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Fractional Modelling for COVID-19 in the World and Iraq

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Abstract
This article talks about a model of fractional differential equations to describe how COVID-19 is spread in the world in general and Iraq in particular. The model contains five fractional differential equations. Moreover, we have proven the existence and uniqueness of the solution of the model, found the equilibrium points of the model and checked its stability. Then we solved it using a fractional linear multi-step method. When we compare the results with the data documented by the World Health Organisation (WHO), we find that the total number of active cases in the world are equal to 584498294 on 3/8/2022, and it is similar to what was done with the system. The number of those who have recovered and died from the disease has also been calculated. For Iraq, the total number of active cases is 2448484 and the total number of active cases calculated by the model is 2628000. In comparison, the calculated number is slightly higher than what is given in the data. This is quite normal because not all infected patients go to the health centres and it is difficult to record them as active cases.

Keywords: Caputo Fractional Derivative, Riemann-Liouville Fractional Derivative, Fractional COVID-19 Model, Numerical Simulations, Fractional Linear Multi-Step Method.

1. Introduction
In 2019, a new respiratory disease emerged that caused the death of a large number of people around the world, starting. It started in Wuhan and spread to every country in the world [1]. This disease is caused by a virus called coronavirus. COVID-19 attacks the respiratory system and lowers oxygen levels in the blood, causing problems. These problems vary from person to person. WHO classified COVID-19 as an epidemic on 11 March 2020 due to its rapid spread and high mortality rate. COVID-19 became the focus of the world, especially researchers, including mathematicians [2-4]. To find solutions and strategies that can help contain the spread of this virus. With the help of ordinary or fractional order system differential equations, delays and stochastic models. Many researchers have been able to construct and develop mathematical models to describe the dynamics of the disease, e.g. SIR, SEIR, SIS, SEIRF…etc [5-8]. Shah and Mittal [9] described the basics of epidemiologic compartmental models and the necessary analyses that contains a large number of studies on the corona epidemic, using various models of systems of ordinary differential equations that describe the spread of the corona epidemic and calculate the

active case in different countries of the world. Higazy M, et al, [10] modified the conventional SEIR model and used a fractional differential equation with Caputo derivative to represent the dynamics of COVID-19 transmission infected individual's ABO blood groups. Shahram Rezapour, et al, [11] use the Caputo fractional derivative to build an SEIR epidemic fractional model of the COVID-19 spread. They want an approximate solution to the system by using the fractional Euler method. They offer a numerical simulation based on actual data to predict COVID-19 transmission in the world and Iran.

This article is based on a model of fractional equations with Caputo derivation. He chose the world and Iraq as the study areas and used a model that divides the community into five classes. The first equation is the disease susceptible class, the second equation is the exposed individuals, the third equation is the infected patients, the fourth is the survivors of the disease and the fifth equation is the dead of the disease and where is the population size. We proved the existence and uniqueness of the solution of the model and found the equilibrium points and checked their stability. Then find the solution numerically using a fractional linear multi-step method and compare it with the data given in [12].

This article consists of five sections: The first section is the introduction. The next section contains the basic concepts used in this article. The third section is about the mathematical model of COVID-19 and how to prove the existence and uniqueness of the solution of the model and determine the equilibrium points. In the last part, a numerical simulation was carried out in the world and in Iraq. Finally: conclusions.

2. Background

This section introducing the basic concepts that is needed in this article.

2.1 Caputo Fractional Derivatives [13,14]

It is defined by:

$$\begin{cases} D_*^\alpha f(t) = \frac{1}{\Gamma(\eta-\alpha)}\int_a^t \frac{f^{(\eta)}(\tau)}{(t-\tau)^{(\alpha-\eta+1)}}d\tau\,, & \eta\text{-}1<\alpha<\eta\in N, \\ \frac{d^\eta}{dt^\eta}f(t) & \alpha=\eta\in N, \end{cases} \tag{1}$$

where is the order of the derivatives, $t>a,\ \alpha, a, t \in R$, and η is the smallest integer greater than α.

2.2 Riemann-Liouville fractional derivative [15]

The derivative of order α is defined as:

$$aD_t^\alpha f(t) = \frac{d^m}{dx^m}\int_a^x \frac{(x-t)^{m-n-1}f(t)}{\Gamma(m-n)}dt\,, \tag{2}$$

where $m-1\le n<m, m\in N$.

2.3 Relation between Caputo and Riemann-Liouville Fractional Derivative [16,17]

Let $\alpha>0$ be a fractional order derivative of the function f and δ=⌈α⌉ which mean δ is smallest natural number greater than α. Suppose that f is such that both $D_{*a}^\alpha f$ and D_a^α f exist, then:

$$D_{*a}^\alpha f(x) = D_a^\alpha f(x) - \sum_{k=0}^{\delta-1}\frac{D^k f(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}, \tag{3}$$

where D_{*a}^α is the Caputo differential operator of order α and D_a^α is the Riemann-Liouville differential operator of order α.

2.4 Equilibrium point [18,19]

Let $\dot{X}(t)=F(X(t))$ be a dynamical system. A point X_0 is called an equilibrium point if $F(X_0)=0$.

2.5 Fractional multi-step method [20,21]

Let:

$$\begin{cases} D_{t_0}^{\alpha} g(t) = f(t, g(t) \\ g(t_0) = g_0, g'(t_0) = g_0^{(1)}, \ldots, g^{(m-1)}(t_0) = g_0^{(m-1)} \end{cases}$$

where α is a fractional order derivative and $f(t, g)$ is assumed to be continuous and $g_0, g_0^{(1)}, \ldots, g_0^{(m-1)}$ are the assigned values of the derivatives at t_0.

Therefore, fractional multi-step approaches are convolution quadrature formulas, which may be expressed generally as follows:

$$g_n = \phi_n + \sum_{j=0}^{n} u_{n-j} f_j, \qquad f_j = f(t_j, g_n).$$

where ϕ_n and u_{n-j} are known coefficients and $t_n = t_0 + nh$ is an assigned grid, with the constant step-size $h > 0$.

3. Mathematical Model

The researchers divided the society into five categories. The first category is the disease-sensitive $S(t)$, the next one is the exposed individuals $E(t)$, the equation of the infected patients is given by $I(t), R(t)$ presents the disease survivors and $F(t)$ gives the equation of the disease die. In addition, all equations respect time.

Form this standpoint, the fractional mathematical model can be formulated as follows:

$$\begin{aligned} &D_*^{\alpha} S(t) = -\frac{\beta}{N} SI, \\ &D_*^{\alpha} E(t) = \frac{B}{N} SI - \omega E, \\ &D_*^{\alpha} I(t) = \omega \vartheta E - (l + \nu) I, \\ &D_*^{\alpha} R(t) = lI, \\ &D_*^{\alpha} F(t) = \nu I. \end{aligned} \tag{4}$$

The coefficients of the model parameters are given in **Table 1**.

Table 1: Description of the model.

Parameters	Description
α	fractional order derivative
$\mathcal{B}$	Measures the coefficient of human-to-human transmission per unit time (days) per person.
$\mathcal{N}$	Population Size.
ω	Is the rate at which a person becomes infectious and leaves the exposed class.
ϑ	Is the percentage of people who advance from E to I.
l	Recovery rate.
ν	Death rate.

3.1 Existence and Uniqueness solution of the model [22,23]

We rewrite the system Eq.(4) in matrix form as follows:

$$\text{Let } M = \begin{bmatrix} S \\ E \\ I \\ R \\ F \end{bmatrix} \; and \; M_*^\alpha = \begin{bmatrix} D_*^\alpha S(t) = -\frac{\beta}{N} SI \\ D_*^\alpha E(t) = \frac{\beta}{N} SI - \omega E \\ D_*^\alpha I(t) = \omega\vartheta E - (l+\nu) I \\ D_*^\alpha R(t) = lI \\ D_*^\alpha F(t) = \nu I \end{bmatrix}$$

By using the relation between Caputo derivative and Riemann derivative to the Model:

$$M_*^\alpha = M^\alpha - \begin{bmatrix} \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} S^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} E^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} I^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} R^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} F^{(k)}(0) \end{bmatrix}$$

$$M^\alpha = M_*^\alpha + \begin{bmatrix} \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} S^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} E^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} I^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} R^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} F^{(k)}(0) \end{bmatrix} \; let \;\; L = M_*^\alpha + \begin{bmatrix} \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} S^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} E^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} I^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} R^{(k)}(0) \\ \sum_{k=0}^{n-1} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} F^{(k)}(0) \end{bmatrix}$$

$M^\alpha = L(S, E, I, R, F)$ and $M^\alpha(t_0) = M_0$ is an initial value problem. In addition, L is continues for all variables and t, L has bounded partial derivatives then according to the existence and uniqueness theorem. That means there exist unique solution to system Eq.(4).

Proof: Let $u_t^{(\alpha)}(t)$ be the fractional derivative of $u(t)$ of order α. The substitution

$\xi(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \xi_0$

reduce $u^{(\alpha)}(t) to\ u'(\xi)$ as follows:

$u_t^{(\alpha)}(t) = u_t^{(\alpha)}(\xi) = u'(\xi)\, D_t^\alpha \xi = u'(\xi) \frac{1}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\alpha+1)} t^{\alpha-\alpha} = u'(\xi)$

Therefore the system Eq.(4)

$M^{(\alpha)}(t) = M'(\xi),$

$$M_*^{(\alpha)}=\begin{bmatrix} D_*^{\alpha}S(t)=-\frac{\beta}{N}SI \\ D_*^{\alpha}E(t)=\frac{\beta}{N}SI-\omega E \\ D_*^{\alpha}I(t)=\omega\vartheta E-(l+\nu)I \\ D_*^{\alpha}R(t)=lI \\ D_*^{\alpha}F(t)=\nu I \end{bmatrix}$$

$M'_{5*1}(\xi)=M_*^{(\alpha)}+L\doteqdot f(t,S,E,I,R,F), M(0)=M_0$ and according to the theorem 1 in [24]
Then there exist unique solution of Eq.(4)

3.2 The equilibrium points of the model [25]

To find the equilibrium points of the model Eq.(4), equalizing the equations to zero

$$D_*^{\alpha}S(t)=D_*^{\alpha}E(t)=D_*^{\alpha}I(t)=D_*^{\alpha}R(t)=D_*^{\alpha}F(t)=0 \quad (5)$$

Then the equilibrium point: $(S,E,I,R,F)=(N,0{,}0{,}0{,}0)$ (6)

Now to find the stability of the equilibrium point

$$\begin{aligned} D_*^{\alpha}S(t)&=-\frac{\beta}{N}SI=f_{1,}\\ D_*^{\alpha}E(t)&=\frac{\beta}{N}SI-\omega E=f_2,\\ D_*^{\alpha}I(t)&=\omega\vartheta E-(l+\nu)I=f_3,\\ D_*^{\alpha}R(t)&=lI=f_4,\\ D_*^{\alpha}F(t)&=\nu I=f_5. \end{aligned} \quad (7)$$

Expanding f_1, f_2, f_3, f_4, f_5 by Tayler series about $(N,0{,}0{,}0{,}0)$.

$$D_*^{\alpha}S(t)=\frac{\partial f_1}{\partial S}|_{(\mathcal{N},0,0,0,0)}(S-\mathcal{N})+\frac{\partial f_1}{\partial E}|_{(\mathcal{N},0,0,0,0)}E+\frac{\partial f_1}{\partial I}|_{(\mathcal{N},0,0,0,0)}I+\frac{\partial f_1}{\partial R}|_{(\mathcal{N},0,0,0,0)}R+\frac{\partial f_1}{\partial F}|_{(\mathcal{N},0,0,0,0)}F+\cdots$$

$$D_*^{\alpha}E(t)=\frac{\partial f_2}{\partial S}|_{(\mathcal{N},0,0,0,0)}(S-\mathcal{N})+\frac{\partial f_2}{\partial E}|_{(\mathcal{N},0,0,0,0)}E+\frac{\partial f_2}{\partial I}|_{(\mathcal{N},0,0,0,0)}I+\frac{\partial f_2}{\partial R}|_{(\mathcal{N},0,0,0,0)}R+\frac{\partial f_2}{\partial F}|_{(\mathcal{N},0,0,0,0)}F+\cdots$$

$$D_*^{\alpha}I(t)=\frac{\partial f_3}{\partial S}|_{(\mathcal{N},0,0,0,0)}(S-\mathcal{N})+\frac{\partial f_3}{\partial E}|_{(\mathcal{N},0,0,0,0)}E+\frac{\partial f_3}{\partial I}|_{(\mathcal{N},0,0,0,0)}I+\frac{\partial f_3}{\partial R}|_{(\mathcal{N},0,0,0,0)}R+\frac{\partial f_3}{\partial F}|_{(\mathcal{N},0,0,0,0)}F+\cdots \quad (8)$$

$$D_*^{\alpha}R(t)=\frac{\partial f_4}{\partial S}|_{(\mathcal{N},0,0,0,0)}(S-\mathcal{N})+\frac{\partial f_4}{\partial E}|_{(\mathcal{N},0,0,0,0)}E+\frac{\partial f_4}{\partial I}|_{(\mathcal{N},0,0,0,0)}I+\frac{\partial f_4}{\partial R}|_{(\mathcal{N},0,0,0,0)}R+\frac{\partial f_4}{\partial F}|_{(\mathcal{N},0,0,0,0)}F+\cdots$$

$$D_*^{\alpha}F(t)=\frac{\partial f_5}{\partial S}|_{(\mathcal{N},0,0,0,0)}(S-\mathcal{N})+\frac{\partial f_5}{\partial E}|_{(\mathcal{N},0,0,0,0)}E+\frac{\partial f_5}{\partial I}|_{(\mathcal{N},0,0,0,0)}I+\frac{\partial f_5}{\partial R}|_{(\mathcal{N},0,0,0,0)}R+\frac{\partial f_5}{\partial F}|_{(\mathcal{N},0,0,0,0)}F+\cdots$$

Then the coefficients matrix is:

$$\begin{bmatrix} 0 & 0 & -\beta & 0 & 0 \\ 0 & -\omega & \beta & 0 & 0 \\ 0 & \omega\vartheta & -(l+v) & 0 & 0 \\ 0 & 0 & l & 0 & 0 \\ 0 & 0 & v & 0 & 0 \end{bmatrix}$$

And the eigenvalues of this matrix are:

$\lambda_1=0, \lambda_2=0, \lambda_3=0,$

$$\lambda_4=-\frac{l}{2}-\frac{\nu}{2}-\frac{\omega}{2}+\frac{\sqrt{4\omega\vartheta\beta+l^2+2l\nu-2\omega l+\nu^2-2\omega\nu+\omega^2}}{2},$$

$$\lambda_5 = -\frac{l}{2} - \frac{\nu}{2} - \frac{\omega}{2} - \frac{\sqrt{4\omega\vartheta\beta + l^2 + 2l\nu - 2\omega l + \nu^2 - 2\omega\nu + \omega^2}}{2}.$$

and when substituting the value of $\beta = 1.55, \omega = 0.25, \vartheta = 0.58, l = 0.27, \nu = 0.01$ get:

$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0.2093, \lambda_5 = -0.7393.$

hence the equilibrium point is unstable [26].

4. Simulation results

Adopting the system equations given by Eq.(4) and the toolbox of MATLAB [27-29] “flmm” allows us to resolve fractional linear multi-step method for the numerical solution of differential equations [30], to represent the pandemic in the world and Iraq. And the number of parameters in the model is equal to:

$\beta = 1.55, \omega = 0.25, \vartheta = 0.58, l = 0.27, \nu = 0.01.$

4.1 World model

The five classes $S(t), E(t), I(t), R(t)$ *and* $F(t)$ associated of the world. The population size of the world is $N = 7.753 * 10^9$ with an initial values [12] $S(0) = 7.753 * 10^9 - 5320948, E(0) = 0, I(0) = 5320948, R(0) = 0, F(0) = 0.$

different values of α are shown respectively in **Figure1**.

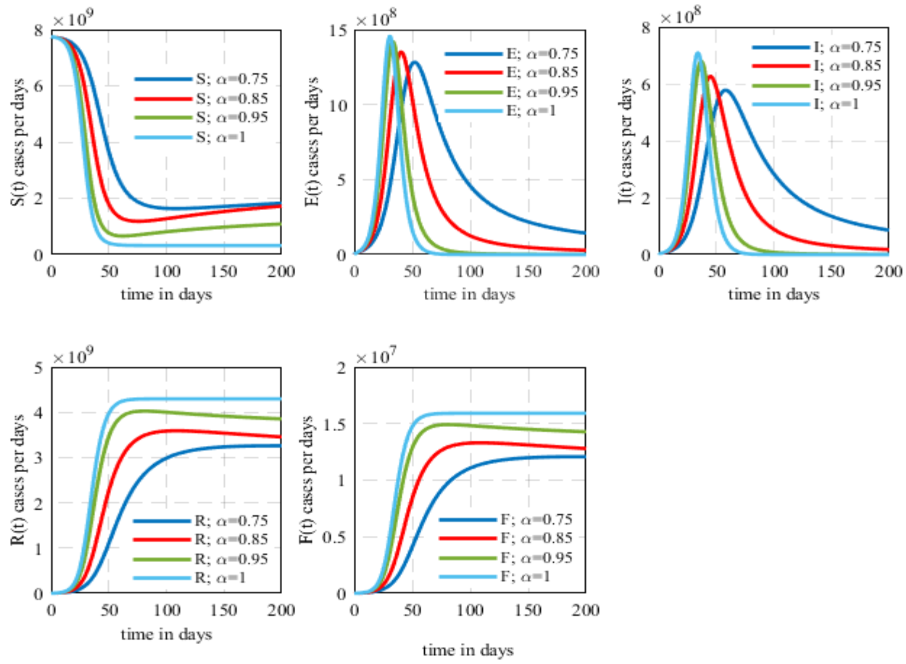

Figure 1: World model for different values of α.

Figure 1 shows the effect of the fractional derivative on spread dynamics of coronavirus in the world. Through it, we conclude that the beast value of the fractional derivative to describe the spread of the virus in the world at $\alpha = 0.75$, as shown in the figure:

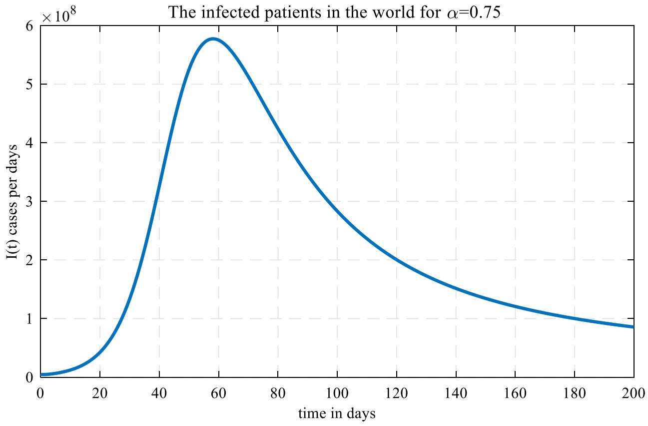

Figure 2: The infected cases in the word at $\alpha = 0.75$

Figure 2 shows the number of infected cases in the world. It is clear that the disease does not stop permanently, but gradually declines and levels off at a certain limit. This is the result of the discovery of vaccines, adherence to isolation policies and countries and individuals taking the necessary measures to control the disease.

4.2 Iraqi Model

Now we are going to apply the model Eq.(4) to Iraq. The population size of Iraq is $N = 40 * 10^6$ with an initial values [12] $S(0) = 40 * 10^6 - 4520, E(0) = 0, I(0) = 4520, R(0) = 0, F(0) = 0$. **Figure 3** shows the stability of the total number of infections and the variance in convergence speed for various values of α. In **Figure 4,** we see that infected populations decline and eventually gravitate to zero over time. Additionally, the convergence to the steady state occurs more slowly the lower the level of differentiation.

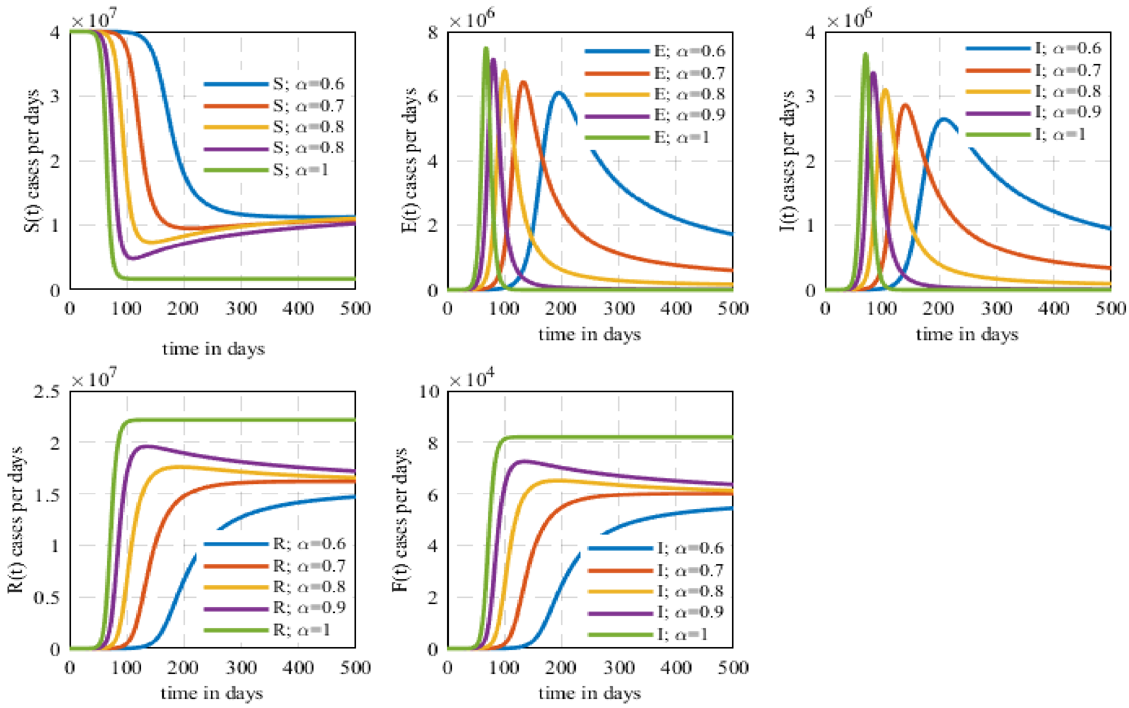

Figure 3: Iraqi model for different values of α

For α=0.6, we represent the infected patients which are given in **Figure 4**.

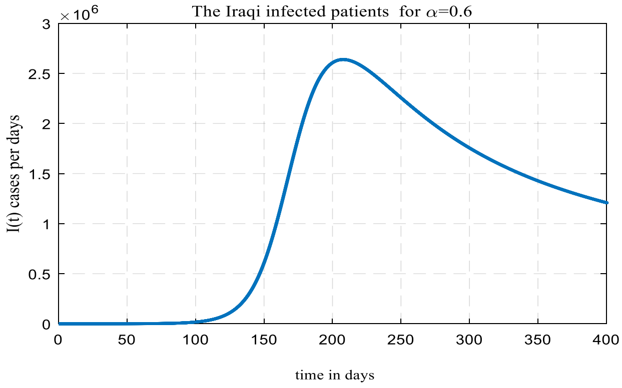

Figure 4: The Iraqi infected cases I(t) of α=0.6.

5. Discussions

The fractional COVID-19 model Eq.(4) was solved using the fractional linear multistep method for the numerical solution of differential equations. The figures show the effects of the fractional derivatives on the solution of the model in a more detailed sense (see **Figures 1** and **3**).

6. Conclusions

In this article, the detail was analyzed in a fractional-order COVID-19 model. The model consists of five differential equations of the fractional derivative of Caputo. We proved the existence and uniqueness of the solution for the system and then calculated the equilibrium points of the model and checked their stability. Finally, the model was solved in the fractional linear multistep method.

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Conflict of Interest

There are no conflicts of interest.

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