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A New Iterative Algorithms for Finite Family of Resolvent Operators

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Abstract
Depending on the needs and requirements of keeping up with the scientific procession, researchers tend to find new recurrence schemes or develop previous recurrence schemes that will help researchers reach the fixed point and solution of variational inequality .The objective of this article is to provide novel approaches to finding a common fixed point of different types of important mappings and the set of zeros of maximal monotone operators .Also, we studied the convergence weakly and convergence strongly of the proposed iterative method under some suitable conditions. To achieve this goal, we will introduce a new technical method of resolvent operators and metric projection using different types of function sequences, including sequence of maximal monotone operators, sequence of $\mathcal{K}$-strictly pseudo-contractive mappings, and sequence of nonexpansive mappings defined on nonempty convex-closed subset of Hilbert space .
Keywords : Nonexpansive Mapping , Metric Projection , Strictly Pseudo Contraction Mapping , Strongly Pseudo-Contractive , Fixed Point .

1. Introduction

Let $\mathcal{H}$ be the Hilbert space and let $\mathcal{K}$ be a convex closed subset of $\mathcal{H}$. The metric projection of $\mathcal{H}$ onto $\mathcal{A}$ is denoted by $\mathcal{P}_{\mathbb{C}}$, while the set of fixed points of $\mathcal{T}$ is denoted as $\mathcal{F}(\mathcal{T})$. Keep in mind, if a constant $\mathcal{K} \in [0,1)$ exists, then there is a mapping $\mathcal{T}: \mathcal{K} \to \mathcal{H}$ that is $\mathcal{K}$ - strictly pseudo-contractive .

$$\|\mathcal{T}_{\omega} - \mathcal{T}_{w}\|^2 \leq \|\omega - w\|^2 + \mathcal{K}\|(I - \mathcal{T})_{\omega} - (I - \mathcal{T})_{w}\|^2 \;, \forall\, w \in \mathcal{K} \qquad (1)$$

When $\mathcal{K} = 0$, $\mathcal{T}$ is nonexpansive, and when $\mathcal{K} = 1$, $\mathcal{T}$ is pseudo-contractive. A strong pseudo-contractive $\mathcal{T}$ is one for which there exists a positive constant ζ between 0 and 1 such that $\mathcal{T} - \zeta I$ is pseudo-contractive. Without a doubt, the $\mathcal{K}$ - strictly pseudo-contractive class is a subset of the larger class that contains both pseudo-contractions and nonexpansive mapping. Furthermore, we highlight the fact that the class $\mathcal{K}$ - strictly pseudo-contractive mappings is unique from the other classes of strictly pseudo-contractive mappings(see [1-3]).
In [4] suggested the now-common Mann's iterative approach. Since then, researchers have looked at the $\mathcal{K}$ - strictly pseudo-contractive mapping strategy and the standard iterative process for constructing fixed points for nonexpansive mapping .When used iteratively, Mann's method yields the sequence ω_n, as seen below:

$\forall\, \omega_1 \in \mathcal{A}\,, \omega_{n+1} = (1-\sigma_n)\omega_n + \sigma_n \mathcal{T}\omega_n\,, n \geq 1$ (2)

Lots of studies have been rolled out on this subject, see[5-25] . In [3] established the first convergence result for $\mathcal{K}$ - strictly pseudo-contractive mapping self mapping in concrete Hilbert space. In subsequent work, [21] considered a varying control sequence denoted by the symbol $\{\sigma_n\}$ to provide a limited extension of the applicable result in[5]. Using the conditions $\sigma_1 = 1, 0 < \sigma_n < 1$, $\sum_{n=1}^{\infty} \sigma_n = \infty$, and the $\lim_{n\to\infty} sup \sigma_n = \sigma < 1 - \mathcal{K}$, he proved a convergence theorem using an algorithm (2). This operation was performed assuming that R's domain is compact and convex. This means that the compact condition on the domain of mapping $\mathcal{T}$ is required to derive the convergence results from Rhoades's convergence theorem.

Recently, a weak convergence theorem was shown [26] using an algorithm .

$\omega_{n+1} = \sigma_n\omega_n + (1-\sigma_n)\sum_{i=1}^{\mathbb{N}} \zeta_i \mathcal{T}_i \omega_n$ (3)

Despite the examples in [27] and [28] , is nevertheless shown that this convergence is frequently weak. In order to get high convergence, it is necessary to make adjustments to Mann's scheme (2). We shall restate the modification proposed by Nakajo and Takahashi for a nonexpansive mapping $\mathcal{T}$.

Consider the algorithm

$$\begin{cases} \omega_0 \in \mathbb{C} \\ \mathcal{W}_n = \sigma_n\omega_n + (1-\sigma_n)\mathcal{A}\omega_n \\ \mathbb{E}_n = \{s \in \mathbb{C}: \|\mathcal{W}_n - s\| \leq \|\omega_n - s\|\} \\ \mathcal{O}_n = \{s \in \mathbb{C}: \langle \omega_n - s, \omega_0 - \omega_n\rangle \geq 0\} \\ \omega_{n+1} = \mathcal{P}_{\ \mathbb{E}_n \cap \mathcal{O}_n} w_0 \end{cases} \quad (4)$$

Where $\mathcal{P}_{\mathbb{C}}$ stands for the metric projection from $\mathcal{H}$ onto $\mathbb{C}$.The convergence of the sequence$\{\omega_n\}$ generated by algorithm (3) to a fixed point of $\mathcal{T}$ is shown by Nakajo and Takahashi, under the condition that the control sequence $\{\sigma_n\}_{n=0}^{\infty}$ is selected so that$sup_{n\geq 0}\sigma_n < 1$ (i.e., $\{\sigma_n\}$ is limited away above from 0 and 1), where T is a fixed point of the Similar large convergence findings were published in the works [25-32].

2 . Preliminaries

If $\mathcal{H}$ is a real Hilbert space and that $\mathbb{C}$ is a nonempty closed convex subset of H.Then we give some necessary lemmas:

Lemma 2.1 [28] The following identities hold

(i) $\|\omega \mp w\|^2 = \|\omega\|^2 \mp 2\langle \omega, w\rangle + \|\omega\|^2$, $\forall \omega\,, w \in \mathcal{H}$

(ii) $\|t\omega + (1-t)w\|^2 = t\|\omega\|^2 + (1-t)\|w\|^2 - t(1-t)\|\omega - w\|^2$, $\forall\, t \in [0,1]$, $\forall\, \omega\,, w \in \mathcal{H}$

Lemma 2.2 [28] : If $z \in \mathcal{H}$ and $w \in \mathbb{C}$. Then

$w = \mathcal{P}_{\mathbb{C}}\omega$ iff there satisfies $\langle \omega - w\,, w - s\rangle \geq 0\,, \forall s \in \mathbb{C}$.

Lemma 2.3 [33] : Let $\{\omega_n\}$ be a sequence of nonnegative real numbers that satisfies the condition

$\alpha_{n+1} \leq (1-t_n)\alpha_n + \mathfrak{b}_n + 0(\mathfrak{e}_n), n \geq 1$

where $\{\mathfrak{e}_n\}$ satisfies the restrictions :

(i) $\mathfrak{e}_{n} \to 0 (n \to \infty)$
(ii) $\sum_{n=1}^{\infty} \mathfrak{b}_{n} < \infty$
(iii) $\sum_{n=1}^{\infty} \mathfrak{e}_{n} = \infty$

3. Main Result

In this section, we introduce a new technique f-point, and prove its strong-weak convergence

Theorem 3.1 : Let $\mathcal{M}_i$ be M.M. operator and $\langle \mathcal{T}^{i} \rangle$ be a sequence of $\mathcal{K}$ - strictly pseudo-contractive map on $\mathbb{C}$. Define the Technique as:

$$\begin{cases} \omega_0 = \omega \in \mathbb{C} \\ w_n = \sum_{\mathrm{i}=0}^{\mathcal{K}} \mathcal{J}_{r_{n,\mathrm{i}}}(\omega_n) \\ \omega_{n+1} = (\mathfrak{a}_n - \mathfrak{b}_n)\mathcal{F}(\omega) + (1-\mathfrak{a}_n)\sum_{\mathrm{i}=0}^{\mathcal{K}} \mathcal{T}^{i}(\omega_n) + \mathfrak{b}_n \mathcal{F}(w_n) \end{cases}$$

Where $\mathcal{F}$ is nonexpansive,$\langle r_n \rangle$ be a sequence in $(0,\infty)$, such that $\langle \mathfrak{a}_n \rangle$, $\langle \mathfrak{b}_n \rangle$ are sequence in $(0,\infty]$ and $\mathfrak{a}_n + \mathfrak{b}_n = 1$. If the following condition are satisfies :

(i) $\|w_n\|^2 \le \psi_n + \sum_{\mathrm{i}=1}^{\mathcal{K}} \left\|\mathcal{J}_{r_{n,\mathrm{i}}}(\omega_n)\right\|^2$ and $\sum_{i=0}^{\infty} \left(\mathfrak{b}_n \psi_n + \sum_{\mathrm{i}=1}^{\mathcal{K}} \left\|\omega_n - \mathcal{T}^{\mathrm{i}}\right\|^2 \right) < \infty$.

(ii) $\sum_{i=0}^{\infty} \mathfrak{a}_n = \infty$, $\sum_{\mathrm{i}=1}^{\infty} \|w_n\|^2 \le \infty$, *and* $\lim_{n\to\infty} \frac{\mathfrak{a}_n - \mathfrak{b}_n}{\mathfrak{a}_n} = 0$.

(iii) $\mathcal{M}_i^{-1}(0) \cap (\cap_{\mathrm{i}=1}^{\infty} \mathrm{fi}x(\mathcal{T}^{i})) \neq \emptyset$ *and* $\mathfrak{a}_n \ge \mathfrak{b}_n$ .Then the Technique f-point $\langle \omega_n \rangle$ converge to point $\mathcal{M}_i^{-1}(0) \cap \left(\cap_{\mathrm{i}=1}^{\infty} \mathrm{fi}x(\mathcal{T}^{i})\right)$.

Proof : Let $\mathfrak{r} \in \mathcal{M}_i^{-1}(0) \cap (\cap_{i=1}^{\infty} \mathrm{fi}x(\mathcal{T}^{i}))$

$\|\omega_1 - \mathfrak{r}\|^2 = \left\|(\mathfrak{a}_0 - \mathfrak{b}_0)\mathcal{F}(\omega) + (1-\mathfrak{a}_0)\sum_{\mathrm{i}=0}^{\mathcal{K}} \mathcal{T}^{i}(\omega_0) + \mathfrak{b}_0 \mathcal{F}(w_0) - \mathfrak{r}\right\|^2$

where $r = \left((\mathfrak{a}_0 - \mathfrak{b}_0) + (1-\mathfrak{a}_0) + \mathfrak{b}_0\right) r$

$\le (\mathfrak{a}_0 - \mathfrak{b}_0)\|\mathcal{F}(\omega) - \mathfrak{r}\|^2 + (1-\mathfrak{a}_0)\sum_{\mathrm{i}=0}^{\mathcal{K}} \left\|\mathcal{T}^{i}(\omega_0) - \mathfrak{r}\right\|^2 + \mathfrak{b}_0 \left\|\mathcal{F}(w_0) - \left((\mathfrak{a}_0 - \mathfrak{b}_0) + (1-\mathfrak{a}_0) + \mathfrak{b}_0\right) r\right\|^2$

$\le (\mathfrak{a}_0 - \mathfrak{b}_0)\|\mathcal{F}(\omega) - \mathfrak{r}\|^2 + (1-\mathfrak{a}_0)\sum_{\mathrm{i}=0}^{\mathcal{K}} \left\|\mathcal{T}^{i}(\omega_0) - \mathfrak{r}\right\|^2 + \mathfrak{b}_0 \|\mathcal{F}(w_0) - \mathfrak{r}\|^2$

But $\mathcal{T}^{i}$ *is* $\mathcal{K}$ - strictly pseudo-contractive

$\le (\mathfrak{a}_0 - \mathfrak{b}_0)\|\mathcal{F}(\omega) - \mathfrak{r}\|^2 + (1-\mathfrak{a}_0)\sum_{\mathrm{i}=0}^{\mathcal{K}} \|\omega_0 - \mathfrak{r}\|^2 + (1-\mathfrak{a}_0)\mathcal{K}\sum_{\mathrm{i}=0}^{\mathcal{K}} \left\|\omega_0 - \mathcal{T}^{i}(\omega_0) - \mathfrak{r} - (\mathcal{T}^{i}(\mathfrak{r})\right\|^2 + \mathfrak{b}_0 \psi_n + \mathfrak{b}_0 \sum_{\mathrm{i}=0}^{\mathcal{K}} \left\|\mathcal{J}_{r_{n,\mathrm{i}}}(\omega_0) - \mathfrak{r}\right\|^2$

Where $\mathcal{K} \in [0,1)$

$\le (1-\mathfrak{b}_0)\sum_{\mathrm{i}=0}^{\mathcal{K}} \|\omega - \mathfrak{r}\|^2 + (1-\mathfrak{a}_0)\sum_{\mathrm{i}=1}^{\mathcal{K}} \left\|\omega_0 - \mathcal{T}^{i}(\omega_0)\right\|^2 + \mathfrak{b}_0 \psi_n + \mathfrak{b}_0 \sum_{\mathrm{i}=0}^{\mathcal{K}} \|\omega_o - \mathfrak{r}\|^2$

$\le \sum_{\mathrm{i}=0}^{\mathcal{K}} \|\omega - \mathfrak{r}\|^2 + \mathfrak{b}_0 \psi_n + \sum_{\mathrm{i}=1}^{\mathcal{K}} \left\|\omega_0 - \mathcal{T}^{\mathrm{i}}(\omega_0)\right\|^2$

If $n = \mathcal{K}$, we have

$\|\omega_{\mathcal{K}} - \mathfrak{r}\|^2 < \sum_{i=0}^{\mathcal{K}} \|\omega - \mathfrak{r}\|^2 + \sum_{i=0}^{k-1} \mathfrak{b}_i \psi_i + \sum_{i=1}^{\mathcal{K}-1} \left\|\omega_i - \mathcal{T}^{\mathrm{i}}(\omega_{\mathrm{i}})\right\|^2$

If $n = \mathcal{K} + 1$, we have

$\|\omega_{\mathcal{K}+1} - \mathfrak{r}\|^2 = \left\|(\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\mathcal{F}(\omega) + (1-\mathfrak{a}_{\mathcal{K}})\sum_{\mathrm{i}=0}^{\mathcal{K}} \mathcal{T}^{i}(\omega_{\mathcal{K}}) + \mathfrak{b}_{\mathcal{K}} \mathcal{F}(w_{\mathcal{K}}) - \mathfrak{r}\right\|^2$

$\le (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\|\mathcal{F}(\mathfrak{a}) - \mathfrak{r}\|^2 + (1-\mathfrak{a}_{\mathcal{K}})\sum_{i=0}^{\mathcal{K}} \left\|\mathcal{F}^{\mathrm{i}}(w_{\mathcal{K}}) - \mathfrak{r}\right\|^2 + \mathfrak{b}_{\mathcal{K}} \|\mathcal{F}(w_{\mathcal{K}}) - \mathfrak{r}\|^2$

$\|\omega_{\mathcal{K}+1} - \mathfrak{r}\|^2 \le (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\|\omega - \mathfrak{r}\|^2 + (1 - \mathfrak{a}_{\mathcal{K}})\sum_{i=0}^{\mathcal{K}}\left[\|\omega_{\mathcal{K}} - \mathfrak{r}\|^2 + \mathcal{K}\left\|\omega_{\mathcal{K}} - \mathcal{T}^i(\omega_{\mathcal{K}}) - (\mathfrak{r} - \mathcal{T}^i(\mathfrak{r})\right\|^2\right] + \mathfrak{b}_{\mathcal{K}}\psi_{\mathcal{K}} + \mathfrak{b}_k\sum_{i=0}^{\mathcal{K}}\left\|\mathcal{J}_{r_{n,i}}(\omega_{\mathcal{K}}) - \mathfrak{r}\right\|^2$

$< (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\|\omega - \mathfrak{r}\|^2 + (1 - \mathfrak{a}_{\mathcal{K}})\sum_{i=0}^{\mathcal{K}}\|\omega_{\mathcal{K}} - \mathfrak{r}\|^2 + (1 - \mathfrak{a}_{\mathcal{K}})\sum_{i=0}^{\mathcal{K}}\left\|\omega_{\mathcal{K}} - \mathcal{T}^i(\omega_{\mathcal{K}})\right\|^2 + \mathfrak{b}_{\mathcal{K}}\psi_{\mathcal{K}} + \mathfrak{b}_{\mathcal{K}}\sum_{i=0}^{k}\|\omega_{\mathcal{K}} - \mathfrak{r}\|^2$

$< (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\|\omega - \mathfrak{r}\|^2 + \mathfrak{b}_{\mathcal{K}}\psi_k + \sum_{i=0}^{\mathcal{K}}\left\|\omega_k - \mathcal{T}^i(\omega_{\mathcal{K}})\right\|^2 + (1 - (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\sum_{i=0}^{\mathcal{K}}\|\omega_{\mathcal{K}} - \mathfrak{r}\|^2$

$< (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\|\omega - \mathfrak{r}\|^2 + \mathfrak{b}_{\mathcal{K}}\psi_k + \sum_{i=0}^{\mathcal{K}}\left\|\omega_k - \mathcal{T}^i(\omega_{\mathcal{K}})\right\|^2 + (1 - (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\sum_{i=0}^{\mathcal{K}}\|\omega_{\mathcal{K}} - \mathfrak{r}\|^2 + (1 - (\mathfrak{a}_{\mathcal{K}} - \mathfrak{b}_{\mathcal{K}})\ \sum_{i=0}^{\mathcal{K}-1}\mathfrak{b}_i\psi_i + \left\|\omega_i - \mathcal{T}^i(\omega_i)\right\|^2$

$\|\omega_{\mathcal{K}+1} - \mathfrak{r}\|^2 \le \sum_{i=0}^{\mathcal{K}}\|\omega - \mathfrak{r}\|^2 + \sum_{i=0}^{\mathcal{K}-1}\mathfrak{b}_i\psi_i + \sum_{i=1}^{\mathcal{K}-1}\left\|\omega_i - \mathcal{T}^i(\omega_i)\right\|^2$

But $\sum_{i=0}^{\mathcal{K}}\ (\mathfrak{b}_i\psi_i + \left\|\omega_i - \mathcal{T}^i(\omega_i)\right\|^2) < \infty$.

$\langle\omega_n\rangle$is bounded sequence .

Since $\langle\mathcal{J}_{r_{n,i}}(\omega_n)\rangle$ also bounded then there exist subsequence $\langle\mathcal{J}_{r_{n_{\mathcal{K}},i}}(\omega_{n_{\mathcal{K}}})\rangle$ of $\langle\mathcal{J}_{r_{n,i}}(\omega_n)\rangle$ converge weakly to $\mathcal{v}$.

Now, since $\mathcal{N}_{r_{n,i}}(\omega_n) = \frac{(\mathrm{I} - \mathcal{J}_{r_{n,i}})(\omega_n)}{r_n}$

$\lim_{n\to\infty}\left\|\mathcal{N}_{r_{n,i}}(\omega_n)\right\| = \lim_{n\to\infty}\left\|\frac{\omega_n - \mathcal{J}_{r_{n,i}}(\omega_n)}{r_n}\right\| = 0 \quad as\ r_n \to \infty$

And $\mathcal{N}_{r_{n,i}}(\omega_n) \in \mathcal{M}_i(\mathcal{J}_{r_{n,i}}(\omega_n))$,so,

$\langle z - \mathcal{J}_{r_{n_j,i}}(\omega_{n_j}), z' - \mathcal{N}_{r_{n_j,i}}(\omega_{n_j})\rangle \ge 0\ , \acute{z} \in \mathcal{M}_i(z)$

$\langle z - \mathcal{v}, z' - 0\rangle \ge 0\ , \quad \acute{z} \in \mathcal{M}_i(s) \quad n_j \to \infty$

Since $\mathcal{M}_i$ be M.M. operator,so

$0 \in \mathcal{M}_i(\mathcal{v}) \Rightarrow \mathcal{v} \in \mathcal{M}_i^{-1}(0) \Rightarrow \mathcal{v} \in \mathfrak{fix}(\mathcal{J}_{r_{n,i}})$. But $\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\| \to 0\ as\ n \to \infty$

Therefore, $\mathcal{v} \in (\bigcap_{i=1}^{\infty}\mathfrak{fix}(\mathcal{T}^i))$

$\|\omega_{n+1} - \mathcal{v}\|^2 = \left\|(\mathfrak{a}_n - \mathfrak{b}_n)\mathcal{F}(\omega) + (1 - \mathfrak{a}_n)\sum_{i=0}^{k}\mathcal{T}^i(\omega_n) + \mathfrak{b}_n\mathcal{T}(w_n) - \mathcal{v}\right\|^2$

$\le (\mathfrak{a}_n - \mathfrak{b}_n)\|\mathcal{F}(\omega) - \mathcal{v}\|^2 + (1 - \mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\left\|\mathcal{T}^i(\omega_n) - \mathcal{v}\right\|^2 + \mathfrak{b}_n\|\mathcal{F}(w_n) - \mathcal{v}\|^2$

$\le (\mathfrak{a}_n - \mathfrak{b}_n)\|\omega - \mathcal{v}\|^2 + (1 - \mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathcal{v}\|^2 + (1 - \mathfrak{a}_n)\mathcal{K}\sum_{i=0}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega_n) - (\mathcal{v} - \mathcal{T}^i(\mathcal{v})\right\|^2 + \mathfrak{b}_n\psi_n + \mathfrak{b}_n\|w_n - \mathcal{v}\|^2$

Hence, $\|\omega_{n+1} - \mathcal{v}\|^2 \le (1 - \mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathcal{v}\|^2 + \mathfrak{a}_n\left(\frac{\mathfrak{a}_n - \mathfrak{b}_n}{\mathfrak{a}_n}\|\omega - \mathcal{v}\|^2\right) + \|w_n - \mathcal{v}\|^2$

we get, $\|\omega_n - \mathcal{v}\| \to 0\ , n \to \infty$.And hence the Technique f-point converge strongly to $\mathcal{v}$ in $\mathcal{M}_i^{-1}(0) \cap (\bigcap_{i=1}^{\infty}\mathfrak{fix}(\mathcal{T}^i))$.

Theorem (2.2) : Let $\mathcal{M}_i$ be M.M. operator, $\langle\mathcal{T}^i\rangle$ be a sequence of $\mathcal{K}$ - strictly pseudo-contractive map on $\mathbb{C}$, $\langle\mathcal{F}_n\rangle$ be a sequence of nonexpansive mapping and $\langle\mathfrak{a}_n\rangle$,$\langle\mathfrak{b}_n\rangle$ are sequence in $(0, \infty]$ such that $\mathfrak{a}_n + \mathfrak{b}_n = 1$ and $\mathfrak{a}_n \ge \mathfrak{b}_n$. Define the Technique f-point as:

$$\begin{cases} w_n = \sum_{i=0}^{\mathcal{K}}\mathcal{J}_{r_{n,i}}(\omega_n) \\ \omega_{n+1} = (\mathfrak{a}_n - \mathfrak{b}_n)\mathcal{F}(\omega) + (1 - \mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\mathcal{T}^i(\omega_n) + \mathfrak{b}_n\mathcal{F}(w_n) \end{cases}$$

satisfies :

(i) $\|w_n\|^2 \leq \psi_n + \sum_{i=1}^{\mathcal{K}} \left\|\mathcal{J}_{r_{n,i}}(\omega_n)\right\|^2$ and $\sum_{i=0}^{\infty}\left(\mathfrak{b}_n\psi_n + \sum_{i=1}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2\right) < \infty$,where $\langle\psi_n\rangle$ be a sequence in $(0,\infty)$

(ii) $\mathcal{M}_i^{-1}(0) \cap (\cap_{i=1}^{\infty} \mathrm{fix}(\mathcal{T}^i)) \cap (\cap_{i=1}^{\infty} \mathrm{fix}(\mathcal{F}_n)) \neq \emptyset$ and $\mathfrak{a}_n \geq \mathfrak{b}_n$.

Then $\langle P_{\mathcal{M}_i^{-1}(0)}\omega_n\rangle$ converge strongly to point v in $\mathcal{M}_i^{-1}(0)$ and

$\lim_{n\to\infty}\|\omega_n - v\| = \inf\left\{\lim_{n\to\infty}\|\omega_n - \mathfrak{r}\| \ , \mathfrak{r} \in \mathcal{M}_i^{-1}(0)\right\}$.

Proof : Let $\mathfrak{r} \in (\cap_{i=1}^{\infty} \mathrm{fix}(\mathcal{T}^i)) \cap \mathcal{M}_i^{-1}(0)$

$\|\omega_{n+1} - \mathfrak{r}\|^2 = \left\|(\mathfrak{a}_n - \mathfrak{b}_n)\mathcal{F}(\omega) + (1-\mathfrak{a}_0)\sum_{i=0}^{\mathcal{K}}\mathcal{T}^i(\omega_n) + \mathfrak{b}_n\mathcal{F}(w_n) - \mathfrak{r}\right\|^2$

$\leq (\mathfrak{a}_n - \mathfrak{b}_n)\|\mathcal{T}(\omega_n) - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\left\|\mathcal{T}^i(\omega_n) - \mathfrak{r}\right\|^2 + \mathfrak{b}_n\|\mathcal{F}(w_n) - \mathfrak{r}\|^2$

$\|\omega_{n+1} - \mathfrak{r}\|^2 \leq (\mathfrak{a}_n - \mathfrak{b}_n)\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\mathcal{K}\left\|\omega_n - \mathcal{T}^i(\omega_n) - (\mathfrak{r} - \mathcal{T}^i(\mathfrak{r})\right\|^2 + \mathfrak{b}_n\|w_n - \mathfrak{r}\|^2$, wherever $\mathcal{K} = sup\{\ \mathcal{K}_i \in [0,1)\ , i \in \mathbb{N}$

$\leq (1-\mathfrak{b}_n)\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\mathcal{K}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2 + \mathfrak{b}_n\psi_n + \mathfrak{b}_n\sum_{i=0}^{\mathcal{K}}\left\|\mathcal{J}_{r_{n,i}}(\omega_n) - \mathfrak{r}\right\|^2$

$\leq (1-\mathfrak{b}_n)\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2 + \mathfrak{b}_n\psi_n + \sum_{i=0}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2 + \mathfrak{b}_n\psi_n + \mathfrak{b}_n\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2$

$\|\omega_{n+1} - \mathfrak{r}\|^2 < \sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2 + \mathfrak{b}_n\psi_n + \sum_{i=0}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2$

Since $\sum_{i=0}^{\infty}(\mathfrak{b}_n\psi_n + \left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2) < \infty$

we get , $\mathcal{R}(\mathfrak{r}) = \lim_{n\to\infty}\|\omega_n - \mathfrak{r}\|$ exist .This is $\langle\omega_n\rangle$ is bounded sequence .

Put $\mathcal{L} = \inf\left\{\mathcal{R}(\mathfrak{r}), \mathfrak{r} \in (\cap_{i=1}^{\infty} \mathrm{fix}(\mathcal{T}^i)) \cap \mathcal{M}_i^{-1}(0)\right\}$and $\mathbb{K} = \left\{\mathcal{W} \in (\cap_{i=1}^{\infty} \mathrm{fix}(\mathcal{T}^i)) \cap \mathcal{M}_i^{-1}(0) : \mathcal{R}(\mathcal{W}) = \mathcal{L}\right\}$

$\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\| \leq \|\omega_n - v\|$, $\forall v \in \mathcal{K}$

$\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\| \leq \mathcal{L}$ for all $n \in \mathbb{N}$.

To prove that $\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\| = \mathcal{L}$

Suppose that $\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\| < \mathcal{L}$. This implies

$\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 < \mathcal{L}^2$ .Then there exist $\sigma > 0$, such that

$\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 < \mathcal{L}^2 - \sigma\ \ \forall\, n \geq m\ , m \in \mathbb{N}$

$\left\|\omega_{n+\hbar+1} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 \leq \sum_{i=0}^{\mathcal{K}}\|\omega_{n+\hbar} - \mathfrak{r}\|^2 + \mathfrak{b}_{n+\hbar}\psi_{n+\hbar} + (\sum_{i=0}^{\mathcal{K}}\left\|\omega_{n+\hbar} - \mathcal{T}^i(\omega_{n+\hbar})\right\|^2)$

$\leq \left[\sum_{i=0}^{\mathcal{K}}\left\|\omega_{n+\hbar-1} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 + \mathfrak{b}_{n+\hbar-1}\psi_{n+\hbar-1} + (\sum_{i=0}^{\mathcal{K}}\left\|\omega_{n+\hbar-1} - \mathcal{T}^i(\omega_{n+\hbar-1})\right\|^2)\right] + \mathfrak{b}_{n+\hbar}\psi_{n+\hbar} - \mathcal{T}^i(\omega_{n+\hbar})$

$\leq \sum_{i=0}^{\mathcal{K}}\left\|\omega_{n+\hbar-2} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 + \mathfrak{b}_{n+\hbar-2}\psi_{n+\hbar-2} + (\sum_{i=0}^{\mathcal{K}}\left\|\omega_{n+\hbar-2} - \mathcal{T}^i(\omega_{n+\hbar-2})\right\|^2) + \mathfrak{b}_{n+\hbar-1}\psi_{n+\hbar-1} - \mathcal{T}^i(\omega_{n+\hbar-1})$

$\leq \left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 + \sum_{i=n}^{n+\hbar}(\mathfrak{b}_i\psi_i + \left\|\omega_i - \mathcal{T}^i\omega_i\right\|^2)$

This implies , for each $n \geq m$, $\hbar \in \mathbb{N}$ the following satisfied

$\mathcal{L}^2 \leq \lim_{n\to\infty}\left(\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 + \sum_{\mathrm{i}=n}^{n+\hbar}\left(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^{\mathrm{i}}\omega_{\mathrm{i}}\|^2\right)\right)$

$= \lim_{n\to\infty}\left(\left\|\omega_{n+\hbar+1} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 + \sum_{\mathrm{i}=n}^{n+\hbar}\left(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^{\mathrm{i}}\omega_{\mathrm{i}}\|^2\right)\right)$

$\leq \lim_{n\to\infty}\left(\mathcal{L}^2 - \sigma + \sum_{\mathrm{i}=n}^{n+\hbar}\left(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^{\mathrm{i}}\omega_{\mathrm{i}}\|^2\right)\right)$

But $\sum_{\mathrm{i}=n}^{\infty}\left(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^{\mathrm{i}}\omega_{\mathrm{i}}\|^2\right) < \infty, \forall\, n \geq m$

And $\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\|^2 < \mathcal{L}^2 - \sigma$.So, $\mathcal{L}^2 \leq \mathcal{L}^2 - \sigma < \mathcal{L}^2$, which is a contradiction

So , $\lim_{n\to\infty}\left\|\omega_n - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\right\| = \mathcal{L}$

Now,to prove that $\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n) \to v$

If not then there exists $\mathfrak{E} > 0$ such that , $\forall\, \hbar \in \mathbb{N}$,we have

$\left\|\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n) - v\right\| \geq \mathfrak{E}$, for some $\acute{\hbar} \geq \hbar$. If $\mathfrak{b} \geq 0$ such that $\mathfrak{b} < \sqrt{\mathcal{L}^2 + \frac{\mathfrak{E}^2}{8}} - \mathcal{L}$, $\hbar \in \mathbb{N}$

$\mathfrak{b} + \mathcal{L} < \sqrt{\mathcal{L}^2 + \frac{\mathfrak{E}^2}{8}} \implies (\mathfrak{b} + \mathcal{L})^2 < \mathcal{L}^2 + \frac{\mathfrak{E}^2}{8} \implies (\mathfrak{b} + \mathcal{L})^2 - \frac{\mathfrak{E}^2}{8} < \mathcal{L}^2$

$\sum_{\mathrm{i}=n}^{\infty}\left(\mathfrak{b}_n\psi_n + \|\omega_n - \mathcal{T}^i\omega_n\|^2\right) \leq \frac{\mathfrak{E}^2}{8}$

$\left\|\omega_{\acute{\hbar}} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})\right\| \leq \mathcal{L} + \mathfrak{b}$ and $\|\omega_{\acute{\hbar}} - v\| \leq \mathcal{L} + \mathfrak{b}$

$\left\|\omega_{n+\acute{\hbar}+1} - \frac{\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})+v}{2}\right\|^2 \leq \left\|\omega_{\acute{\hbar}} - \frac{\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})+v}{2}\right\|^2 + \sum_{\mathrm{i}=\acute{\hbar}}^{n+\acute{\hbar}}\left(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^{\mathrm{i}}(\omega_{\mathrm{i}})\|^2\right)$

$= \left\|\frac{2\omega_{\acute{\hbar}} - \left(\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})\right)+v}{2}\right\|^2 + \sum_{\mathrm{i}=\acute{\hbar}}^{n+\acute{\hbar}}(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^i(\omega_{\mathrm{i}})\|^2)$

$= \left\|\frac{\omega_{\acute{\hbar}} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})}{2} + \frac{\omega_{\acute{\hbar}} - v}{2}\right\|^2 + \sum_{\mathrm{i}=\acute{\hbar}}^{n+\acute{\hbar}}(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^i(\omega_{\mathrm{i}})\|^2)$

$= 2\left\|\frac{\omega_{\acute{\hbar}} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})}{2}\right\|^2 + 2\left\|\frac{\omega_{\acute{\hbar}} - v}{2}\right\|^2 - \left\|\frac{\omega_{\acute{\hbar}} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})}{2} + \frac{\omega_{\acute{\hbar}} - v}{2}\right\| + \sum_{\mathrm{i}=\acute{\hbar}}^{n+\acute{\hbar}}(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^i(\omega_{\mathrm{i}})\|^2)$

$= \frac{1}{2}\left\|\omega_{\acute{\hbar}} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{\hbar}})\right\|^2 + \frac{1}{2}\|\omega_{\acute{\hbar}} - v\|^2 - \frac{1}{4}\left\|P_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{h}}) - v\right\|^2 + \sum_{\mathrm{i}=\acute{\hbar}}^{n+\acute{\hbar}}(\mathfrak{b}_{\mathrm{i}}\psi_{\mathrm{i}} + \|\omega_{\mathrm{i}} - \mathcal{T}^i(\omega_{\mathrm{i}})\|^2)$

$\left\|\omega_{n+\acute{\hbar}+1} - \frac{P_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{h}})+v}{2}\right\|^2 \leq \frac{1}{2}(\mathcal{L} + \mathfrak{b})^2 + \frac{1}{2}(\mathcal{L} + \mathfrak{b})^2 - \frac{1}{4}\mathfrak{E}^2 + \frac{\mathfrak{E}^2}{8} = (\mathcal{L} + \mathfrak{b})^2 + \frac{\mathfrak{E}^2}{8}$

As $n \to \infty$ we get

$\mathcal{L}^2 \leq \lim_{n\to\infty}\left\|\omega_{n+\acute{\hbar}+1} - \frac{P_{\mathcal{M}_i^{-1}(0)}(\omega_{\acute{h}})+v}{2}\right\|^2 \leq (\mathcal{L} + \mathfrak{b})^2 - \frac{\mathfrak{E}^2}{8} < \mathcal{L}^2$

Which is a contraction .So, $\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n) \to v$.That is $\langle\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)\rangle$ converge strongly to point in $\mathcal{M}_i^{-1}(0)$

Theorem(2.3): Let $\mathbb{C}$, $\mathcal{M}_{\mathrm{i}}$, $\langle\mathcal{F}_n\rangle$, $\langle\mathcal{T}^i\rangle$ and $\langle\omega_n\rangle$ as in theorem 2.2 , $\langle\mathfrak{b}_n\rangle$ be a sequence in (0,1] and $\langle\mathfrak{a}_n\rangle$ be a sequence in $[\mathfrak{a}, \mathfrak{b}]$ such that $o < \mathfrak{a} < \mathfrak{b} < 1$ *and* $\mathfrak{a}_n + \mathfrak{b}_n = 1$. If $\mathfrak{a}_n \geq \mathfrak{b}_n$ and $\lim_{n\to\infty}\mathfrak{b}_n = 0$ then the technique f-point $\langle\omega_n\rangle$ define

$$\begin{cases} w_n = \sum_{i=0}^{\mathcal{K}} \mathcal{J}_{r_{n,i}}(\omega_n) \\ \omega_{n+1} = (\mathfrak{a}_n - \mathfrak{b}_n)\mathcal{F}(\omega_n) + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}} \mathcal{T}^i(\omega_n) + \mathfrak{b}_n \mathcal{F}(w_n) \end{cases}$$

With two conditions :

(i) $\|w_n\|^2 \le \psi_n + \sum_{i=1}^{k} \left\|\mathcal{J}_{r_{n,i}}(\omega_n)\right\|^2$ and $\sum_{i=0}^{\infty}\left(\mathfrak{b}_n\psi_n + \sum_{i=1}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2\right) < \infty$,where $\langle\psi_n\rangle$ be a sequence in $(0,\infty)$.

(ii) $\mathcal{M}_i^{-1}(0) \cap (\cap_{i=1}^{\infty} \mathfrak{fix}(\mathcal{T}^i)) \cap (\cap_{i=1}^{\infty} \mathfrak{fix}(\mathcal{F}_n)) \ne \emptyset$ *and* $\mathfrak{a}_n \ge \mathfrak{b}_n$.
has converges weakly to a point $v \in \mathcal{M}_i^{-1}(0)$ where $\lim_{n\to 0} \langle \mathcal{P}_{\mathcal{M}_i^{-1}(0)} x_n \rangle = v$.

Proof : Let $\mathfrak{r} \in \mathcal{M}_i^{-1}(0) \cap (\cap_{i=1}^{\infty} \mathfrak{fix}(\mathcal{T}^i)) \cap (\cap_{i=1}^{\infty} \mathfrak{fix}(\mathcal{F}_n))$

$\|\omega_{n+1} - \mathfrak{r}\|^2 = \left\|(\mathfrak{a}_n - \mathfrak{b}_n)\mathcal{F}(\omega_n) + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}} \mathcal{T}^i(\omega_n) + \mathfrak{b}_n\mathcal{F}(w_n) - \mathfrak{r}\right\|^2$

$\le (\mathfrak{a}_n - \mathfrak{b}_n)\|\mathcal{F}(\omega_n) - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}}\left\|\mathcal{T}^i(\omega_n) - \mathfrak{r}\right\|^2 + \mathfrak{a}_n\|\mathcal{F}(w_n) - \mathfrak{r}\|^2$

$\|\omega_{n+1} - \mathfrak{r}\|^2 \le (\mathfrak{a}_n - \mathfrak{b}_n)\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{k}\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}} \mathcal{K}\big\|\omega_n - \mathcal{T}^i(\omega_n) - (\mathfrak{r} - \mathcal{T}^i(\mathfrak{r})\big\|^2 + \mathfrak{b}_{\mathcal{K}}\|w_n - \mathfrak{r}\|^2$, where $\mathcal{K} = sup\ \{\ \mathcal{K}_i \in [0,1)\ , i \in \mathbb{N}$

$\le (1-\mathfrak{b}_n)\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\sum_{i=0}^{\mathcal{K}} \mathcal{K}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2 + \mathfrak{b}_n\psi_n + \mathfrak{b}_n\sum_{i=0}^{\mathcal{K}}\left\|\mathcal{J}_{r_{n,i}}(\omega_n) - \mathfrak{r}\right\|^2$

$\le (1-\mathfrak{b}_n)\sum_{i=0}^{k}\|\omega_n - \mathfrak{r}\|^2 + \mathfrak{b}_n\psi_n + \sum_{i=0}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2 + \mathfrak{b}_n\psi_n + \mathfrak{b}_n\sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2$

$\|\omega_{n+1} - \mathfrak{r}\|^2 < \sum_{i=0}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2 + \mathfrak{b}_n\psi_n + \sum_{i=0}^{\mathcal{K}}\left\|\omega_n - \mathcal{T}^i(\omega)\right\|^2$

Since $\sum_{i=0}^{\infty}(\mathfrak{b}_n\psi_n + \sum_{i=0}^{\infty}\left\|\omega_n - \mathcal{T}^i(\omega_n)\right\|^2) < \infty$,we get , $\lim_{n\to\infty}\|\omega_n - \mathfrak{r}\|$ exist .

So that, the technique f-point $\langle\omega_n\rangle$ is bounded ,so $\exists\ \langle\omega_{n_\mathcal{K}}\rangle$ subsequence of $\langle\omega_n\rangle$, such that , $\omega_{n_\mathcal{K}} \rightharpoonup v$.

Now, put $\mathcal{Q}_n = \{\|\omega_n - \mathfrak{r}\|.\|w_n - \mathfrak{r}\|, n \in \mathbb{N}\}$ and $\mathcal{K} \in [0,1)$ s.t

$(1-\mathcal{K})\mathfrak{a}_n\|\omega_n - w_n\|^2 \le (1-\mathfrak{a}_n)\|\omega_n - w_n\|^2$

$\le (1-\mathfrak{a}_n)\|\omega_n - \mathfrak{r}\|^2 + (1-\mathfrak{a}_n)\|w_n - \mathfrak{r}\|^2 + 2(1-\mathfrak{a}_n)\mathcal{Q}_n$

$\le \mathfrak{b}_n\|\omega_n - \mathfrak{r}\|^2 + \mathfrak{b}_n\left[\psi_n + \sum_{i=o}^{\mathcal{K}}\left\|\mathcal{J}_{r_{n,i}}(\omega_n) - \mathfrak{r}\right\|^2\right] + 2\mathfrak{b}_n\mathcal{Q}_n$

$\le \mathfrak{b}_n\|\omega_n - \mathfrak{r}\|^2 + \mathfrak{b}_n\left[\psi_n + \sum_{i=o}^{\mathcal{K}}\|\omega_n - \mathfrak{r}\|^2\right] + 2\mathfrak{b}_n\mathcal{Q}_n$

$(1-\mathcal{K})\mathfrak{a}_n\|\omega_n - w_n\|^2 \le \frac{\mathfrak{b}_n}{\mathfrak{a}_n}\{\sum_{i=o}^{k}\|\omega_n - \mathfrak{r}\|^2 + \psi_n + 2\mathfrak{b}_n\mathcal{Q}_n\}$

So, $\|\omega_n - w_n\| \to 0$ as $n \to \infty$,since $\omega_{n_\mathcal{K}} \rightharpoonup v$. Then $w_{n_\mathcal{K}} \rightharpoonup v$

Now, since $w_n = \sum_{i=0}^{\mathcal{K}} \mathcal{J}_{r_{n,i}}(\omega_n)$, so $\langle z - \mathcal{J}_{r_{n\mathcal{K},i}}(\omega_{n\mathcal{K}}), z' - \mathbb{N}_{r_{n\mathcal{K},i}}(\omega_{n\mathcal{K}})\rangle \ge 0$, $\acute{z} \in \mathcal{M}_i(z)$

$\lim_{n\to\infty}\left\|\mathbb{N}_{r_{n,i}}(\omega_n)\right\| = \lim_{n\to\infty}\left\|\frac{\omega_n - \mathcal{J}_{r_{n,i}}(\omega_n)}{r_n}\right\| = 0\quad as\ r_n \to \infty$

$\langle z - v, \acute{z}\rangle \ge 0$, $\acute{z} \in \mathcal{M}_i(z)$. as $\mathcal{K} \to \infty$.Then, $\mathcal{P}_{\mathcal{M}_i^{-1}(0)}\omega_n \to \acute{v} \in \mathcal{M}_i^{-1}$

Hence,$\langle\omega_{n\mathcal{K}} - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{n\mathcal{K}}), t - \mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_{n\mathcal{K}})\rangle \le 0$, $t \in \mathcal{M}_i^{-1}(0)$

So, $\langle v - \acute{v}, t - \acute{v}\rangle \le 0$, $t \in \mathcal{M}_i^{-1}(0)$.But $v \in \mathcal{M}_i^{-1}(0)$. Then

$\langle v - \acute{v}, t - \acute{v}\rangle \le 0 \Longrightarrow \|v - \acute{v}\|^2 \le 0$. That is $v = \acute{v}$

Therefore, the technique f-point $\langle\omega_n\rangle$ has convergence-w to the limit point of $\mathcal{P}_{\mathcal{M}_i^{-1}(0)}(\omega_n)$.

4. Conclusion

1. A new technical methods of resolvent operators and metric projection of strictly pseudo contraction mapping are introduced
2. The convergence weakly and convergence strongly of the proposed iterative method under some suitable conditions are studied.
3. A common fixed points of different types of important mappings and the set of zeros of maximal monotone operators are found.

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Conflict of Interest

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