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The Continuous Classical Optimal Control Problems for Quaternary Elliptic Partial Differential Equations

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Abstract

In this paper, the Quaternary Continuous Classical Optimal Control Problem (QCCOCP) for the Quaternary Linear Elliptic Partial Differential Equations (QLEPDEqs) is studied. The mathematical model for the proposed problem is formulated, and it consists of the QLEPDEqs, the Objective Function (OF), and the set of state controls. The method of Galerkin (MG) is used to prove the existence theorem of a unique state vector solution (QSVS) of the Weak Form (WF) for the QLEPDEqs when the Quaternary Classical Continuous Control Vector (QCCCV) is fixed. Furthermore, the existence of a Quaternary Classical Continuous Optimal Control Vector (QCCOCV) ruled by the QLEPDEqs is stated and proved. The Quaternary Adjoint Equations (QAJEqs) associated with the QLEPDEqs are formulated and then studied. The Fréchet Derivative (FD) for the OF is derived. Finally, the necessary condition theorem (NCTH) for the optimality of the QCCOCP is proved.

Keywords: Quaternary Continuous Classical Optimal Control Vector, Quaternary Partial Differential Equations, Objective Function, and Adjoint Equations.

1. Introduction

Optimal control problems have an essential role in important areas of applied mathematics that relate to many important aspects of life. One of the important applications of life is in medicine [1,2], aircraft [3,4], economics [5,6], robotics [7,8], weather conditions [9,10], biology [11,12], Aerospace [13-14], Electrical Machines [15-16], and many other important applications[17-21].

Because of this importance, many researchers have been interested in studying optimal control problems related to Nonlinear Ordinary Differential Equations (NLODEs) [22], or those related to different types of NLPDEs like parabolic, hyperbolic, and elliptic [23–25], or those related to NLOPDEs of a couple of these three kinds [26]. In the current work, the study of the optimal control problem is motivated to deal with the study of the QCCOCP related to the QLEPDEqs. The mathematical model for the proposed problem is formulated; the MG is used to study and prove the existence theorem for a unique QSVS for the Wf of the QLEPDEqs for fixed CCOCV. The existence theorem for a CCOCV associated with the QLEPDEqs is stated and proved. The

QAEqs related to the QLEPDEqs are formulated and then studied. The FD of the OF is derived; finally, the NCTH is stated and proved.

2. Description of the Problem

Let Ω be an abounded and open connected subset in R^2 with a Lipschitz (LIP) boundary $\partial\Omega$ in the QCCOCP, including the QLEPDEqs:

$$-\Delta y_1 + y_1 + y_2 + y_3 - y_4 = b_1(x) + u_1 \quad (1)$$

$$-\Delta y_2 - y_1 + y_2 + y_3 - y_4 = b_2(x) + u_2 \quad (2)$$

$$-\Delta y_3 - y_1 - y_2 + y_3 - y_4 = b_3(x) + u_3 \quad (3)$$

$$-\Delta y_4 + y_1 + y_2 + y_3 + y_4 = b_4(x) + u_4 \quad (4)$$

With a Drichlet Boundary Condition(DBC)

$$y_i(x) = 0 \ , \qquad \forall\, i = 1,2,3,4 \qquad in\ \partial\Omega \quad (5)$$

$\vec{y} = (y_1, y_2, y_3, y_4) \in \left(H_0^2(\Omega)\right)^4$ is the QSVS , $\vec{u} = (u_1, u_2, u_3, u_4) \in \left(L^2(\Omega)\right)^4$ is the QCCCV and $b_i(x) \in (L^2(\Omega))^4\ \forall\, i = 1,2,3,4$, is give, $\forall x = (x_1, x_2) \in \Omega$.

The set of ACV is $\vec{U} \subset (L^2(\Omega))^4$, s.t.

$$\vec{U} = \left\{(\vec{u} \in (L^2(\Omega))^4 \mid (u_1, u_2, u_3, u_4) \in \vec{U} \subset R^4\ a.e\ in\ \Omega\right\}.$$

Where $\overrightarrow{U} = U_1 \times U_2 \times U_3 \times U_4$ is a convex set

The OF is

$$J_0(\vec{u}) = \frac{1}{2}\sum_{i=1}^{n}\|y_i - y_{id}\|_0^2 + \frac{\alpha}{2}\sum_{i=1}^{n}\|u_i\|_0^2 \quad \forall \vec{u} \in \overrightarrow{U}. \quad (6)$$

Where α is a positive real number, $\vec{y}$ is the QSVS of (1-5) corresponding to the QCCCV $\vec{u}$ and $(y_{1d}, y_{2d}, y_{3d}, y_{4d})$ is the desired data.

The QCCOCP is: $J_0\left(\vec{\bar{u}}\right) = \underset{\vec{u}\in\vec{U}}{Min}\ J_0(\vec{u})$.

3. The WF of the QLEPDEqs:

To obtain the WF of problem (1-5) consider:

$$\boldsymbol{W} = W_1 \times W_2 \times W_3 \times W_4 = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) = \left(H_0^1(\Omega)\right)^4$$

$$= \left\{\vec{w}: \vec{w} = (w_1, w_2, w_3, w_4) \in \left(H_0^1(\Omega)\right)^4\ with\ w_i = 0\ \ on\ \partial\Omega, \forall i = 1,2,3,4\right\}.$$

MBS of (1-4) by $w_i \in W_i$ ($i = 1,2,3,4$) resp., then, integrating w.r.t. x . And finally using the generalized Greens theorem for the first term in the L.H.S of the four obtained equations, to get:

$$(\nabla y_1, \nabla w_1) + (y_1, w_1) + (y_2, w_1) + (y_3, w_1) - (y_4, w_1) = (b_1, w_1) + (u_1, w_1) \quad (7)$$

$$(\nabla y_2, \nabla w_2) - (y_1, w_2) + (y_2, w_2) + (y_3, w_2) - (y_4, w_2) = (b_2, w_2) + (u_2, w_2) \quad (8)$$

$$(\nabla y_3, \nabla w_3) - (y_1, w_3) - (y_2, w_3) + (y_3, w_3) - (y_4, w_3) = (b_3, w_3) + (u_3, w_3) \quad (9)$$

$$(\nabla y_4, \nabla w_4) + (y_1, w_4) + (y_2, w_4) + (y_3, w_4) + (y_4, w_4) = (b_4, w_4) + (u_4, w_4) \quad (10)$$

By blending to gather equations (7-10), one gets:

$$B(\vec{y}, \vec{w}) = \check{A}(\vec{w}) \quad (11)$$

Where $B(.,.): \overrightarrow{W} \times \overrightarrow{W} \to R$ is a bilinear form and $\check{A}(.): \overrightarrow{W} \to R$ is a linear from, such that (s.t.)

$B(\vec{y},\vec{w}) = (\nabla y_1, \nabla w_1) + (y_1, w_1) + (y_2, w_1) + (y_3, w_1) - (y_4, w_1) + (\nabla y_2, \nabla w_2) - (y_1, w_2) + (y_2, w_2) + (y_3, w_2) - (y_4, w_2) + (\nabla y_3, \nabla w_3) - (y_1, w_3) - (y_2, w_3) + (y_3, w_3) - (y_4, w_4) + (\nabla y_4, \nabla w_4) + (y_1, w_4) + (y_2, w_4) + (y_3, w_4) + (y_4, w_4)$, and

$\check{A}(\vec{w}) = (b_1 + u_1, w_1) + (b_2 + u_2, w_2) + (b_3 + u_3, w_3) + (b_4 + u_4, w_4).$

The following hypotheses (HYPs) are required in the study of the existence of a unique QSVS of the WF (11).

3.1 HYPs

1) $B(.,.)$ satisfies the following:

a) $B(\vec{y},\vec{w})$ is coercive, i.e.

$\frac{B(\vec{y},\vec{y})}{\|\vec{y}\|_1} = \|\vec{y}\|_1 > 0, \quad \vec{y} \in \vec{W} \Rightarrow B(\vec{y},\vec{y}) = \|\vec{y}\|_1^2 \to \infty \quad as \ \|\vec{y}\|_1 \to \infty.$

b) $B(\vec{y},\vec{w})$ is continuous, i.e. $\exists\ \epsilon_1 > 0$, s.t. $|B(\vec{y},\vec{w})| \le \epsilon_1 \|\vec{y}\|_1 \|\vec{w}\|_1, \forall \vec{y}, \vec{w} \in \vec{W}$.

2) $\check{A}(\vec{w})$ is a bounded on $\vec{W}$, where $\vec{u} \in (L^2(\Omega))^4$ is bounded, i.e. $\exists\ \epsilon_2 > 0\ s.t.$ $|\check{A}(\vec{w})| \le \epsilon_2 \|w\|_1, \forall\ \vec{w} \in \vec{W}$.

The MG is used to find the approximation solution (app. sl.) of the WF (11) which is found through choosing a finite subspace $\vec{W_n} \subset \vec{W}$ (where W_n be the set piece wise affine functions (PWAFs) in Ω), therefore (11), will reduced to the following app. problem (app. pro.)

$$B(\vec{y}_n, \vec{w}) = \check{A}(\vec{w}), \forall\ \vec{y}_n, \vec{w} \in \vec{W_n} \tag{12}$$

Theorem 3.1 [27] For each $\vec{w} \in \vec{W}$, there is a sequence $\{ \vec{\psi}_n \}$, with $\vec{\psi}_n \in \vec{W_n}$ for each n, s.t $\vec{\psi}_n \to \vec{w}$ strongly (ST) in $\vec{W}$.

3.2 Existence and Uniqueness Solution of the WF Theorem 3.2

For every fixed QCCCV $\vec{u} \in (L^2(\Omega))^4$, the WF (12) has a unique app. sl. $\vec{y}_n \in \vec{W_n}$.

Proof: Let $\{\vec{\psi}_1, \vec{\psi}_2 \dots \dots \vec{\psi}_n\}$ be a basis of $\vec{W_n}$ for each n, with $n = 4N$ and let

$$\vec{y}_n = \vec{y}_n(x_1,x_2) = \sum_{j=1}^n C_j \vec{\psi}_j(x_1,x_2)$$
$$= \left(\sum_{j=1}^n C_j \vec{\psi}_{1j}, \sum_{j=1}^n C_j \vec{\psi}_{2j}, \sum_{j=1}^n C_j \vec{\psi}_{3j}, \sum_{j=1}^n C_j \vec{\psi}_{4j}\right) \tag{13}$$

Where $\vec{\psi}_j = (a_1\psi_k, a_2\psi_k, a_3\psi_k, a_4\psi_k)$.

Where $a_1 = \left(1 - \frac{L+(2Lmod3)}{3}\right)$, $a_2 = \left(\frac{1}{2}(Pmod3)(Lmod3)\right)$, $a_3 = \left(\frac{1+(Lmod3)-(Pmod3)}{3}\right)$, $a_4 = \left(\frac{(2Pmod3)+P}{3} - 1\right)$, for $L = 0,1,2,3$, $P = L + 1 = 1,2,3,4$, $K = 1,2, \dots \dots N$

$\hat{\jmath} = K + N\left[\left((P-1)L\right)mod4\right] + N\left[\frac{L(L-1)}{2}\right]$, and C_j is an unknown constant for each $j = 1,2, \dots, n$

By substituting $\vec{y}_n$ from (13), with $\vec{w} = \vec{\psi}_i$ in (11), to get

$$B\left(\sum_{j=1}^n C_j \vec{\psi}_j, \vec{\psi}_i\right) = \check{A}(\vec{\psi}_i) \qquad \forall\ i = 1,2, \dots n \tag{14}$$

Equation (14) can be rewritten as the following linear system:

$$AC = b \tag{15}$$

Where $A = (a_{ij})_{n\times n}$, $a_{ij} = B(\psi_j, \psi_i)$, $\forall i,j = 1,2 \dots, N$, $b = (b_1, b_2, \dots, b_n)^T$, $b_i = \check{A}(\vec{\psi}_i)$, $\forall i = 1,2, \dots, N$, and $C = (c_1, c_2, \dots, c_n)^T$.

Now, let

$AC = 0 = 0 \Rightarrow \sum_{j=1}^n C_j a_{ij} = 0, \forall i = 1,2, \dots n,$

$$\Rightarrow B\left(\sum_{j=1}^N C_j \vec{\psi}_j, \vec{\psi}_i\right) = 0\ , \quad \forall i = 1,2, \dots n \tag{16}$$

From HYP 3.1(1-a), once get that:

$$\left\|\sum_{j=1}^{n} C_j\vec{\psi}_j\right\|_1^2 = B(\sum_{j=1}^{n} C_j\vec{\psi}_j , \sum_{j=1}^{n} C_j\vec{\psi}_j) = B(\sum_{j=1}^{n} C_j\vec{\psi}_j , \sum_{j=1}^{n} C_i\vec{\psi}_i)$$

$$= \sum_{j=1}^{n} C_i B\,(\sum_{j=1}^{n} C_j\vec{\psi}_j, \vec{\psi}_i) = 0 \text{ , by (16)}$$

$$\Rightarrow \sum_{j=1}^{n} C_j\vec{\psi}_j = 0.$$

But $\{ \vec{\psi}_1, \vec{\psi}_2, \dots\dots\dots \vec{\psi}_n \}$ are linearly independent, thus there exists $C_j = 0, \quad \forall\, j = 1,2, \dots n$, which means equation (15) has a unique solution.

Now, from the WF (12) and theorem (3.2), one gets that there exists a sequence of the WF,

$$B(\vec{y}_n, \vec{\psi}_n) = \check{A}(\vec{\psi}_n), \qquad \forall\, \vec{y}_n , \vec{\psi}_n \in \vec{W}_n ,\ \forall\, n \tag{17}$$

Which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$.

Theorem 3.3 :(Existence and Uniqueness Solution of the WF)

The sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$ (of the sequence of WF (17)) converges strongly to $\vec{y}$ (the unique solution of (11)).

Proof: Since$\vec{y}_n$ is a solution of (17), then from hypo.3.1 (1-a and 2), one gets:

$$\|\vec{y}_n\|_1^2 = B(\vec{y}_n, \vec{y}_n) \le \left|\check{A}(\vec{y}_n)\right| \le \epsilon_2 \|\vec{y}_n\|_1.$$

$$\therefore \|\vec{y}_n\|_1 \le \epsilon_3 \ where\ \epsilon_3 = \frac{\epsilon_2}{\epsilon} > 0 \quad \forall n$$

i.e. $\{\vec{y}_n\}$ is bounded in $\vec{W}$, ∀ n then by the Alaglou theorem, there exists a subsequence of $\{\vec{y}_n\}$ (for simplicity say again $\{\vec{y}_n\}$), such that $\vec{y}_n \to \vec{y}$ weakly (WK) in $\vec{W}$)

Now, we have the following two steps:

First, since $\vec{y}_n \to \vec{y}$ WK in $\vec{W}$ and $\vec{\psi}_n \to \vec{w}$ ST in $\vec{W}$ then

$$\left|B(\vec{y}_n, \vec{\psi}_n) - B(\vec{y}, \vec{w})\right| \le \left|B(\vec{y}_n, \vec{\psi}_n - \vec{w}\right| + |B(\vec{y}_n - \vec{y}, \vec{w})|$$

$$\le \epsilon_1 \|\vec{y}_n\|_1 \left\|\vec{\psi}_n - \vec{w}\right\|_1 + |B(\vec{y}_n - \vec{y}, \vec{w})| \to 0$$

$$\Rightarrow B(\vec{y}_n, \vec{\psi}_n) \to B(\vec{y}, \vec{w}).$$

Second, since $\vec{\psi}_n \to \vec{w}$ WK in $\vec{W}$ **,** then $\check{A}(\vec{\psi}_n) \to \check{A}(\vec{w})$.

From the above two steps, we conclude that $B(\vec{y}, \vec{w}) = \check{A}(\vec{w}) \quad \forall \vec{w} \in \vec{W}$.

Thus$\vec{y}$ is solution of (11) .

To prove $\vec{y}_n \to \vec{y}$ ST in $\vec{W}$, from HYP 3.1 (1-a), one has

$$\|\vec{y} - \vec{y}_n\|_1^2 = B(\vec{y} - \vec{y}_n, \vec{y} - \vec{y}_n) = B(\vec{y} - \vec{y}_n, \vec{y}) - B(\vec{y}, \vec{y}_n) + B(\vec{y}_n, \vec{y}_n)$$

$$= B(\vec{y} - \vec{y}_n, \vec{y}) + \check{A}(\vec{y}) - \check{A}(\vec{y}_n) \to 0$$

i.e. $\|\vec{y} - \vec{y}_n\| = 0$. Thus $\vec{y}_n \to \vec{y}$.

The Uniqueness of the Solution:

Let $\vec{y}$, $\vec{y}_n$ be two solutions of (11), i.e.

$$B(\vec{y}_1, \vec{w}) = \check{A}(\vec{w}) , \qquad \forall \vec{w} \in \vec{W}$$

$$B(\vec{y}_2, \vec{w}) = \check{A}(\vec{w}), \qquad \forall \vec{w} \in \vec{W}$$

Subtract the second above equation from the first one, and then setting $\vec{w} = \vec{y}_1 - \vec{y}_2$, one gets

$B(\vec{y}_1 - \vec{y}_2, \vec{y}_1 - \vec{y}_2) = 0 , \quad \forall\, \vec{w} \in \vec{W}$, i.e.

From HYP 3.1 (1-a), one obtains: $\vec{y}_1 = \vec{y}_2$.

4. Existence of a QCCOCV

Lemma 4.1: The operator $\vec{u} \to \vec{y}_{\vec{u}}$ from $(L^2(\Omega))^4$ is LIP continuous, i.e.

$\left\|\overrightarrow{\delta y}\right\|_0 \le \hat{c} \left\|\overrightarrow{\delta u}\right\|_0$, for $\hat{c} > 0$.

Proof: Let $\vec{ú} = (ú_1, ú_2, ú_3, ú_4)$ be a given QCCCV of the WF (7-10) and $\vec{ý} = (ý_1, ý_2, ý_3, ý_4)$ its corresponding QSVS, i.e.

$$(\nabla ý_1, \nabla w_1) + (ý_1, w_1) + (ý_2, w_1) + (ý_3, w_1) - (ý_4, w_1) = (b_1, w_1) + (ú_1, w_1) \quad (18)$$

$$(\nabla ý_2, \nabla w_2) - (ý_1, w_2) + (ý_2, w_2) + (ý_3, w_2) - (ý_4, w_2) = (b_2, w_2) + (ú_2, w_2) \quad (19)$$

$$(\nabla ý_3, \nabla w_3) - (ý_1, w_3) - (ý_2, w_3) + (ý_3, w_3) - (ý_4, w_3) = (b_3, w_3) + (ú_3, w_3) \quad (20)$$

$$(\nabla ý_4, \nabla w_4) + (ý_1, w_4) + (ý_2, w_4) + (ý_3, w_4) + (ý_4, w_4) = (b_4, w_4) + (ú_4, w_4) \quad (21)$$

By subtracting equations (7 -10) from (18-21) resp. then substituting $\delta y_i = ý_i - y_i$, $\delta u_i = ú_i - u_i$, $\forall i = 1,2,3,4$ in the obtained equations, we get:

$$(\nabla \delta y_1, \nabla w_1) + (\delta y_1, w_1) + (\delta y_2, w_1) + (\delta y_3, w_1) - (\delta y_4, w_1) = (\delta u_1, w_1) \quad (22)$$

$$(\nabla \delta y_2, \nabla w_2) - (\delta y_1, w_2) + (\delta y_2, w_2) + (\delta y_3, w_2) - (\delta y_4, w_2) = (\delta u_2, w_2) \quad (23)$$

$$(\nabla \delta y_3, \nabla w_3) - (\delta y_1, w_3) - (\delta y_2, w_3) + (\delta y_3, w_3) - (\delta y_4, w_3) = (\delta u_3, w_3) \quad (24)$$

$$(\nabla \delta y_4, \nabla w_4) + (\delta y_1, w_4) + (\delta y_2, w_4) + (\delta y_3, w_4) + (\delta y_4, w_4) = (\delta u_4, w_4) \quad (25)$$

Blending together these equalities, setting $w_i = \delta y_i, \forall i = 1,2,3,4$ in (22-25) resp., applying HYP3.1(1-a), then using the Cauchy Schwarz inequality (C-S-I) to the R.H.S. to obtain:

$$\|\overrightarrow{\delta y}\|_1^2 \le \|\delta u_1\|_0 \|\delta y_1\|_0 + \|\delta u_2\|_0 \|\delta y_2\|_0 + \|\delta u_3\|_0 \|\delta y_3\|_0 + \|\delta u_4\|_0 \|\delta y_4\|_0 \quad (26)$$

Since $\|\delta y_i\|_0 \le \|\overrightarrow{\delta y}\|_0 \le c\|\overrightarrow{\delta y}\|_1$ *and* $\|\delta u_i\|_0 \le \|\overrightarrow{\delta u}\|_0 \quad \forall i = 1,2,3,4$, then (26) gives, $\|\overrightarrow{\delta y}\|_1 \le \check{c}\|\overrightarrow{\delta u}\|_0$, with $\hat{c} = \frac{3c}{\epsilon}$.

So the operator $\vec{u} \to \vec{y}_{\vec{u}}$ is (LICS) on $\left(L^2(\Omega)\right)^4$.

Lemma 4.2[28]: The norm $\|.\|_0$ is W L Sc.

Lemma 4.3: The OF in (6) is W L Sc.

Proof: since $\vec{u}_n \to \vec{u}$ WK in $\left(L^2(\Omega)\right)$ then (by lemma 4.1), $\vec{y}_n \to \vec{y}$ WK in $\left(L^2(\Omega)\right)$ which gives by lemma 4.2, $\|\vec{y} - \vec{y}_n\|_0^2$ is W L Sc.

i .e $J_0(\vec{u})$ is W L Sc.

Lemma 4.4[11]: The norm $\|.\|_0^2$ is strictly convex.

Remark 4.1: From Lemma 4.4, one can conclude that $J_0(\vec{u})$ is strictly convex.

Theorem 4.1: If $J_0(\vec{u})$ is coercive, then there exists a QCCOCV for the CCOCVP.

Proof: From the convexity of $\vec{U}$, and the coercivity of $J_0(\vec{u})$, with $J_0(\vec{u}) \ge 0$ there exist a minimizing sequence $\{\vec{u}_n\} \in \vec{U}\ \forall n$ s.t.: $\lim_{n\to\infty} J_0(\vec{u}_n) = \inf_{\vec{u}\in\vec{U}} J_0(\vec{u})$.

Hence, there exists a constant $\epsilon > 0$, s.t.

$$\|\vec{u}_n\| \le \epsilon \quad \forall n \quad (27)$$

Then by ALTH, there exists a subsequence of $\{\vec{u}_n\}$ s.t $\vec{u}_n \to \vec{u}$ WK in $\left(L^2(\Omega)\right)^4$.

But for each QCCCV $\vec{u}_n$ $(\forall n)$ the SVEs has a unique QSVS $\vec{y}_n$.

Now, using (27), HYPs 3. (1-a and 2) and the C-S-I, it yields:

$$\|\vec{y}_n\|_1^2 = B(\vec{y}_n, \vec{y}_n) = \breve{A}(\vec{y}_n)$$
$$\le \|b_1\|_0\|y_{1n}\|_0 + \|u_{1n}\|_0\|y_{1n}\|_0 + \|b_2\|_0\|y_{2n}\|_0 + \|u_{2n}\|_0\|y_{2n}\|_0 + \|b_3\|_0\|y_{3n}\|_0 + \|u_{3n}\|_0\|y_{3n}\|_0 + \|b_4\|_0\|y_{4n}\|_0 + \|u_{4n}\|_0\|y_{4n}\|_0$$
$$\le h_1\|y_{1n}\|_0 + \epsilon_1\|y_{1n}\|_0 + h_2\|y_{2n}\|_0 + \epsilon_2\|y_{2n}\|_0 + h_3\|y_{3n}\|_0 +$$

$$\epsilon_3\|y_{3n}\|_0 + h_4\|y_{4n}\|_0 + \epsilon_4\|y_{4n}\|_0$$
$$\leq 2(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)\|\vec{y}_n\|_1 = \varpi\|\vec{y}_n\|_1.$$

Where $\gamma_1 = max(h_1, \epsilon_1)$, $\gamma_2 = max(h_2, \epsilon_2)$, $\gamma_3 = max(h_3, \epsilon_3)$, $\gamma_4 = max(h_4, \epsilon_4)$ and $\varpi = max(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$.

Then $\|\vec{y}_n\|_1 \leq \gamma$, for each n, with $\gamma = \frac{\varpi}{\epsilon} > 0$.

By ALTH there exists a subsequence of $\{\vec{y}_n\}$ s.t $\vec{y}_n \to \vec{y}$ WK in $\overrightarrow{W}$

Since for each n, $\vec{y}_n$ satisfies the WF (11), then $\forall \vec{w} = (w_1, w_2, w_3, w_4) \in \overrightarrow{W}, \forall n$

$$(\nabla y_{1n}, \nabla w_1) + (y_{1n}, w_1) + (y_{2n}, w_1) + (y_{3n}, w_1) - (y_{4n}, w_1) + (\nabla y_{2n}, \nabla w_2) - (y_{1n}, w_2) + (y_{2n}, w_2) + (y_{3n}, w_2) - (y_{4n}, w_2) + (\nabla y_{3n}, \nabla w_3) - (y_{1n}, w_3) - (y_{2n}, w_3) + (y_{3n}, w_3) - (y_{4n}, w_3) + (\nabla y_{4n}, \nabla w_4) + (y_{1n}, w_4) + (y_{2n}, w_4) + (y_{3n}, w_4) + (y_{4n}, w_4) = (b_{1n}, w_1) + (u_{1n}, w_1) + (b_{2n}, w_2) + (u_{2n}, w_2) + (b_{3n}, w_3) + (u_{3n}, w_3) + (b_{4n}, w_4) + (u_{4n}, w_4) \quad (28)$$

To show (28) converges to the following equations:

$$(\nabla y_1, \nabla w_1) + (y_1, w_1) + (y_2, w_1) + (y_3, w_1) - (y_4, w_1) + (\nabla y_2, \nabla w_2) - (y_1, w_2) + (y_2, w_2) + (y_3, w_2) - (y_4, w_2) + (\nabla y_3, \nabla w_3) - (y_1, w_3) - (y_2, w_3) + (y_3, w_3) - (y_4, w_3) + (\nabla y_4, \nabla w_4) + (y_1, w_4) + (y_2, w_4) + (y_3, w_4) + (y_4, w_4) = (b_1, w_1) + (u_1, w_1) + (b_2, w_2) + (u_2, w_2) + (b_3, w_3) + (u_3, w_3) + (b_4, w_4) + (u_4, w_4), \ \vec{w} \in \overrightarrow{W}, \forall n \quad (29)$$

First

Since $y_{in} \to y_i$ WK in $\overrightarrow{W}$ then from theorem 3.2, $y_{in} \to y_i$ ST in $\overrightarrow{W}$, which gives $y_{in} \to y_i$, and $\frac{\partial y_{in}}{\partial x_i} \to \frac{\partial y_i}{\partial x_i}$ ST in $L^2(\Omega)$, and by using the C-S-I and HYP 3.1 (1-b)

$$|(\nabla y_{1n}, \nabla w_1) + (y_{1n}, w_1) + (y_{2n}, w_1) + (y_{3n}, w_1) - (y_{4n}, w_1) + (\nabla y_{2n}, \nabla w_2) - (y_{1n}, w_2) + (y_{2n}, w_2) + (y_{3n}, w_2) - (y_{4n}, w_2) + (\nabla y_{3n}, \nabla w_3) - (y_{1n}, w_3) - (y_{2n}, w_3) + (y_{3n}, w_3) - (y_{4n}, w_3) + (\nabla y_{4n}, \nabla w_4) + (y_{1n}, w_4) + (y_{2n}, w_4) + (y_{3n}, w_4) + (y_{4n}, w_4) - (\nabla y_1, \nabla w_1) - (y_1, w_1) - (y_2, w_1) - (y_3, w_1) + (y_4, w_1) - (\nabla y_2, \nabla w_2) + (y_1, w_2) - (y_2, w_2) - (y_3, w_2) + (y_4, w_2) - (\nabla y_3, \nabla w_3) + (y_1, w_3) + (y_2, w_3) - (y_3, w_3) + (y_4, w_3) - (\nabla y_4, \nabla w_4) - (y_1, w_4) - (y_2, w_4) - (y_3, w_4) - (y_4, w_4)|$$
$$\leq \|\nabla y_{1n} - \nabla y_1\|_0\|w_1\|_0 + \|y_{1n} - y_1\|_0\|w_1\|_0 + \|y_{2n} - y_2\|_0\|w_1\|_0 + \|y_{3n} - y_3\|_0\|w_1\|_0 + \|y_{4n} - y_4\|_0\|w_1\|_0 + \|\nabla y_{2n} - \nabla y_2\|_0\|w_2\|_0 + \|y_{1n} - y_1\|_0\|w_2\|_0 + \|y_{2n} - y_2\|_0\|w_2\|_0 + \|y_{3n} - y_3\|_0\|w_2\|_0 + \|y_{4n} - y_4\|_0\|w_2\|_0 + \|\nabla y_{3n} - \nabla y_3\|_0\|w_3\|_0 + \|y_{1n} - y_1\|_0\|w_3\|_0 + \|y_{2n} - y_2\|_0\|w_3\|_0 + \|y_{3n} - y_3\|_0\|w_3\|_0 + \|y_{4n} - y_4\|_0\|w_3\|_0 + \|\nabla y_{4n} - \nabla y_4\|_0\|w_4\|_0 + \|y_{1n} - y_1\|_0\|w_4\|_0 + \|y_{2n} - y_2\|_0\|w_4\|_0 + \|y_{3n} - y_3\|_0\|w_4\|_0 + \|y_{4n} - y_4\|_0\|w_4\|_0 \longrightarrow 0 .$$

Second, the convergence for the R.H.S of (28) to the L.H.S of (29) is obtained through $u_{in} \to u_i \quad \forall i = 1,2,3,4$ WK in $L^2(\Omega)$.

Then from these two steps of convergences, (28) converges to (29).

Since $J_0(\vec{u})$ is WLSc (by Lemma 4.3) and $\vec{u}_n \to \vec{u}$ WK in $(L^2(\Omega))^4$, then

$$J_0(\vec{u}) \leq \lim_{n\to\infty} \inf_{\vec{u}_n \in \vec{U}} J_0(\vec{u}_n) = \lim_{n\to\infty} J_0(\vec{u}_n) = \inf_{\vec{u}\in\vec{U}} J_0(\vec{u}) \Rightarrow J_0(\vec{u}) = \min_{\vec{u}\in\vec{U}} J_0(\vec{u})$$

$\Rightarrow \overrightarrow{u}$ is QCCOCV.

To prove the uniqueness:

Let $\vec{u}_1, \vec{u}_2 \in \overrightarrow{U}$ be two QCCOCV of $J_0(\vec{u})$, then $\frac{\vec{u}_1}{2} + \frac{\vec{u}_2}{2} \in \vec{U}$, and

$J_0\left(\frac{\vec{u}_1}{2} + \frac{\vec{u}_2}{2}\right) \leq \frac{1}{2} J_0(\vec{u}_1) + \frac{1}{2} J_0(\vec{u}_2) = J_0(\vec{u})$, C!

Then the uniqueness is obtained from lemma 4.4.

5. The NCTh for Optimality

Theorem 5.1: Consider the OF (2.6) and the QAEqs (z_1, z_2, z_3, z_4) of the QLEPDEq (2.1-2.5) are given by:

$$-\Delta z_1 + z_1 - z_2 - z_3 + z_4 = (y_1 - y_{1d}) \quad (30)$$

$$-\Delta z_2 + z_1 + z_2 - z_3 + z_4 = (y_2 - y_{2d}) \quad (31)$$

$$-\Delta z_3 + z_1 + z_2 + z_3 + z_4 = (y_3 - y_{3d}) \quad (32)$$

$$-\Delta z_1 - z_1 - z_2 - z_3 + z_4 = (y_4 - y_{4d}) \quad (33)$$

$$z_i = 0 \quad \forall i = 1,2,3,4 \quad on\ \partial\Omega \quad (34)$$

Then the FD of J_0 is given by $\left(\acute{J}_0(\vec{u}), \overrightarrow{\delta u}\right) = \left(\vec{z} + \overrightarrow{\alpha u}, \overrightarrow{\delta u}\ \right)$.

Proof: Rewriting the QAEqs ((30) -(34)) by their following WF

$$(\nabla z_1, \nabla w_1) + (z_1, w_1) - (z_2, w_1) - (z_3, w_1) + (z_4, w_1) = (y_1 - y_{1d}, w_1) \quad (35)$$

$$(\nabla z_2, \nabla w_2) + (z_1, w_2) + (z_2, w_2) - (z_3, w_2) + (z_4, w_2) = (y_2 - y_{2d}, w_2) \quad (36)$$

$$(\nabla z_3, \nabla w_3) + (z_1, w_3) + (z_2, w_3) + (z_3, w_3) + (z_4, w_3) = (y_3 - y_{3d}, w_3) \quad (37)$$

$$(\nabla z_4, \nabla w_4) - (z_1, w_4) - (z_2, w_4) - (z_3, w_4) + (z_4, w_4) = (y_4 - y_{4d}, w_4) \quad (38)$$

By blending (35-38) together, we get

$$(\nabla z_1, \nabla w_1) + (z_1, w_1) - (z_2, w_1) - (z_3, w_1) + (z_4, w_1) + (\nabla z_2, \nabla w_2) + (z_1, w_2) + (z_2, w_2) - (z_3, w_2) + (z_4, w_2) + (\nabla z_3, \nabla w_3) + (z_1, w_3) + (z_2, w_3) + (z_3, w_3) + (z_4, w_3) + (\nabla z_4, \nabla w_4) - (z_1, w_4) - (z_2, w_4) - (z_3, w_4) + (z_4, w_4)$$
$$= (y_1 - y_{1d}, w_1) + (y_2 - y_{2d}, w_2) + (y_3 - y_{3d}, w_3) + (y_4 - y_{4d}, w_4) \quad (39)$$

The WF (39) has a unique solution $(z_1, z_2, z_3, z_4) = (z_{1u1}, z_{2u2}, z_{3u3}, z_{4u4}) \in \overrightarrow{W}$ (this can be proved by the same way used in the proof of theorem 3.2).

Now, substituting $w_i = \delta z_i$ in ((35) – (38)) $\forall i = 1,2,3,4$, then subtracting each obtained equations from those each obtained from substituting $w_i = z_i$ in ((22) – (25)), we get :

$$(z_2, \delta y_1) + (z_3, \delta y_1) + (\delta y_2, z_1) + (\delta y_3, z_1) - (\delta y_4, z_1) - (z_4, \delta y_1) =$$
$$(\delta u_1, z_1) - (y_1 - y_{1d}\ , \delta y_1) \quad (40)$$

$$-(\delta y_1, z_2) - (z_1, \delta y_2) + (\delta y_3, z_2) + (z_3, \delta y_2) - (\delta y_4, z_2) - (z_4, \delta y_2) =$$
$$(\delta u_2, z_2) - (y_2 - y_{2d}, \delta y_2) \quad (41)$$

$$-(\delta y_1, z_3) - (\delta y_2, z_3) - (\delta y_4, z_3) - (z_1, \delta y_3) - (z_2, \delta y_3) - (z_4, \delta y_3) =$$
$$(\delta u_3, z_3) - (y_3 - y_{3d}, \delta y_3) \quad (42)$$

$$(\delta y_1, z_4) + (\delta y_2, z_4) + (\delta y_3, z_4) + (z_1, \delta y_4) + (z_2, \delta y_4) + (z_3, \delta y_4) =$$
$$(\delta u_4, z_4) - (y_4 - y_{4d}, \delta y_4) \quad (43)$$

Blending together the above quaternary equations, we get:

$$(\delta u_1, z_1) + (\delta u_2, z_2) + (\delta u_3, z_3) + (\delta u_4, z_4) =$$
$$(y_1 - y_{1d}, \delta y_1) + (y_2 - y_{2d}, \delta y_2) + (y_3 - y_{3d}, \delta y_3) + (y_4 - y_{4d}, \delta y_4) \quad (44)$$

On the other hand, the OB becomes:

$$J_0\left(\vec{u} + \overrightarrow{\delta u}\right) = \frac{1}{2} \iint_\Omega \sum_{i=1}^4 (y_i + \delta y_i - y_{id})^2\, dx + \frac{\alpha}{2} \iint_\Omega \sum_{i=1}^4 (u_i - u_{id})^2\, dx$$

But by using (44), we have:

$$J_0(\vec{u}+\overrightarrow{\delta u})-J_0(\vec{u})=(z_1+\alpha u_1,\delta u_1)+(z_2+\alpha u_2,\delta u_2)+(z_3+\alpha u_3,\delta u_3)$$
$$+(z_4+\alpha u_4,\delta u_4)+\tfrac{1}{2}\left\|\overrightarrow{\delta y}\right\|_0^2+\tfrac{\alpha}{2}\left\|\overrightarrow{\delta u}\right\|_0^2 \qquad (45)$$

Using lemma 4.1, we get:

$\frac{1}{2}\left\|\overrightarrow{\delta y}\right\|_0^2+\frac{\alpha}{2}\left\|\overrightarrow{\delta u}\right\|_0^2=\epsilon(\overrightarrow{\delta u})\left\|\overrightarrow{\delta u}\right\|_0$, where $\epsilon(\overrightarrow{\delta u}) \to 0 \quad as \ \left\|\overrightarrow{\delta u}\right\|_0 \to 0$.

Hence (45), becomes:

$$J_0(\vec{u}+\overrightarrow{\delta u})-J_0(\vec{u})=(\vec{z}+\alpha\vec{u},\overrightarrow{\delta u})+\epsilon(\overrightarrow{\delta u})\left\|\overrightarrow{\delta u}\right\|_0$$

where $\epsilon(\overrightarrow{\delta u}) \to 0 \quad as \ \left\|\overrightarrow{\delta u}\right\|_0 \to 0$.

From the FD for J_0 , one concludes that

$(\acute{J}_0(\vec{u}),\overrightarrow{\delta u})=\ (\vec{z}+\overrightarrow{bu},\overrightarrow{\delta u})$.

Theorem5.2: If the QCCCV of (1-5) is optimal, the $\acute{J}_0(\vec{u})=\vec{z}+\alpha\vec{u}=0$ with $\vec{y}=\vec{y}_{\vec{u}}$ and $\vec{z}=\vec{z}_{\vec{u}}$.

Proof: If $\vec{u}$ is an QCCOCV of the problem, then

$J_0(\vec{u})=\min_{\vec{n}\in\vec{U}} J_0(\vec{v})\ ,\forall\vec{v}\in(L^2(\Omega))^4,\quad$ i .e $\acute{J}_0(\vec{u})=0$

$\Rightarrow \ \vec{z}+\alpha\,\vec{u}=0 \qquad \Rightarrow \quad \vec{u}=-\frac{\vec{z}}{b(x)}$, with $\overrightarrow{\delta u}=\vec{v}-\vec{u}$

Then the NCO is $(\acute{J}_0\,(\vec{u})\,,\overrightarrow{\delta u}\,)\ \geq 0$, $\Rightarrow \ (\vec{z}+\alpha\,\vec{u}\,,\overrightarrow{\delta u})\ \geq 0$

$\Rightarrow(\vec{z}+\alpha\,\vec{u},\vec{u})\ \leq\ (\vec{z}+\alpha\vec{u},\vec{v})\quad \forall\vec{v}\ \in(L^2(\Omega))^4$.

6. Conclusion

The mathematical model for the "new" proposed problem is formulated. The existence and uniqueness theorem for a QSVS of the WF from the QLEPDEqs is stated and proved successfully by using the GM when the QCCCV is given. Furthermore, the existence of a QCCOCV ruled by the QLEPDEqs is stated and proven. The mathematical formulations for the QAEqs, which are related to the QLEPDEqs, are formulated and then studied. The FD for the OF is derived. Finally, the NCTH "for optimality" is proved for this problem.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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