\hat{\varphi}_{\text{MLE}}$	MSE
15	2.0065	0.2619	3.009	0.6111	4.9586	1.6449
25	1.9961	0.1630	2.9995	0.3544	5.0096	0.9754
50	1.9972	0.0788	2.9964	0.178	5.0122	0.5065
75	1.9951	0.0518	2.9998	0.1158	5.0068	0.3405
100	2.0006	0.0401	3.0043	0.0904	4.992	0.2462

For the results listed in **Table 1**, which represented by the estimated values of the maximum likelihood estimator ($\hat{\varphi}_{\text{MLE}}$) and MSE's .we see that the values of MSE are

- increased when the true value parameter φ is increased with fixed the sample size(n).
- decreased when the sample size(n) is increased with fixed value for the true value parameter φ .

Table 2. The estimated values of the Bayes estimators ($\hat{\varphi}_{\text{GELF}}$) and MSE's for the estimators of the inverse ayleigh distribution based on the GELF, under different priors with the true value ($\varphi = 2$) and r=5000.

	true value ($\varphi = 2$)	a= -1		a= 1		a= 2	
n	Bayes under	$\hat{\varphi}_{\text{GELF}}$	MSE	$\hat{\varphi}_{\text{GELF}}$	MSE	$\hat{\varphi}_{\text{GELF}}$	MSE
15	inverse gamma ($\lambda = 4, \eta = 3$)	2.0057	0.2039	1.8943	0.193	1.8438	0.1967
	inverse chi-square ($k = 4$)	2.0061	0.2302	1.8881	0.2164	1.8349	0.2198
	the standard Levy ($\omega = 0.6$)	2.0964	0.2895	1.9611	0.2467	1.9008	0.2402
25	inverse gamma ($\lambda = 4, \eta = 3$)	1.9964	0.1398	1.9251	0.1356	1.8916	0.1372
	inverse chi-square ($k = 4$)	1.9962	0.1507	1.9223	0.1458	1.8877	0.1474
	the standard Levy ($\omega = 0.6$)	2.0491	0.1721	1.9687	0.1576	1.9312	0.1555
50	inverse gamma ($\lambda = 4, \eta = 3$)	1.9973	0.0729	1.9596	0.0718	1.9414	0.0723
	inverse chi-square ($k = 4$)	1.9972	0.0758	1.9588	0.0746	1.9403	0.0751
	the standard Levy ($\omega = 0.6$)	2.0234	0.081	1.9833	0.0776	1.964	0.0771
75	inverse gamma ($\lambda = 4, \eta = 3$)	1.9952	0.0491	1.9697	0.0488	1.9572	0.0491
	inverse chi-square ($k = 4$)	1.9952	0.0504	1.9693	0.05	1.9566	0.0503
	the standard Levy ($\omega = 0.6$)	2.0125	0.0526	1.9859	0.0513	1.9728	0.0511
100	inverse gamma ($\lambda = 4, \eta = 3$)	2.0005	0.0385	1.9811	0.0381	1.9716	0.0382
	inverse chi-square ($k = 4$)	2.0006	0.0393	1.9809	0.0389	1.9713	0.039
	the standard Levy ($\omega = 0.6$)	2.0136	0.0407	1.9936	0.0397	1.9837	0.0395

Table 3. The estimated values of the Bayes estimators $(\hat{\varphi}_{GELF})$ and MSE's for the estimators of the inverse Rayleigh distribution based on the GELF, under different priors with the true value $(\varphi = 3)$ and r=5000.

	true value $(\varphi = 3)$	**a= -1**		**a=1**		**a= 2**	
n	Bayes under	$\hat{\varphi}_{GELF}$	MSE	$\hat{\varphi}_{GELF}$	MSE	$\hat{\varphi}_{GELF}$	MSE
15	inverse gamma $(\lambda = 4, \eta = 3)$	2.8903	0.4877	2.7297	0.4974	2.6569	0.5197
	inverse chi-square $(k = 4)$	2.946	0.5399	2.7727	0.5274	2.6945	0.5426
	the standard Levy $(\omega = 0.6)$	3.1335	0.6717	2.9313	0.5769	2.8411	0.5628
25	inverse gamma $(\lambda = 4, \eta = 3)$	2.9255	0.3094	2.821	0.3146	2.7719	0.3248
	inverse chi-square $(k = 4)$	2.9611	0.3292	2.8514	0.326	2.8000	0.3330
	the standard Levy $(\omega = 0.6)$	3.073	0.3744	2.9525	0.3429	2.8962	0.3386
50	inverse gamma $(\lambda = 4, \eta = 3)$	2.9581	0.1663	2.9023	0.1679	2.8753	0.171
	inverse chi-square $(k = 4)$	2.9769	0.1716	2.9196	0.171	2.892	0.1731
	the standard Levy $(\omega = 0.6)$	3.0328	0.1826	2.9727	0.1752	2.9437	0.1742
75	inverse gamma $(\lambda = 4, \eta = 3)$	2.9738	0.1105	2.9357	0.1112	2.9171	0.1126
	inverse chi-square $(k = 4)$	2.9866	0.1129	2.9478	0.1126	2.9289	0.1135
	the standard Levy $(\omega = 0.6)$	3.0239	0.1179	2.9839	0.1145	2.9643	0.114
100	inverse gamma $(\lambda = 4, \eta = 3)$	2.9846	0.0871	2.9557	0.0872	2.9414	0.0878
	inverse chi-square $(k = 4)$	2.9944	0.0886	2.965	0.0881	2.9506	0.0885
	the standard Levy $(\omega = 0.6)$	3.0224	0.0918	2.9924	0.0895	2.9776	0.0891

For the results listed in **Table 2** and **Table 3** which represented by the estimated values of the estimated values of the Bayes estimators $(\hat{\varphi}_{GELF})$ and MSE's for the estimators of the inverse Rayleigh distribution based on the GELF, under different priors .We see that the best Bayes estimates $(\hat{\varphi})$ of φ based on GELF with the shape parameter (a) where

- a=-1 , under the inverse gamma $(\lambda = 4, \eta = 3)$ and inverse chi-square $(k = 4)$, for all samples sizes(n).
- a=1 and a=2 , under the inverse gamma $(\lambda = 4, \eta = 3)$, inverse chi-square $(k = 4)$ and the standard Levy $(\omega = 0.6)$, for all samples sizes(n).

According to the smallest mean square error (MSE) compared with the MLE estimators.

Table 4. The estimated values of the Bayes estimators ($\hat{\varphi}_{\text{GELF}}$) and MSE's for the estimators of the inverse ayleigh distribution based on the GELF, under different priors with the true value ($\varphi = 5$) and r=5000.

	true value ($\varphi = 5$)	**a=-1**		**a=1**		**a=2**	
n	Bayes under	$\hat{\varphi}_{\text{GELF}}$	MSE	$\hat{\varphi}_{\text{GELF}}$	MSE	$\hat{\varphi}_{\text{GELF}}$	MSE
15	inverse gamma ($\lambda = 4, \eta = 3$)	4.6106	1.431	4.3544	1.5579	4.2383	1.6613
	inverse chi-square ($k = 4$)	4.7737	1.4955	4.4929	1.5365	4.3663	1.6098
	the standard Levy ($\omega = 0.6$)	5.1503	1.7811	4.818	1.572	4.6697	1.5547
25	inverse gamma ($\lambda = 4, \eta = 3$)	4.7867	0.8817	4.6157	0.9252	4.5354	0.9666
	inverse chi-square ($k = 4$)	4.8938	0.913	4.7126	0.9188	4.6277	0.945
	the standard Levy ($\omega = 0.6$)	5.1241	1.031	4.9231	0.9434	4.8294	0.9312
50	inverse gamma ($\lambda = 4, \eta = 3$)	4.8964	0.4789	4.804	0.4891	4.7593	0.5003
	inverse chi-square ($k = 4$)	4.9532	0.4889	4.8579	0.4884	4.8119	0.4947
	the standard Levy ($\omega = 0.6$)	5.0689	0.5214	4.9686	0.4974	4.9201	0.4932
75	inverse gamma ($\lambda = 4, \eta = 3$)	4.9287	0.3281	4.8655	0.3328	4.8346	0.3381
	inverse chi-square ($k = 4$)	4.9673	0.3326	4.9027	0.3324	4.8712	0.3354
	the standard Levy ($\omega = 0.6$)	5.0445	0.347	4.9776	0.3365	4.945	0.3346
100	inverse gamma ($\lambda = 4, \eta = 3$)	4.9333	0.241	4.8854	0.2451	4.8619	0.2489
	inverse chi-square ($k = 4$)	4.9624	0.2427	4.9137	0.244	4.8898	0.2464
	the standard Levy ($\omega = 0.6$)	5.0201	0.249	4.9701	0.2446	4.9456	0.2443

For the results listed in table 4 which represented by the estimated values of the estimated values of the Bayes estimators ($\hat{\varphi}_{\text{GELF}}$) and MSE's for the estimators of the inverse Rayleigh distribution based on the GELF, under different priors .We see that the best Bayes estimates ($\hat{\varphi}$) of φ based on GELF with the shape parameter (a) where

- a=-1 , under the inverse gamma ($\lambda = 4, \eta = 3$) and inverse chi-square ($k = 4$), for all samples sizes(n) .
- a=1 , under the inverse gamma ($\lambda = 4, \eta = 3$), inverse chi-square ($k = 4$) and the standard Levy ($\omega = 0.6$), for all samples sizes(n) .
- a=2 , under the inverse gamma ($\lambda = 4, \eta = 3$), inverse chi-square ($k = 4$) and the standard Levy ($\omega = 0.6$) ,for ($n \geq 50$)

According to the smallest mean square error (MSE) compared with the MLE estimators.

6. Conclusion

In this study, we derived Bayesian estimation of unknown scale parameter φ of the inverse Rayleigh distribution (IRD) based on general entropy loss function (GELF) under three different priors represented by the inverse Gamma, the inverse chi-square and the standard Levy as priors. In addition to the maximum likelihood estimator (MLE) of the scale parameter of IRD.

From the simulation study, we noted that Bayes estimators under the based on general entropy loss function (GELF) showed better performance than the maximum likelihood estimator (MLE) using the mean square error criterion (MSE). Also, the results showed that the posterior distribution obtained under the inverse gamma prior based on GELF with a=-1, the standard Levy prior based on GELF with a=2, and the inverse chi-squared prior based on GELF with a=1 corresponding to the true values of IRD provided more accurate results in terms of minimum MSE for all samples sizes(n).

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Conflict of Interest

No conflicts of interest.

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References

1. Soliman, A.; Essam, A. A.; Alaa, A.A. Estimation and prediction from inverse Rayleigh distribution based on lower record values. *Applied Mathematical Sciences*. **2010,** *4(62)*, 3057 - 3066.
2. Dey, S. Bayesian estimation of the parameter and reliability function of an inverse Rayleigh Distribution. *Malaysian Journal of Mathematical Sciences*. **2012**, *6(1)*, 113-124.
3. Shawky, A.I.; Badr, M.M. Estimations and prediction from the inverse Rayleigh model based on lower record statistics. *Life Science Journal*. **2012**, *9(1)*,985-990.
4. Sindhu, T.N.; Aslam, M.; Feroze, N. Bayes estimation of the parameters of the inverse Rayleigh distribution for left censored data. *Prob. Stat. Forum.* **2013**, *6*, 42-59.
5. Khan, M.S. Modified inverse Rayleigh distribution. *International Journal of Computer Applications* (0975 – 8887), **2014**, *87*(*13*),28-33. https://doi.org/10.17713/ajs.v44i3.21.
6. Fan, G. Bayes estimation for inverse Rayleigh model under different loss functions. *Research Journal of Applied Sciences, Engineering and Technology* **2015**, *9(12),* 1115-1118. https://doi.org/10.19026/rjaset.9.2605.
7. Malik,A.S.; Ahmad, S.P. A new inverse Rayleigh distribution: properties and application. *International Journal of Scientific Research in Mathematical and Statistical Sciences* **2018**,*5*(*5*),92-96. https://doi.org/10.26438/ijsrmss/v5i5.9296.
8. Pandey, T. N.; Sanat, P. Minimum risk estimation of scale parameter of inverse Rayleigh distribution under asymmetric loss function. *Journal of Computer and Mathematical Sciences* **2019**, *10(12)*,1665-1672.
9. Mudasir, S.; Ahmad, S.P.; Rehman, S. A. A comparative study for weighted Rayleigh distribution. *Journal of Reliability and Statistical Studies* **2021**,*14*(*1*), 243–262. https://doi.org/10.13052/jrss0974-8024.14112.
10. Mohsin, M.; Shahbaz, M.Q. Comparison of Negative Moment Estimator with Maximum Likelihood Estimator of Inverse Rayleigh Distribution. *PJSOR.* **2005**, *1(1)*,45-48. https://doi.org/10.18187/pjsor.v1i1.115.
11. Robert, C. P. The Bayesian Choice: From Decision-Theoretic Motivations to Computational Implementation. *2nd paperback ed., New York: Springer* **2007**.
12. Ahmad, A.; Ahmad, S.P. Weighted analogue of inverse gamma distribution: statistical properties, estimation and simulation study. *Pak.j.stat.oper.res*.**2019**, *15*(*1*) (1),25-37. https://doi.org/10.18187/pjsor.v15i1.2238.
13. Zhang, L.; Zhang, Y.-Y. The Bayesian Posterior and Marginal Densities of the Hierarchical Gamma–Gamma, Gamma–Inverse Gamma, Inverse Gamma–Gamma, and Inverse Gamma–Inverse Gamma Models with Conjugate Priors, *Mathematics*, **2022**,*10*(*21*), 4005. https://doi.org/10.3390/math10214005.

14. Kaewprasert, T.; Niwitpong, S.A.; Niwitpong, S. Confidence Intervals for the Ratio of the Coefficients of Variation of Inverse Gamma Distributions, *Applied Science and Engineering Progress* **2023**,*16(1)*, 5660. https://doi.org/10.14416/j.asep.2021.12.002.
15. Bickel, P.J.; Doksum, K. A. Mathematical Statistics: Basic Ideas and Selected Topics, *Holden- Day, Inc., San Francisco* **1977.**
16. Bickel, P.J.; Doksum, K. A. Mathematical Statistics: Basic Ideas and Selected Topics, *New York, Chapman and Hall/CRC, I*, 2nd ed. **2015**, https://doi.org/10.1201/b18312.
17. Puza, B. Bayesian methods for statistical analysis, The Australian National University Acton ACT 2601, Australia **2015**. http://doi.org/10.22459/BMSA.10.2015.
18. Bayes, R. T. Mathematical Statistics with Applications in R. Chapter 10, Bayesian estimation and inference, Copyright © *Elsevier Inc. All rights reserved* **2021.**
19. Ramos, P. L.; Mota, A.L.; Ferreira, P.H.; Ramos, E.; Tomazella, V. L. D.; Louzada F. Bayesian analysis of the Inverse Generalized Gamma Distribution using Objective Priors, *Journal of Statistical Computation and Simulation* **2020.** https://doi.org/10.1080/ 00949655.2020.1830991.
20. Lee, P. M.; Bayesian Statistics: An Introduction, 4th. Wiley, New York **2012** .
21. Ahmad, S.P.; Sultan, R.; Sultan, H. Bayesian Analysis of Normal Distribution-A Simulation Study, *International journal of advanced scientific and technical research* **2014**, *1(4)*.
22. Achcar, JA.; Coelho-Barros, EA.; Cuevas, JRT; Mazucheli, J. Use of Lèvy distribution to analyze longitudinal data with asymmetric distribution and presence of left censored data. *Communications for Statistical Applications and Methods* **2018**, *25(1)*, 43–60.doi: https://doi.org/10.29220/CSAM.2018.25.1.043.
23. Mobolaji, A. T.; Dibal, N. P.; Musa, Y. A. An Estimation of Unknown Variance of a Normal Distribution: Application to Borno State Rainfall Data, *International Journal of Data Science and Analysis* **2019**,*5(1)*, 1-5.https://doi.org/10.11648/j.ijdsa.20190501.11.
24. Dorniani, S.; Mohammadpour, A.; Nematollahi, N. Estimation of the Parameter of Levy Distribution Using Ranked Set Sampling, *AUT J. Math. Com.* **2021**,*2(1)*, 53-60. https://doi.org/10.22060/ajmc.2020.18499.1037.
25. Calabria, R.; Pulcini, G. An engineering approach to bayes estimation for the weibull distribution. *Micro-Electronics Reliability* **1994**,*34(5)*, 789–802. https://doi.org/10.1016/0026-2714(94)90004-3.
26. Kumar, D.; Kumar, P.; Singh, S. K.; Singh, U. A New Asymmetric Loss Function: Estimation of Parameter of Exponential Distribution, *Journal of Statistics Applications and Probability Letters* **2019**,*6(1)*, 37-50. https://doi.org/10.18576/jsapl/060105.
27. Kumari, R.; Tripathi, Y.M.; Sinha, R.K.; Wang, L. Comparison of Estimation Methods for Reliability Function for Family of Inverse Exponentiated Distributions under New Loss Function, *Axioms* **2023**,*12(12)*, 1096. https://doi.org/10.3390/axioms12121096.
28. Adegoke, T. M.; Obisesan, K.O.; Oladoja, O. M.; Adegoke, G.K.; Bayesian and Classical Estimations of Transmuted Inverse Gompertz Distribution, *RT and A* **2023**,*18(2)*,(*73*), 207-222. https://doi.org/10.24412/1932-2321-2023-273-207-222.
29. Ali S.; Aslam M.; Ali Kazmi S. M. A study of the effect of the loss function on Bayes Estimate, posterior risk and hazard function for Lindley distribution. *Applied Mathematical Modelling*, **2013,***37*(*8*), 6068–6078. https://doi.org/10.1016/j.apm.2012.12.008.
30. Goyal, T.; Rai, P.K.; Maurya, S.K.; Bayesian Estimation for Exponentiated Inverted Weibull distribution under Different Loss Functions, *International Journal of Pure and Applied Researches* **2019**, *2(1)*, 1-13.