IHJPAS.37 (2) 2024
Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407

Statistical Propoerties and Application for [0,1] Truncated Nadarajah-Haghighi Exponential Distribution

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Received 19 March 2023 Accepted 14 May 2023 Published 20 April 2024
doi.org/10.30526/37.2.3349

Abstract
There is a desperate need for extended versions of the classical distributions. There have been attempts to find novel families of probability distributions that widen existing families and provide great flexibility in data modeling in a number of application areas, including lifetime analysis, finance, and insurance. In this paper, we introduce a new family of distributions based on [0,1] Truncated and propose a new extension for the exponential distribution. The new distribution is called Truncated Nadarajah-Haghighi Distribution, symbolized with {[0,1]TNHE}. This study aims to derive some statistical properties for the new distribution, such as the quantile function, the mixture representation for the probability density function, the moments, the incomplete moments, the stress strength, the Rényi entropy, and the Shannon entropy. In addition, we estimated the parameters using the maximum likelihood method and proposed the simulation and application of the selected parameters using the statistical software R.
Keywords: exponential distribution, entropy, MLE, moments, Nadarajah-Haghighi.

1. Introduction

Statistical distributions are important part of our lives. They enable us to understand the world around us and make informed decisions. They also help us to recognize trends and opportunities. In recent years, the modeling of lifetime data has become an important research topic. Numerous studies have been published on this topic with the aim of introducing new statistical methods for dealing with lifetime phenomena. Several families of statistical distributions have been used in the last decades in a variety of fields, such as engineering, economics, medicine, demography, etc. In this paper, we propose a new extension of the exponential distribution based on the family of [0,1] truncated Nadarajah-Haghighi G distributions. The family of truncated Nadarajah-Haghighi-G distributions is proposed. The generated families generalize and extend most of the formal distributions. Some of the generators are Beta-G [1] and Exponential-G [2]. The Weibull-G family was proposed by [3] and the generalized transmuted-G was studied and introduced by [4]. The Gompertez-G family from [5]. the generalized odd Lindley-G family was proposed by [6], while the generalized odd gamma-G family was introduced by [7], and the Marshal-Olkin alpha-power family was proposed in [8]. The Gamma-Kumaraswamy G- family of distributions was introduced

by [9]. The Marshall-Olkin Topp Leone-G family was proposed by [10]. The Marshal-Olkin-Weibull-H family was introduced by [11]. The Odd-Chen family was introduced by [12]. Researchers have derived a truncated distribution from a parent distribution, such as a normal or exponential distribution, by bounding the random variable from below, from above, or from both. [13] The authors discussed the [0, 1] truncated inverted inverted gamma distribution as a special case of the CDF, moments, mean, variance, skewness, kurtosis, median, and characteristic function. Following the same method, [14] introduced the [0,1]-truncated inverse Weibull family. More recently, [15] introduced and studied a new distribution called [0, 1]-truncated exponential Gompertz distribution. In addition, many researchers have introduced new extensions to the exponential distribution, such as Gupta et. al. (2001) [16], Generalized exponential distribution. [17], a new generalization of the exponential Pareto distribution [18], the Weibull exponential distribution [19] and the beta exponential distribution. [20]. We use the modified exponential distribution ([0,1]) centered on the Nadarajah-Haghighi distribution to generate a new family of distributions and achieve greater flexibility than the existing submodels.
The article is organized as follows: Section 2 presents a useful [0, 1] truncated K-MD. Section 3 introduces the [0, 1] Truncated Nadarajah-Haghighi-M family of distributions. In addition, Section 4 contains the statistical properties of the [0, 1] TNHE distribution. To estimate the parameters of this new distribution, the MLE method is used, which is presented in Section 5. In Section 6, the estimates are validated by the simulation process of the TNHE distribution. In Section 7, a real data set is used to illustrate the effectiveness of the TNHE distribution. The conclusions are presented in Section 8.

2. [0,1] Truncated K-MD

In this paper, we have generated a new family of continues distributions based on [0,1] truncated CDF K-M, named [0,1] K-M as noted by [0,1] TK-M distributions.

Suppose that $M(x), m(x)$ is any continuous CDF and PDF, respectively of the random variable X, and assume that $K(.), k(.)$, respectively represent the CDF and PDF of any continuous distribution on the interval $[0, \infty)$. The suggested general formula of CDF for this class depends on the synthesis of K with M is

$$F(x)_{TK-M} = \frac{K[M(x)] - K(0)}{K(1) - K(0)} \quad (1)$$

Now, let $K(0) = 0$, then the CDF in (1) can be written as:

$$F(x)_{TK-M} = \frac{K[M(x)]}{K(1)} \quad (2)$$

And its associated PDF will be

$$f(x)_{TK-M} = \frac{k[M(x)]m(x)}{K(1)} \quad (3)$$

3. [0,1] Truncated Nadarajah-Haghighi-M family

Here we will propose a new family of [0,1] Truncated based on Nadarajh-Haghighi distribution. The Nadarajh-Haghighi distribution was introduced by [21] as an extention of the exponential distribution. The N-H distribution has the CDF and PDF, as folllows:

$$K(X) = 1 - e^{1-[1+bx]^{a}} \qquad x > 0, \quad a, b > 0 \quad (4)$$

$$k(x) = ab[1 + bx]^{a-1} e^{1-[1+bx]^{a}} \qquad x > 0, \quad a, b > 0 \quad (5)$$

Then, $K[M(x)] = 1 - e^{1-[1+bM(x)]^a}$, $k[M(x)] = ab[1+bx]^{a-1} e^{1-[1+bM(x)]^a}$

So that equations (2), (3) can be rewritten as follows:

$$F(x) = \frac{1- e^{1-[1+bM(x,\varphi)]^a}}{1- e^{1-[1+b]^a}} \quad (6)$$

$$f(x) = \frac{ab[1+bM(x,\varphi)]^{a-1} e^{1-[1+bM(x,\varphi)]^a} m(x,\varphi)}{1- e^{1-[1+b]^a}} \quad (7)$$

The equations (6), (7), respactively represent the CDF, PDF of [0,1] Truncated Nadarajah-Haghighi-M family of distributions, where $M(x,\varphi), m(x,\varphi)$ are the CDF and PDF of the baseline distribution with vector of parameters φ.

The exponential distribution was introduced by the random variable X provided that the following CDF and PDF:

$$M(x,\lambda) = 1 - e^{-\lambda x} \quad (8)$$

$$m(x,\lambda) = \lambda e^{-\lambda x} \quad (9)$$

Now, from substituting equations (8) and (9) in (6) and (7), we have obtained the CDF, PDF of [0,1] Truncated Nadarajah-Haghighi Exponential distribution, as follows:

$$F(x)_{[0,1]TNHE} = \frac{1- e^{1-\left[1+b\left(1-e^{-\lambda x}\right)\right]^a}}{1- e^{1-[1+b]^a}} \quad (10)$$

$$f(x)_{[0,1]TNHE} = \frac{ab\lambda\left[1+b\left(1-e^{-\lambda x}\right)\right]^{a-1} e^{1-\left[1+b\left(1-e^{-\lambda x}\right)\right]^a} e^{-\lambda x}}{1- e^{1-[1+b]^a}} \quad (11)$$

Then, equations (10), (11) respectivelly represent the CDF and PDF of [0,1] Truncated Nadarajah-Haghighi Exponentail distribution.

According to equations (10), (11), we can obtain the survival and hazard functions of [0,1] TNHEdistribuion, as follows:

The survival function of [0,1] TNHE distribution:

$$S(x,a,b,\lambda)_{[0,1]TNHE} = \frac{e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^a} - e^{1-(1+b)^a}}{1 - e^{1-(1+b)^a}} \quad (12)$$

Hazard rate function of [0, 1] TNHE distribution is [22]

$$(x,a,b,\lambda)_{[0,1]TNHE} = \frac{ab\lambda\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a-1} e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^a} e^{-\lambda x}}{e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^a} - e^{1-(1+b)^a}} \quad (13)$$

4. Statistical properties of [0,1] Truncated Nadarajah-Haghighi Exponential distribution

In this section, we introduce some of important Statistical Properties of [0,1] Truncated Nadarajah-Haghighi Exponential distribution like mixture representation, quantile function, moments, incomplete moments, moment generated function, Rényi entropy, Shannon entropy, $c-$entropy, and order statistics.

4.1 Mixture Representation

The mixture representation of the PDF is essential in the derivation of the statistical properties of [0,1] Truncated Nadarajah-Haghighi Exponential distribution. The mixture representation on the [0,1] Truncated Nadarajah-Haghighi Exponential distribution PDF can be written as follows:

$$f(x)_{[0,1]TNHE} = \frac{ab\lambda\left(1+b(1-e^{-\lambda x})\right)^{a-1} e^{1-\left(1+b(1-e^{-\lambda x})\right)^a} e^{-\lambda x}}{1-e^{1-(1+b)^a}}$$

Using the expansion of exponential, we get that

$$e^{1-\left(1+b(1-e^{-\lambda x})\right)^{a}} = \sum_{s=0}^{\infty}\frac{1}{s!}\left(1-\left(1+b(1-e^{-\lambda x})\right)^{a}\right)^{s}$$

Then

$$f(x)_{[0,1]TNHE} = \frac{ab\lambda\left(1+b(1-e^{-\lambda x})\right)^{a-1}\ e^{-\lambda x}}{1-e^{1-[1+b]^{a}}}\sum_{s=0}^{\infty}\frac{1}{s!}\left(1-\left(1+b(1-e^{-\lambda x})\right)^{a}\right)^{s}$$

Now, by the generalized binomial theorem, we get that

$$\left(1-\left(1+b(1-e^{-\lambda x})\right)^{a}\right)^{s} = \sum_{n=0}^{\infty}\binom{s}{n}(-1)^{n}\left(1+b(1-e^{-\lambda x})\right)^{an}$$

So that

$$f(x)_{[0,1]TNHE} = \frac{ab\lambda\ e^{-\lambda x}}{1-e^{1-(1+b)^{a}}}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{s!}\binom{s}{n}(-1)^{n}\left(1+b(1-e^{-\lambda x})\right)^{an+a-1}$$

Again, by the generalized binomial theorem

$$\left(1+b(1-e^{-\lambda x})\right)^{an+a-1} = \sum_{d=0}^{\infty}\binom{an+a-1}{d}\ b^{d}\left((1-e^{-\lambda x})\right)^{d}$$

Then

$$f(x)_{[0,1]TNHE} = \frac{ab\lambda\ e^{-\lambda x}}{1-e^{1-(1+b)^{a}}}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\sum_{d=0}^{\infty}\frac{(-1)^{n}}{s!}\binom{s}{n}\binom{an+a-1}{d}b^{d}\left((1-e^{-\lambda x})\right)^{d}$$

By the same way, we obtain that

$$\left((1-e^{-\lambda x})\right)^{d} = \sum_{k=0}^{\infty}\binom{d}{k}(-1)^{k}e^{-\lambda kx}$$

Then

$$f(x)_{[0,1]TNHE} = \frac{ab\lambda}{1-e^{1-(1+b)^{a}}}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{d}{k}\frac{1}{s!}\binom{s}{n}(-1)^{n+k}\ e^{-\lambda x(k+1)}$$

Moreover, this equation can be rewritten as follow:

$$f(x)_{[0,1]TNHE} = \Omega_{s,n,k,d}\, e^{-\lambda x(k+1)} \qquad (14)$$

Where

$$\Omega_{s,n,k,d} = \frac{ab\lambda}{1-e^{1-(1+b)^{a}}}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{d}{k}\frac{1}{s!}\binom{s}{n}(-1)^{n+k}$$

4.2 Quantile function

The [0,1] Truncated Nadarajah-Haghighi Exponential distribution quantile function can be obtained by inverting the CDF which is defined in (10), as follow [23]:

$$Q(u) = F^{-1}{}_{[0,1]TNHExpo}(x)$$

$$Q(u) = G^{-1}\left(\frac{-1}{\lambda}\left(\ln\left\{1-\frac{1}{b}\left(\left(1-\ln\left\{1-u\left(1-e^{1-(1+b)^{a}}\right)\right\}\right)^{\frac{1}{a}}-1\right)\right\}\right)\right) \qquad (15)$$

Now, according to eq. (15), the median (M) of [0.1] TNHE distribution can be obtained by instead *u = 0.5*.

4.3 Moments

The r^{th} moments of [0,1] Truncated Nadarajah-Haghighi Exponential distribution is [24], [25].

$$\mu_r = \Omega_{s,n,k,d}\left(\frac{1}{\lambda(k+1)}\right)^{r+1}\Gamma(r+1)$$

We can obtain r^{th} moment of a random variable X given by the following relation:

$$\mu_r = \int_{-\infty}^{\infty} x^{r} f(x)dx$$

Where $f(x)$ is given in (14), so that

$$\mu_r = \int_0^\infty x^r \Omega_{s,n,k,d}\, e^{-\lambda x(k+1)} dx$$

Let $y = \lambda x(k+1) \Rightarrow x = y\frac{1}{\lambda(k+1)} \Rightarrow dx = \frac{1}{\lambda(k+1)}\, dy$

That is,

$$\mu_r = \Omega_{s,n,k,d} \int_0^\infty \left(y\frac{1}{\lambda(k+1)}\right)^r e^{-y} \frac{1}{\lambda(k+1)}\, dy$$

$$= \Omega_{s,n,k,d} \left(\frac{1}{\lambda(k+1)}\right)^{r+1} \int_0^\infty (y)^r\, e^{-y}\, dy \qquad (16)$$

The mean and variance of [0,1] Truncated Nadarajah-Haghighi Exponential distribution can be obtained in equation (15), as follow:

$$\mu_1 = E(X)_{[0,1]TNHE} = \Omega_{s,n,k,d} \left(\frac{-1}{\lambda(k+1)}\right)^2 \qquad (17)$$

$$\mu_2 = E(X^2)_{[0,1]TNHE} = \Omega_{s,n,k,d} \frac{-2}{\left(\lambda(k+1)\right)^3}$$

$$\mu_3 = E(X^3)_{[0,1]TNHE} = \Omega_{s,n,k,d} \frac{6}{\left(\lambda(k+1)\right)^4} \qquad (18)$$

$$\mu_4 = E(X^4)_{[0,1]TNHE} = \Omega_{s,n,k,d} \frac{-24}{\left(\lambda(k+1)\right)^5} \qquad (19)$$

(20)

Moreover, the variance can be found using the following form:

$$Var(X) = E(X^2) - \left(E(X)\right)^2 \qquad (21)$$

So that

$$Var(X)_{[0,1]TNHE} = D_{j,k,m} \frac{-2}{\left(\lambda(k+1)\right)^3} - D_{j,k,m} \frac{1}{\left(\lambda(k+1)\right)^4} \qquad (22)$$

In addition, measures of skewness and kurtosis of [0,1] TNHE distribution based on the above equations can be obtained according to the following relations:

$$Sk = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{\left(\mu_2 - \mu_1^2\right)^{3/2}} \quad , \quad Ku = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4}{\left(\mu_2 - \mu_1^2\right)^2}$$

4.4 Incomplete Moments

The incomplete moments of [0,1] Truncated Nadarajah-Haghighi Exponential distribution can be obtained in the same way of (4.3), as follows:

$$M_r(y) = \int_{-\infty}^{y} x^r f(x)_{[0,1]TNHE}\, dx$$

$$= \Omega_{s,n,k,d} \frac{1}{\left(\lambda(k+1)\right)^{r+1}} \gamma\left(r+2, \lambda y(m+1)\right) \qquad (23)$$

4.5 Moment Generating Functions

The Moment Generating Functions of [0,1] Truncated Nadarajah-Haghighi Exponential distribution can be obtained in the following:

$$M_X(t) = \sum_{l=0}^{\infty} \frac{l^r}{l!} \int_{-\infty}^{\infty} x^l f(x)_{[0,1]TNHE}\, dx$$
$$= \sum_{l=0}^{\infty} \frac{l^r}{l!} \Omega_{s,n,k,d} \int_0^{\infty} x^l e^{-\lambda(k+1)x}\, dx \qquad (24)$$

4.6 Entropy

In this sub-section, we will find three common measures of entropy for a random variable X. These are Rényi entropy, Shannon entropy and delta entropy. Shannon entropy is a special case of Rényi entropy.

4.6.1 Rényi entropy

Rényi entropy of [0,1] Truncated Nadarajah-Haghighi Exponential distribution is defined as follows

$$I_R(c)_{[0,1]TNHE} = \frac{1}{1-c} \log\left\{\int_{-\infty}^{\infty} f^c(x)dx\right\}, c \neq 1, c > 0$$

The Rényi entropy for the random variable X is defined by

$$I_R(c)_{[0,1]TNHE} = \frac{1}{1-c} \log\left\{\int_{-\infty}^{\infty} f^c(x)dx\right\}, c \neq 1, c > 0$$

Now

$$f^c(x) = \left(\frac{ab\lambda\left(1+b(1-e^{-\lambda x})\right)^{a-1} e^{1-\left(1+b(1-e^{-\lambda x})\right)^a} e^{-\lambda x}}{1+b}\right)^c$$
$$= \frac{(ab\lambda)^c e^{-c\lambda x}}{(1+b)^c} \left(\left(1+b(1-e^{-\lambda x})\right)^{c(a-1)} e^{c\left(1-\left(1+b(1-e^{-\lambda x})\right)^a\right)}\right)$$

Now, by use the expansion exponential formula, we get

$$e^{c\left(1-\left(1+b(1-e^{-\lambda x})\right)^a\right)} = \sum_{j=0}^{\infty} \frac{1}{j!} c^j \left(1-\left(1+b(1-e^{-\lambda x})\right)^a\right)^j$$

So that

$$f^c(x) = \frac{(ab\lambda)^c e^{-c\lambda x}}{(1+b)^c} \left(\left(1+b(1-e^{-\lambda x})\right)^{c(a-1)} \sum_{j=0}^{\infty} \frac{1}{j!} c^j \left(1-\left(1+b(1-e^{-\lambda x})\right)^a\right)^j\right)$$

And by the generalized binomial theorem

$$\left(1-\left(1+b(1-e^{-\lambda x})\right)^a\right)^j = \sum_{k=0}^{\infty} \binom{j}{k} (-1)^k \left(1+b(1-e^{-\lambda x})\right)^{aj}$$

Hence

$$f^c(x) = \frac{(ab\lambda)^c e^{-c\lambda x}}{(1+b)^c} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{j}{k} \frac{c^j}{j!} (-1)^k \left(1+b(1-e^{-\lambda x})\right)^{aj+ca-c}$$

In the same way

$$\left(1+b(1-e^{-\lambda x})\right)^{aj+ca-c} = \sum_{r=0}^{\infty} \binom{aj+ca-c}{r} b^r (1-e^{-\lambda x})^r$$

Then

$$f^c(x) = \frac{(ab\lambda)^c e^{-c\lambda x}}{(1+b)^c} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \binom{j}{k} \frac{c^j}{j!} (-1)^k \binom{aj+ca-c}{r} b^r (1-e^{-\lambda x})^r$$

Also

$$(1-e^{-\lambda x})^r = \sum_{s=0}^{\infty} \binom{r}{s} (-1)^s e^{-s\lambda x}$$

Finally

$$f^{c}(x) = \Psi_{j,k,r,s}\, e^{-\lambda x(s+c)}$$

Where

$$\Psi_{j,k,r,s} = \frac{(ab\lambda)^{c}}{(1+b)^{c}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{r}{s}\binom{j}{k}\binom{aj+ca-c}{r} \frac{c^{j}}{j!} (-1)^{k+s}\, b^{r}$$

Now

$$I_R(c)_{[0,1]TNH-Expo} = \frac{1}{1-c}\log\left\{\int_0^{\infty} \Psi_{j,k,r,s}\, e^{-\lambda x(s+c)}\, dx\right\}, c \neq 1, c > 0$$

To find

$$\Psi_{j,k,r,s} \int_0^{\infty} e^{-\lambda x(s+c)}\, dx = \Psi_{j,k,r,s}\, \frac{1}{\lambda(s+c)}$$

Then the final form of Rényi entropy for [0,1] TNHE distribution is

$$I_R(c)_{[0,1]TNH-Expo} = \frac{1}{1-c}\log\left\{\Psi_{j,k,r,s}\, \frac{1}{\lambda(s+c)}\right\}, c \neq 1, c > 0 \quad (25)$$

4.6.2 Shannon entropy

The Shannon entropy of the new distribution is given by

$$\eta_x = E\left(-\log \Omega_{s,n,k,d}\, e^{-\lambda x(k+1)}\right) \quad (26)$$

Shannon entropy, defined as an a random variable X with a PDF $f(x)$, is a special case of the Renyi entropy when c↑ 1 and is defined as follow

$$\eta_x = E(-\log f(x))$$

So, the [0,1]TNHE distribution random variable is given by

$$\eta_x = E\left(-\log \Omega_{s,n,k,d}\, e^{-\lambda x(k+1)}\right)$$

4.6.3 Delta Entropy

The $c -$ entropy of a random variable X is given by

$$H(c) = \frac{1}{1-\delta}\log\left\{1 - \int_{-\infty}^{\infty} f^{c}(x)dx\right\}$$

Hence, [0,1] Truncated Nadarajah-Haghighi Exponential distribution is given by:

$$H(c) = \frac{1}{1-c}\log\left\{1 - \Psi_{j,k,r,s} \int_0^{\infty} e^{-\lambda x(s+c)} dx\right\}$$

$$H(\delta) = \frac{1}{1-\delta}\log\left\{1 - \Psi_{j,k,r,s}\, \frac{1}{\lambda(s+c)}\right\}$$

4.7 Order statistic

Let $X_1, X_2, X_3, \dots, X_n$ have [0,1] Truncated Nadarajah-Haghighi Exponential distribution with CDF, PDF defined in (10), (11), respectively and let $X_{1:n}, X_{2:n}, X_{3:n}, \dots, X_{n:n}$ be the order statistic obtained from this sample. Then, the probability density function of p^{th} order statistic from [0,1]TNHE distribution is obtained as follows [26]:

The PDF of order statistic with order $p, X_{p;n}$ is given by the form:

$$f_{p:n}(x) = \frac{n!}{(\mathrm{p}-1)!(n-p)!} [F(x)]^{p-1}[1 - F(x)]^{n-p} f(x)$$

$$= \sum_{s=0}^{n-p} d(-1)^{s}\binom{n-p}{s}[F(x)]^{p+s-1} f(x) \quad (27)$$

Where $d = \frac{n!}{(\mathrm{p}-1)!(n-p)!}$

Now, substituting (10) , (11) in (27), we have get

$$f_{p:n}(x) = \sum_{s=0}^{n-p} d\,(-1)^s \binom{n-p}{s} \left(\frac{1-e^{1-\left(1+b(1-e^{-\lambda x})\right)}}{1-e^{1-(1+b)^a}} \right)^{p+s-1} \times \left(\frac{ab\lambda\left(1+b(1-e^{-\lambda x})\right)^{a-1} e^{1-\left(1+b(1-e^{-\lambda x})\right)^a} e^{-\lambda x}}{1-e^{1-(1+b)^a}} \right) \quad (28)$$

Now, for $p = 1$, we get the smallest order statistic (least value function):

$$f_{p:n}(x) = \sum_{s=0}^{n-1} d\,(-1)^s \binom{n-1}{s} \left(\frac{1-e^{1-\left(1+b(1-e^{-\lambda x})\right)}}{1-e^{1-(1+b)^a}} \right)^{s} \times \left(\frac{ab\lambda\left(1+b(1-e^{-\lambda x})\right)^{a-1} e^{1-\left(1+b(1-e^{-\lambda x})\right)^a} e^{-\lambda x}}{1-e^{1-(1+b)^a}} \right) \quad (29)$$

And for $p = n$, we get the largest order statistic (big value function):

$$f_{n:n}(x) = \left(\frac{1-e^{1-\left(1+b(1-e^{-\lambda x})\right)}}{1-e^{1-(1+b)^a}} \right)^{n+s-1} \times \left(\frac{ab\lambda\left(1+b(1-e^{-\lambda x})\right)^{a-1} e^{1-\left(1+b(1-e^{-\lambda x})\right)^a} e^{-\lambda x}}{1-e^{1-(1+b)^a}} \right) \quad (30)$$

5. Maximum Likelihood Method

Assume that ,-1., ,x-2.,…, ,x-n. is a random sample of size n from the [0,1] truncated Nadarajah-Haghighi Exponential distribution. The corresponding log-likelihood function is then given by [27], [28] :

$$L(\varphi\backslash X) = \frac{(ab\lambda)^n \; e^{-\lambda\sum_{i=1}^n x_i} \; \sum_{i=1}^n \left(\left(1+b(1-e^{-\lambda x_i})\right)\right)^{a-1} \; e^{\sum_{i=1}^n (1-\left(1+b(1-e^{-\lambda x_i})\right)^a)}}{(1-e^{1-(1+b)^a})^n}$$

Let $l = logL(\varphi\backslash X)$ be the natural logarithm probability function.

$$l = nlog(ab\lambda) - \lambda\sum_{i=1}^n x_i + (a-1)\sum_{i=1}^n log\{\left(1+b(1-e^{-\lambda x_i})\right)\} + \sum_{i=1}^n (1-\left(1+b(1-e^{-\lambda x_i})\right)^a) - nlog\{1-e^{1-(1+b)^a}\}$$

$$l = nlog(a) + nlog(b) + nlog(\lambda) - \lambda\sum_{i=1}^n x_i + \left(a\sum_{i=1}^n log\left\{\left(1+b\left(1-e^{-\lambda x_i}\right)\right)\right\} - \sum_{i=1}^n log\left\{\left(1+b\left(1-e^{-\lambda x_i}\right)\right)\right\}\right) + \sum_{i=1}^n 1 - \sum_{i=1}^n \left(1+b(1-e^{-\lambda x_i})\right)^a) - nlog\left(1-e^{1-(1+b)^a}\right)$$

$$l = nlog(a) + nlog(b) + nlog(\lambda) - \lambda \sum_{i=1}^{n} x_i + \left(a \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} - \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} \right) + n - \sum_{i=1}^{n} (1 + b(1 - e^{-\lambda x_i}))^{a}) - nlog\{1 - e^{1-(1+b)^{a}}\}$$

Now, by taking the first partial derivative of the log likelihood function with respect to the parameters (a, b, λ), we get:

$$\frac{\partial(l)}{\partial a} = \left\{ \frac{n}{a} + \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} + (1 + b(1 - e^{-\lambda x_i}))^{a} \, log\{(1 + b(1 - e^{-\lambda x_i}))^{a}\} - \frac{n\,(1 + b)^{a}\ \ log\{1 + b\} e^{1-(1+b)^{a}}}{1 - e^{1-(1+b)^{a}}} \right\} \quad (31)$$

$$\frac{\partial(l)}{\partial b} = \frac{n}{b} + (a - 1) \sum_{i=1}^{n} \frac{1 - e^{-\lambda x_i}}{1 + b(1 - e^{-\lambda x_i})} - \sum_{i=1}^{n} \left(a(1 - e^{-\lambda x_i}) \left(1 + b(1 - e^{-\lambda x_i})\right)^{a-1} \right) - \frac{na(1 + b)^{a-1} \ \ e^{1-(1+b)^{a}}}{1 - e^{1-(1+b)^{a}}} \quad (32)$$

$$\frac{\partial(l)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (a - 1) \sum_{i=1}^{n} \frac{b\, x_i\, e^{-\lambda x_i}}{1+b(1-e^{-\lambda x_i})} - \sum_{i=1}^{n} ab\, x_i\, e^{-\lambda x_i} \left(1 + b(1 - e^{-\lambda x_i})\right)^{a-1} \quad (33)$$

By setting the above equations (31), (32), and (33) to zero, solving it numerically through using iterative methods, such as Newton-Raphson type algorithms, we can get the estimators of the parameters.

6. Simulation Study

In this section, we have conducted simulation study for $[0,1]$ TNHE distribution. We have generated samples of sizes $n = \{30, 50, 80, 120, 200\}$ from the proposed model and parameters estimated by MLE method, the simulation study is in terms of the averages of the three quantities: absolute bias $|\mathrm{Bais}(\boldsymbol{\tau})| = \frac{1}{\mathrm{N}} \sum_{\mathrm{i=1}}^{\mathrm{N}} |\, \hat{\boldsymbol{\tau}} - \boldsymbol{\tau}|$, mean square error (MSE), $MSE(\boldsymbol{\tau}) = \frac{1}{\mathrm{N}} \sum_{\mathrm{i=1}}^{\mathrm{N}} (\hat{\boldsymbol{\tau}} - \boldsymbol{\tau})^2$, and the mean relative error (MRE), $MRE(\boldsymbol{\tau}) = \frac{1}{\mathrm{N}} \sum_{\mathrm{i=1}}^{\mathrm{N}} |\, \hat{\boldsymbol{\tau}} - \boldsymbol{\tau}| / \boldsymbol{\tau}$. All the computations are made by using *R* Statistical Software. Table 1 shows some simulation results for different values of $\boldsymbol{\tau} = (a, b, \lambda)^T$. Based on the simulation criteria, **Table 1** discovered that the maximum likelihood estimate strategy performs pretty well in estimating the $[0,1]$ TNHE distribution parameters.

7. Application

In this section, we fit the TNHE distribution to a real data set to show that the proposed distribution fits well compared to competing distributions. The statistical software R is used to calculate all results. To obtain the best results, we used the following statistical criteria (-*l*, AIC, AIC, BIC, HQIC) for the proposed model compared to other models such as Beta-Exponential

(BeEx), Kumaraswamy-Exponential (KuEx), Exponential Generalized Exponential (EGEx), Weibull-Exponential (WeEx), Gompertez-Exponential (GoEx), Marshal-Olkin-Ex (MoEx) and Exponential (Ex). This data set is the use of the failure rate data set (103 hours) for the turbocharger for the engine type. For the dataset, we consider the large recorded intensities (on the Richter scale) of earthquakes at seismometer locations in western North America between 1940 and 1980, as in [29], [30], [31].

7.5,8.8,8.9,9.4,9.7,9.7,10.5,10.5,12,12.2,12.8,14.6,14.9,17.6,23.9,25,2.9,3.2,7.6,17,8,10,10,8,19,21,13,22,29,31,5.8,12,12.1,20.5,20.5,25.3,35.9,36.1,36.3,38.5,41.4,43.6,44.4,46.1,47.1,47.7,49.2,53.1,4,10.1,11.1,17.7,22.5,26.5,29,30.9,37.8,48.3,62,50,16,62,1.2,1.6,9.1,3.7,5.3,7.4,17.9,19.2,23.4,30,38.9,10.8,15.7,16.7,20.8,28.5,33.1,40.3,8,32,30,31,16.1,63.6,6.6,9.3,13,17.3,105,112,123,5,23.5,26,0.5,0.6,1.3,1.4,2.6,3.8,4,5.1,6.2,6.8,7.5,7.6,8.4,8.5,8.5,10.6,12.6,12.7,12.9,14,15,16,17.7,18,22,22,23,23.2,29,32,32.7,36,43.5,49,60,64,105,122,141,200,45,130,147,187,197,203,211,17,19.6,20.2,21.1,21.9,66,87,23.4,24.6,25.7,28.6,37.4,46.7,56.9,60.7,61.4,62,64,82,88,91,12,24.2,148,42,85,107,109,156,224,293,359,370,25.4,32.9,92.2,45,145,300.

According to the values shown in **Tables 2** and **3**, it is clear that the TNHE distribution is superior to the comparative distributions. The proposed expanded distribution provides an accurate representation because it has the lowest values according to the statistical and informational criteria and the largest value of the p-value. It is from **Figures 1** and **2,** the $[0,1]$ TNHE model provides the overall best fit and therefore could be chosen as the adequate model for explaining data.

Table 1. Bias, MSE and MRE of parameters of $[0,1]$ TNHE distribution.

$\tau = (a = 0.75, b = 1.75, \lambda = 3)^T$						
Est.	Est. Par.	$n = 30$	$n = 50$	$n = 80$	$n = 120$	$n = 200$
|Bias|	$\hat{a}$	0.87476	0.80613	0.73878	0.68144	0.61788
	$\hat{b}$	1.96507	1.66894	1.38657	1.17777	0.92474
	$\hat{\lambda}$	1.08240	0.90548	0.78722	0.67574	0.55274
MSE	$\hat{a}$	1.23050	1.02999	0.87907	0.71129	0.56899
	$\hat{b}$	7.99706	5.59836	3.67168	2.61345	1.48122
	$\hat{\lambda}$	1.94217	1.34383	0.98659	0.72033	0.47828
MRE	$\hat{a}$	1.16634	1.07483	0.98504	0.90858	0.82383
	$\hat{b}$	1.12290	0.95368	0.79233	0.67301	0.52842
	$\hat{\lambda}$	0.36080	0.30183	0.26241	0.22525	0.18425
$\tau = (a = 1.5, b = 0.5, \lambda = 0.4)^T$						
Est.	Est. Par.	$n = 30$	$n = 50$	$n = 80$	$n = 120$	$n = 200$
|Bias|	$\hat{a}$	0.69942	0.66331	0.64801	0.62884	0.60516
	$\hat{b}$	0.91634	0.81820	0.71024	0.61350	0.50648
	$\hat{\lambda}$	0.11670	0.09739	0.08630	0.07838	0.06481
MSE	$\hat{a}$	0.93774	0.8509	0.82684	0.79344	0.74607
	$\hat{b}$	1.34474	1.07283	0.81895	0.63727	0.44504
	$\hat{\lambda}$	0.02236	0.01483	0.01156	0.00962	0.00685
MRE	$\hat{a}$	0.46628	0.44220	0.43201	0.41923	0.40344
	$\hat{b}$	1.83267	1.63640	1.42047	1.22700	1.01296
	$\hat{\lambda}$	0.29175	0.24347	0.21574	0.19595	0.16203
$\tau = (a = 3, b = 3, \lambda = 1.6)^T$						
Est.	Est. Par.	$n = 30$	$n = 50$	$n = 80$	$n = 120$	$n = 200$
|Bias|	$\hat{a}$	1.62784	1.31950	1.11533	0.94522	0.74159
	$\hat{b}$	0.67291	0.60525	0.54433	0.48626	0.43437
	$\hat{\lambda}$	0.74430	0.62328	0.53197	0.46325	0.38669
MSE	$\hat{a}$	4.13035	2.85928	2.05816	1.51047	0.93697
	$\hat{b}$	0.80849	0.64676	0.50889	0.40259	0.31474
	$\hat{\lambda}$	1.0537	0.63751	0.42608	0.32020	0.22417

MRE	$\hat{a}$	0.54261	0.43983	0.37178	0.31507	0.24720
	$\hat{b}$	0.22430	0.20175	0.18144	0.16209	0.14479
	$\hat{\lambda}$	0.46519	0.38955	0.33248	0.28953	0.24168

Table 2. The K-S value with its corresponding p-value and W value of the data set

Model	W	A	K-S	p-value
[0,1]TNH-Expo	0.2156	1.2964	0.08246	0.1682
BeEx	0.5323	3.0980	0.1149	0.0162
KuEx	0.5134	2.9906	0.1068	0.0313
EGEx	0.53635	3.1207	0.1169	0.0138
WeEx	0.2172	1.3341	0.0908	0.0994
GoEx	0.2172	1.3341	0.0773	0.2259
MoEx	0.5079	2.9607	0.1218	0.0090
Ex	0.5289	3.0797	0.1267	0.0057

Table 3. Represented the values of statistically criteria (-LL, AIC, CAIC, BIC, HQIC).

Model	MLEs	- l	AIC	CAIC	BIC	HQIC
[0,1] TNHE	$\hat{a}$ = 1.154 $\hat{b}$ = 2.388 $\hat{\lambda}$ = 0.010	868.143	1742.286	1742.421	1751.898	1746.183
BeE	$\hat{a}$ = 0.903 $\hat{b}$ = 1.482 $\hat{\lambda}$ = 0.013	876.633	1759.266	1759.401	1768.878	1763.162
KuE	$\hat{a}$ = 0.892 $\hat{b}$ = 2.327 $\hat{\lambda}$ = 0.008	876.051	1758.103	1758.238	1767.715	1762
EGE	$\hat{a}$ = 1.667 $\hat{b}$ = 0.911 $\hat{\lambda}$ = 0.012	876.782	1759.565	1759.7	1769.177	1763.462
WeE	$\hat{a}$ = 0.883 $\hat{b}$ = 0.259 $\hat{\lambda}$ = 0.006	874.402	1745.208	1745.343	1754.82	1749.105
GoE	$\hat{a}$ = 1.006 $\hat{b}$ = 0.171 $\hat{\lambda}$ = 0.028	869.604	1745.208	1745.343	1754.82	1749.105
MoE	$\hat{a}$ = 0.944 $\hat{b}$ = 0.021	876.340	1756.683	1756.75	1763.091	1759.281
E	$\hat{\lambda}$ = 0.021	877.236	1756.473	1756.495	1759.677	1757.772

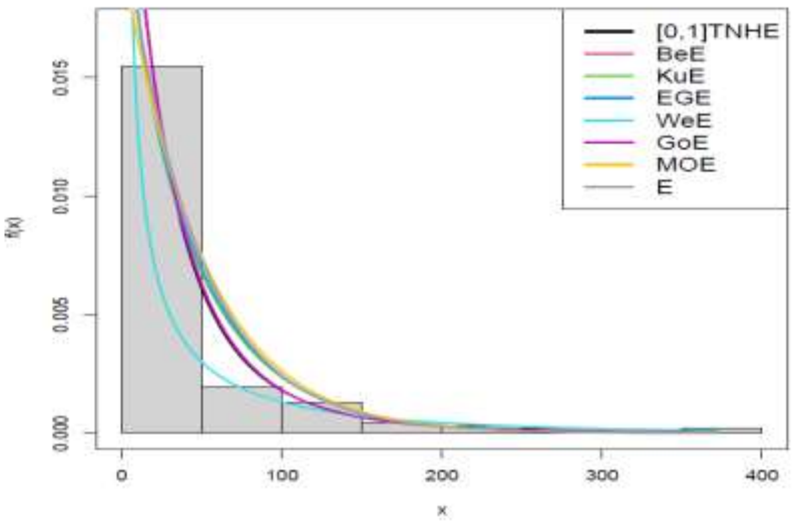

Figure 1. Estimated fitted densities of model for dataset.

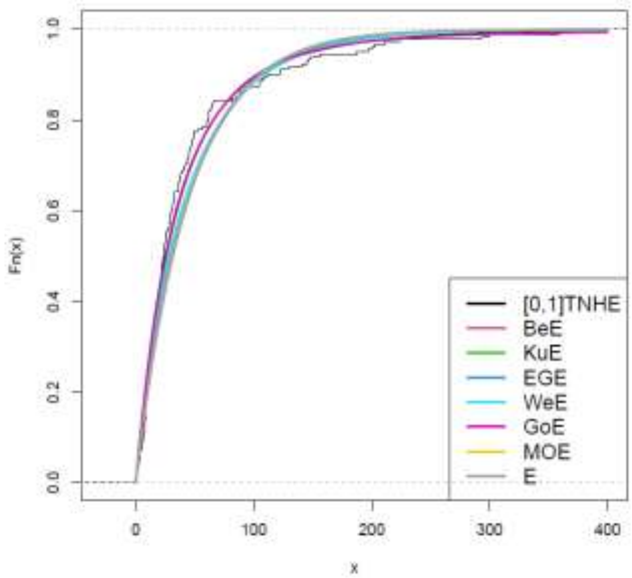

Figure 2. Estimated fitted CDF for data set.

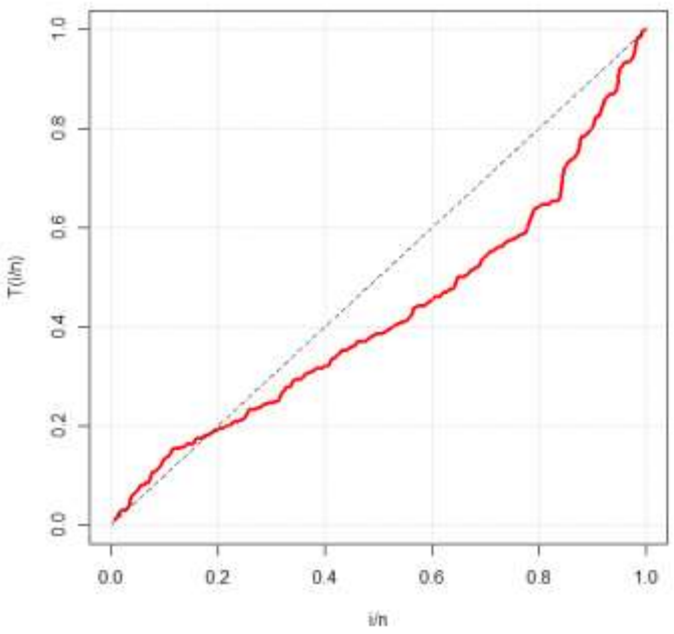

Figure 3. TTT plot of [0,1] TNHE distribution for data set.

5. Conclusion

This paper proposes a new extension of the exponential distribution based on the [0,1] Truncated Nadarajah-Haghighi-G family of distributions called [0,1] Truncated Nadarajah-Haghighi. The exponential distribution, which is a new distribution with three parameters, is more flexible than some other distributions, such as the exponential distribution and the Weibull exponential distribution. Also, we derive some statistical properties for the new distribution, such as the quantile function, moments, incomplete moments and entropy. Finally, we use the maximum likelihood method to estimate the parameters for the new distribution.

Acknowledgment

Our researcher extends his Sincere thanks to the editor and members of the preparatory committee of the Ibn AL-Haitham Journal of Pure and Applied Sciences.

Conflict of Interest

There are no conflicts of interest.

Funding

There is no funding for the article.

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