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Spin Characters' Decomposition Matrices of S_{27}, S_{28} modulo, $p = 13$

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Abstract

In this study, when the field characteristic is 13, we calculate decomposition matrices for the spin characters S_{27} and S_{28} which are broken down into blocks, where the decomposition matrices are connected between irreducible spin characters and irreducible modular spin characters. The technique used in this study is $(r, \bar{r})$-inducing, which produces projective characters for symmetric group S_{27} by projecting S_{26}'s character, and symmetric group S_{28} by projecting S_{27}'s character. We can find it by fixing all bar divisions, finding all irreducible spin characters for S_{27} (S_{28}), $p = 13$, and all irreducible modular spin characters for S_{27} (S_{28}), $p = 13$. In order to explore irreducible modular spin characteristics, general correlations and theorems will be discovered as a result of this research.

Keywords: Decomposition Matrix, Irreducible Modular Spin Character, projective character.

1. Introduction

Symmetric group S_n has a representation group $\bar{S}_n$ with a central $Z = \{-1,1\}$ such that $\bar{S}_n/Z \cong S_n$. The representations which do not have Z in their kernel are called the spin representations of S_n for more information, see [1-3]. The spin characters of the spin representations of S_n are labelled by the distinct parts of the partitions of n and denoted by $\langle\alpha\rangle$. In fact, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is partition of n and $n - m$ is even, then there is one irreducible spin character denoted by $\langle\alpha\rangle^*$ which is self-associate(double), and if $n - m$ is odd, then there are two associate spin characters denoted by $\langle\alpha\rangle$ and $\langle\alpha\rangle'$see [4-6]. The number of rows and columns of decomposition matrix corresponds to the number of projective characters and (p, α)-regular classes, respectively [3]. In this study we found the decomposition matrices of spin characters for S_{27} and S_{28} modulo $p = 13$. The distribution of the spin characters into p-blocks is accomplished using the $(r, \bar{r})$-inducing (restricting) approach [7,8]. Numerous people conduct research on this subject, have contributed to this field of study [9-22]. Before we declare any results, let's define certain notations and terminologies. "p.s." is the principal spin character ("p.i.s." indecomposable), "m.s." is means modular spin character ("i.m.s." irreducible), "d_i" is p.i.s. of S_n, "D_i" is p.i.s. of S_{n-1}, and "$\langle\dots\rangle^{no}$" is the number of i.m.s.

1. Preliminaries

For the study, some important conclusions were needed.

Theorem 2.1. Degree of the spin character $\langle \alpha_1, \dots, \alpha_{\mathrm{m}} \rangle = 2^{[(n-m)/2]} \frac{n!}{\prod_{i=1}^{m} \alpha_i!} \prod_{1 \le i < j \le m} \frac{(\alpha_i - \alpha_j)}{(\alpha_i + \alpha_j)}$ [1].

Theorem 2.2. Given that b is the number of p-conjugate characters to the irreducible ordinary character χ of G and that B is an ablock of defect one, then:

a. $\exists N \in \mathbb{Z}^+$ such that the irreducible ordinary characters lying in the block B can be partitioned into two disjoint classes:B_1={$\chi \in B \;\; |b \deg x \equiv N mod p^a$}, B_2={$\chi \in B \;\; |b \deg x \equiv -N mod p^a$}

b. The block B's decomposition matrix has coefficients that are either 1 or 0 [23].

Theorem 2.3. Let$G$ be a group of order $|G| = m_o p^a$, where $(p, m_o) = 0$. If c is a principal character of sub group H of G, then $\deg c \;\equiv 0 \; mod p^a$ [24,25].

Theorem 2.4. Let p be odd then

1. If n be even, $p \nmid n$, then $\langle n \rangle = \varphi\langle n \rangle$ and $\langle n \rangle' = \varphi\langle n \rangle'$ are distinct irreducible modular spin characters.
2. If n is odd, $p \nmid n$ or $p \nmid (n-1)$, then $\langle n-1{,}1 \rangle$ and $\langle n-1{,}1 \rangle'$ are distinct irreducible modular spin characters of degree $2^{[(n-3)/2]} \times (n-2)$ which are denoted by $\varphi\langle n-1{,}1 \rangle$ and $\varphi\langle n-1{,}1 \rangle'$respectively [2].

2. Decomposition matrix for $\bar{\mathbf{S}}_{\mathbf{27}}$

The decomposition matrix for $\bar{S}_{27}$ of degree (288,253), and it is decomposed in to blocks of character it consists of 69 blocks which $B_1$of defect two, B_2 ,B_3,…,B_{16} are defect one, and the remaining blocks are defect zero, decomposition matrix is equal to $B_1 \oplus B_2 \oplus \ldots \oplus B_{69}$.

Lemma 3.1. Decomposition matrix for the block $\boldsymbol{B_1}$ of type double as shown in the **Table 1**.

Table 1. Block B_1

Spin characters	Decomposition matrix																					
⟨27⟩*	1																					
⟨26,1⟩	1	1																				
⟨24,2,1⟩*		1	1																			
⟨23,3,1⟩*			1	1																		
⟨22,4,1⟩*				1	1																	
⟨21,5,1⟩*					1	1																
⟨20,6,1⟩*						1																
⟨19,7,1⟩*						1	1															
⟨18,8,1⟩*					1	1	1	1														
⟨17,9,1⟩*				1	1			1	1													
⟨16,10,1⟩*			1	1					1	1												
⟨15,11,1⟩*		1	1							1	1											
⟨14,13⟩	1	1										1										
⟨14,12,1⟩*	2	1									1	2	2									
⟨14,11,2⟩*										1	1	1	2	1								
⟨14,10,3⟩*									1	1				1	1							
⟨14,9,4⟩*								1	1						1	1						
⟨14,8,5⟩*							1	1								1	1					
⟨14,7,6⟩*							1										1					
⟨13,11,2,1⟩												1	1	1				1				
⟨13,10,3,1⟩													1	1	1			1	1			
⟨13,9,4,1⟩															1	1			1	1		
⟨13,8,5,1⟩																1	1			1	1	
⟨13,7,6,1⟩																	1				1	
⟨11,10,3,2,1⟩*													1						1			1

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}
$\langle 11,9,4,2,1\rangle^*$																		1	1	1		1	1			
$\langle 11,8,5,2,1\rangle^*$																				1	1		1	1		
$\langle 11,7,6,2,1\rangle^*$																					1			1		
$\langle 10,9,4,3,1\rangle^*$																		1					1		1	
$\langle 10,8,5,3,1\rangle^*$																						1	1	1	1	1
$\langle 10,7,6,3,1\rangle^*$																								1		1
$\langle 9,8,5,4,1\rangle^*$																						1				1
$\langle 9,7,6,4,1\rangle^*$																									1	1
$\langle 8,7,6,5,1\rangle^*$																									1	

Proof: By using $(0,1)$-inducing of p.i.s. method on D_1 in S_{27} we have

$$D_1\uparrow^{(0,1)} S_{27} = \langle 26\rangle + \langle 26\rangle' + \langle 25,1\rangle^* + 2\langle 14,12\rangle^* + \langle 13,1,2,1\rangle + \langle 13,12,1\rangle' \uparrow^{(0,1)} S_{27}$$
$$= 2\langle 27\rangle^* + 2\langle 26,1\rangle + 2\langle 26,1\rangle' + 2\langle 14,13\rangle + 2\langle 14,13\rangle' + 4\langle 14,12,1\rangle^*$$
$$= 2d_1$$

similarly, using $(\mathrm{r},\bar{r})$-inducing of p.i.s. $D_2, D_3, D_4, D_5, D_6, D_{46}, D_{40}, D_{34}, D_{28}, D_{52}, D_{12}, D_{13}, D_{29}, D_{35}, D_{41}, D_{47}, D_{18}, D_{19}, \ldots, D_{26}$ of S_{26} to S_{27} we get on $d_2, d_3, \ldots, d_{26}$ respectively, and on $(13,\alpha)$-regular classes we have

1. $\langle 26,1\rangle = \langle 26,1\rangle'$
2. $\langle 14,13\rangle = \langle 14,13\rangle'$
3. $\langle 13,11,2,1\rangle = \langle 13,11,2,1\rangle'$
4. $\langle 13,10,3,1\rangle = \langle 13,10,3,1\rangle'$
5. $\langle 13,9,4,1\rangle = \langle 13,9,4,1\rangle'$
6. $\langle 13,8,5,1\rangle = \langle 13,8,5,1\rangle'$
7. $\langle 13,7,6,1\rangle = \langle 13,7,6,1\rangle'$
8. $\langle 21,5,1\rangle^* = \langle 20,6,1\rangle^* + \langle 22,4,1\rangle^* - \langle 23,3,1\rangle^* + \langle 24,2,1\rangle^* - \langle 26,1\rangle + \langle 27\rangle^*$
9. $\langle 14,8,5\rangle^* = \langle 14,7,6\rangle^* + \langle 14,9,4\rangle^* - \langle 14,10,3\rangle^* + \langle 14,11,2\rangle^* - \langle 14,12,1\rangle^* + \langle 14,13\rangle + \langle 27\rangle^*$
10. $\langle 13,8,5,1\rangle = \langle 13,7,6,1\rangle + \langle 13,9,4,1\rangle - \langle 13,10,3,1\rangle + \langle 13,11,2,1\rangle - \langle 14,13\rangle + \langle 26,1\rangle$
11. $\langle 11,8,5,2,1\rangle^* = \langle 11,7,6,2,1\rangle^* + \langle 11,9,4,2,1\rangle^* - \langle 11,10,3,2,1\rangle^* - \langle 13,11,2,1\rangle + \langle 14,11,2\rangle^* - \langle 15,11,1\rangle^* + \langle 24,2,1\rangle^*$

Table 2. Block B_2, B_3

Block	Spin characters	Decomposition matrix																				
	$\langle 25,2\rangle$	1																				
	$\langle 25,2\rangle'$		1																			
	$\langle 15,12\rangle$	1		1																		
	$\langle 15,12\rangle'$		1		1																	
	$\langle 13,12,2\rangle^*$			1	1	1	1															
	$\langle 12,10,3,2\rangle$					1		1														
B_2	$\langle 12,10,3,2\rangle'$						1		1													
	$\langle 12,9,4,2\rangle$							1		1												
	$\langle 12,9,4,2\rangle'$								1		1											
	$\langle 12,8,5,2\rangle$									1		1										
	$\langle 12,8,5,2\rangle'$										1		1									
	$\langle 12,7,6,2\rangle$											1										
	$\langle 12,7,6,2\rangle'$												1									
	$\langle 24,3\rangle$													1								
	$\langle 24,3\rangle'$														1							
	$\langle 16,11\rangle$													1		1						
	$\langle 16,11\rangle'$														1		1					
B_3	$\langle 13,11,3\rangle^*$															1	1	1	1			
	$\langle 12,11,3,1\rangle$																	1		1		
	$\langle 12,11,3,1\rangle'$																		1		1	
	$\langle 11,9,4,3\rangle$																			1		1

	d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}	d_{37}	d_{38}	d_{39}	d_{40}	d_{41}	d_{42}	d_{43}	d_{44}	d_{45}	d_{46}	d_{47}	d_{48}	d_{49}	d_{50}
$\langle 11,9,4,3\rangle'$																				1		1		
$\langle 11,8,5,3\rangle$																					1		1	
$\langle 11,8,5,3\rangle'$																						1		1
$\langle 11,7,6,3\rangle$																							1	
$\langle 11,7,6,3\rangle'$																								1

12. $\langle 10,8,5,3,1\rangle^* = \langle 10,7,6,3,1\rangle^* + \langle 10,9,4,3,1\rangle^* + \langle 11,10,3,2,1\rangle^* - \langle 13,10,3,1\rangle + \langle 14,10,3\rangle^* - \langle 16,10,1\rangle^* + \langle 23,3,1\rangle^*$
13. $\langle 9,8,5,4,1\rangle^* = \langle 10,8,5,3,1\rangle^* - \langle 8,7,6,5,1\rangle^* - \langle 11,8,5,2,1\rangle^* + \langle 13,8,5,1\rangle - \langle 14,8,5\rangle^* + \langle 18,8,1\rangle^* - \langle 21,5,1\rangle^*$
14. $\langle 10,9,4,3,1\rangle^* = \langle 8,7,6,5,1\rangle^* + \langle 11,9,4,2,1\rangle^* - \langle 11,10,3,2,1\rangle^* - \langle 13,9,4,1\rangle + \langle 13,10,3,1\rangle + \langle 14,9,4\rangle^* - \langle 14,10,3\rangle^* + \langle 16,10,1\rangle^* - \langle 17,9,1\rangle^* + \langle 22,4,1\rangle^* - \langle 23,3,1\rangle^*$
15. $\langle 10,7,6,3,1\rangle^* = \langle 9,7,6,4,1\rangle^* - \langle 8,7,6,5,1\rangle^* + \langle 11,7,6,2,1\rangle^* - \langle 13,7,6,1\rangle - \langle 14,9,4\rangle^* + \langle 19,7,1\rangle^* - \langle 20,6,1\rangle^*$

then the matrix contains at most 41 columns since the number of the i.m.s. is equal or less than the number of the spin characters, but **Table 1** contains at most 26 columns since there are 15 equations corresponding the spin characters of S_{27} in $\boldsymbol{B_1}$, and because $d_i - d_j$ is not p.s. to S_{27} $\forall 1 \leq i < j \leq 26$, and $d_1, d_2, \cdots, d_{26}$ are linearly independent, then we get **Table 1**.

Lemma 3.2. The blocks $\boldsymbol{B_2}$, $\boldsymbol{B_3}$ of type associate as shown in the **Table 2**.

Proof:. By using $(\mathrm{r}, \bar{r})$-inducing of p.i.s. $D_3, D_{10}, D_{14}, D_{15}, D_{16}, D_{17}, D_2, D_9, D_{12}, D_{168}, D_{169}, D_{20}, D_{21}$ of S_{26} to S_{27}we get on $k_1, k_2, \ldots, k_9$, d_{45},d_{46}, k_{10}, k_{11}respectively. Since $\langle 25,2\rangle \neq \langle 25,2\rangle'$ are distinct irreducible modular spin characters then k_1 must split to d_{27}, d_{28}, also since $\boldsymbol{B_2}$ of defect one then from **(theorem 2.2)** k_2,k_3 must splits to d_{30}, d_{31} and d_{32}, d_{33}, respectively. Since $\langle 12,9,4,2\rangle \neq \langle 12,9,4,2\rangle'$ so k_4 or k_5 is split. If k_4 is split to d_{35}, d_{36},but $\langle 12,8,5\rangle \neq \langle 12,8,5\rangle'$ then k_5 split to, d_{37}, d_{38}. If k_5 is split and from $(13, \alpha)$-regular classes,

$$\langle 12,9,4,2\rangle + \langle 12,7,6,2\rangle - \langle 12,8,5,2\rangle \neq \langle 12,9,4,2\rangle' + \langle 12,7,6,2\rangle' - \langle 12,8,5,2\rangle' \quad (1)$$

then k_4 must split, so in both cases we get k_4 and k_5 are splits. Since$\langle 12,7,6,2\rangle \neq \langle 12,7,6,2\rangle'$ then k_6 must split to d_{37}, d_{38}. For $\boldsymbol{B_3}$ since $\langle 24,3\rangle \neq \langle 24,3\rangle'$ then k_7 must split to d_{39}, d_{40}, also since $\boldsymbol{B_3}$ of defect one then k_8,k_9 must splits to d_{41}, d_{42} and d_{43}, d_{44}, respectively. Since $\langle 11,8,5,3\rangle \neq \langle 11,8,5,3\rangle'$ so k_{10} or k_{11} is split. If k_9 is split to d_{47}, d_{48}, but $\langle 11,7,6,3\rangle \neq \langle 11,7,6,3\rangle'$ then k_{10} split to, d_{49}, d_{50}. If k_{10} is split , from $(13, \alpha)$-regular classes,

$$\langle 11,8,5,3\rangle - \langle 11,7,6,3\rangle \neq \langle 11,8,5,3\rangle' - \langle 11,7,6,3\rangle' \quad (2)$$

then k_9 must split,so in both cases we get k_9 and k_{10} are splits,then we get **Table 2**.

Lemma 3.3. The blocks $\boldsymbol{B_4}$, $\boldsymbol{B_5}$ of type associate as shown in the **Table 3**.

Table 3. BlocksB_4, B_5

Block	Spin characters	Decomposition matrix																							
	⟨23,4⟩	1																							
	⟨23,4⟩′		1																						
	⟨17,10⟩	1		1																					
	⟨17,10⟩′		1		1																				
	⟨13,10,4⟩*			1	1	1	1																		
	⟨12,10,4,1⟩					1		1																	
B_4	⟨12,10,4,1⟩′						1		1																
	⟨11,10,4,2⟩							1		1															
	⟨11,10,4,2⟩′								1		1														
	⟨10,8,5,4, ⟩									1		1													
	⟨10,8,5,4⟩′										1		1												
	⟨10,7,6,4⟩											1													
	⟨10,7,6,4⟩′												1												
	⟨22,5⟩													1											
	⟨22,5⟩′														1										
	⟨18,9⟩													1		1									
	⟨18,9⟩′														1		1								
	⟨13,9,5⟩*															1	1	1	1						
	⟨12,9,5,1⟩																	1		1					
B_5	⟨12,9,5,1⟩′																		1		1				
	⟨11,9,5,2⟩																			1		1			
	⟨11,9,5,2⟩′																				1		1		
	⟨10,9,5,3⟩																					1		1	
	⟨10,9,5,3⟩′																						1		1
	⟨9,7,6,5⟩																							1	
	⟨9,7,6,5⟩′																								1
		d_{51}	d_{52}	d_{53}	d_{54}	d_{55}	d_{56}	d_{57}	d_{58}	d_{59}	d_{60}	d_{61}	d_{62}	d_{63}	d_{64}	d_{65}	d_{66}	d_{67}	d_{68}	d_{69}	d_{70}	d_{71}	d_{72}	d_{73}	d_{74}

Proof:. By using $(\mathrm{r}, \bar{r})$-inducing of p.i.s.$D_3, D_8, D_{170}, D_{171}, D_{13}, D_{23}, D_{24}$,$D_4, D_7, D_{172}, D_{173}$, D_{19}, D_{22}, D_{26} of S_{26} to S_{27}we get on k_1, k_2, d_{55},d_{56}, k_3, k_4, k_5,k_6, k_7, d_{67},d_{68}, k_8, k_9, k_{10} respectively. Since $\langle 23,4\rangle \neq \langle 23,4\rangle'$ then k_1 must split to d_{51}, d_{52}, also since $\boldsymbol{B_4}$ of defect one then k_2 must split to d_{53}, d_{54}. $\langle 11,10,4,2\rangle \neq \langle 11,10,4,2\rangle'$ so k_3 or k_4 is split. If k_3 is split to d_{57}, d_{58}, but $\langle 10,8,5,4\rangle \neq \langle 10,8,5,4\rangle'$ then k_4 split to, d_{59}, d_{60}. If k_4 is split and from $(13, \alpha)$-regular classes,

$$\langle 11,10,4,2\rangle + \langle 10,7,6,4\rangle - \langle 10,8,5,4\rangle \neq \langle 11,10,4,2\rangle' + \langle 10,7,6,4\rangle' - \langle 10,8,5,4\rangle' \qquad (3)$$

then k_3 must split, so in both cases we get k_3 , k_4 are splits. Since$\langle 10,7,6,4\rangle \neq \langle 10,7,6,4\rangle'$ then k_5 split to d_{61}, d_{62}. In $\boldsymbol{B_5}$ $\langle 22,5\rangle \neq \langle 22,5\rangle'$ then k_6 must split to d_{63}, d_{64}, also $\boldsymbol{B_5}$ of defect one then k_7 must split to d_{65}, d_{66}. Since $\langle 11,9,5,2\rangle \neq \langle 11,9,5,2\rangle'$ so k_8 or k_9 is split. If k_8 is split to d_{69}, d_{70}, but $\langle 10,9,5,3\rangle \neq \langle 10,9,5,3\rangle'$ then k_9 split to, d_{71}, d_{72}. If k_9 is split and

$$\langle 11,9,5,2\rangle + \langle 9,7,6,5\rangle - \langle 10,9,5,3\rangle \neq \langle 11,9,5,2\rangle' + \langle 9,7,6,5\rangle' - \langle 10,9,5,3\rangle' \qquad (4)$$

then k_8 must split,so in both cases we get k_8 and k_9 are splits. Finally. Since$\langle 9,7,6,5\rangle \neq \langle 9,7,6,5\rangle'$ then k_{10} must split to d_{73}, d_{74}, so we get **Table 3**.

Lemma 3.4. Blocks $\boldsymbol{B_6}$, $\boldsymbol{B_8}$ of type double andblock $\boldsymbol{B_7}$ of type associateas given in **Table 4**.

Table 4. Blocks B_6**,** B_7, B_8

Block	Spin characters	Decomposition matrix																							
	$\langle 22,3,2\rangle^*$	1																							
	$\langle 16,9,2\rangle^*$	1	1																						
	$\langle 15,9,3\rangle^*$		1	1																					
B_6	$\langle 13,9,3,2\rangle$			1	1																				
	$\langle 12,9,3,2,1\rangle^*$				1	1																			
	$\langle 9,8,5,3,2\rangle^*$					1	1																		
	$\langle 9,7,6,3,2\rangle^*$						1																		
	$\langle 21,6\rangle$							1																	
	$\langle 21,6\rangle'$								1																
	$\langle 19,8\rangle$							1		1															
	$\langle 19,8\rangle'$								1		1														
	$\langle 13,8,6\rangle^*$									1	1	1	1												
	$\langle 12,8,6,1\rangle$											1		1											
B_7	$\langle 12,8,6,1\rangle'$												1		1										
	$\langle 11,8,6,2\rangle$													1		1									
	$\langle 11,8,6,2\rangle'$														1		1								
	$\langle 10,8,6,3\rangle$															1		1							
	$\langle 10,8,6,3\rangle'$																1		1						
	$\langle 9,8,6,4\rangle$																	1							
	$\langle 9,8,6,4\rangle'$																		1						
	$\langle 21,4,2\rangle^*$																			1					
	$\langle 17,8,2\rangle^*$																			1	1				
	$\langle 15,8,4\rangle^*$																				1	1			
B_8	$\langle 13,8,4,2\rangle$																					1	1		
	$\langle 12,8,4,2,1\rangle^*$																						1	1	
	$\langle 10,8,4,3,2\rangle^*$																							1	1
	$\langle 8,7,6,4,2\rangle^*$																								1
		d_{75}	d_{76}	d_{77}	d_{78}	d_{79}	d_{80}	d_{81}	d_{82}	d_{83}	d_{84}	d_{85}	d_{86}	d_{87}	d_{88}	d_{89}	d_{90}	d_{91}	d_{92}	d_{93}	d_{94}	d_{95}	d_{96}	d_{97}	d_{98}

Proof: Since

- degree $\{\langle 16,9,2\rangle^*, \langle 13,9,3,2\rangle + \langle 13,9,3,2\rangle', \langle 9,8,5,3,2\rangle^*\} \equiv 156 \bmod 13^2$
- degree $\{\langle 22,3,2\rangle^*, \langle 15,9,3\rangle^*, \langle 12,9,3,2,1\rangle^*, \langle 9,7,6,3,2\rangle^*\} \equiv -156 \bmod 13^2$,
- degree $\{\langle 21,4,2\rangle^*, \langle 15,8,5\rangle^*, \langle 12,8,4,2,1\rangle^*, \langle 8,7,6,4,2\rangle^*\} \equiv 91 \bmod 13^2$
- degree $\{\langle 17,8,2\rangle^*, \langle 13,8,4,2\rangle + \langle 13,8,4,2\rangle', \langle 10,8,4,3,2\rangle^*\} \equiv -91 \bmod 13^2$,

and by(2,12)-inducing of p.i.s. $D_{39}, D_{41}, D_{43}, D_{45}, D_{47}, D_{49}, D_{51}, D_{53}, D_{55}, D_{57}, D_{59}, D_{61}$ of S_{26} to S_{27}, and on $(13,\alpha)$-regular classes we have:

1. $\langle 13,9,3,2\rangle = \langle 13,9,3,2\rangle'$
2. $\langle 12,9,3,2,1\rangle^* = \langle 9,8,5,3,2\rangle^* - \langle 9,7,6,3,2\rangle^* + \langle 13,9,3,2\rangle - \langle 15,9,3\rangle^* + \langle 16,9,2\rangle^* - \langle 22,3,2\rangle^*$
3. $\langle 13,8,4,2\rangle = \langle 13,8,4,2\rangle'$
4. $\langle 12,8,4,2,1\rangle^* = \langle 10,8,4,3,2\rangle^* - \langle 8,7,6,4,2\rangle^* + \langle 13,8,4,2\rangle - \langle 15,8,4\rangle^* + \langle 17,8,2\rangle^* - \langle 21,4,2\rangle^*$

then each blocks $\boldsymbol{B_6}, \boldsymbol{B_8}$ contains at most 6 columns, so we get $\boldsymbol{B_6}, \boldsymbol{B_8}$. To find block $\boldsymbol{B_7}$ by using $(\mathrm{r}, \bar{r})$-inducing of p.i.s. D_5, D_8, D_{175}, D_{176}, D_{20}, D_{23}, D_{22} of S_{26} to S_{27} get on $k_1, k_2, d_{85}, d_{86}, k_3$, k_4, k_5. Since $\langle 21,6\rangle \neq \langle 21,6\rangle'$ then k_1 split to d_{81}, d_{82}, also since $\boldsymbol{B_7}$ of defect one then k_2 split to d_{83}, d_{84}. Since $\langle 11,8,6,2\rangle \neq \langle 11,8,6,2\rangle'$ so k_3 or k_4 is split. If k_3 is split to d_{87}, d_{88}, but $\langle 10,8,6,3\rangle \neq \langle 10,8,6,3\rangle'$ then k_4 split to, d_{89}, d_{90}. If k_4 is split and

$$\langle 11,8,6,2\rangle + \langle 9,8,6,4\rangle - \langle 10,8,6,3\rangle \neq \langle 11,8,6,2\rangle' + \langle 9,8,6,4\rangle' - \langle 10,8,6,3\rangle' \quad (5)$$

then k_3 split, so in both cases we get k_3 and k_4 are splits. Finally. Since $\langle 9,8,6,4\rangle \neq \langle 9,8,6,4\rangle'$ then k_5 must split to d_{91}, d_{92}, then we get **Table 4**.

Lemma 3.5. Block $\boldsymbol{B_9}$ of type associate, and $\boldsymbol{B_{10}}, \boldsymbol{B_{11}}$ of type double as shown in the **Table 5**.

Table 5. Blocks B_9**,** B_{10}, B_{11}

Block	Spin characters	d_{99}	d_{100}	d_{101}	d_{102}	d_{103}	d_{104}	d_{105}	d_{106}	d_{107}	d_{108}	d_{109}	d_{110}	d_{111}	d_{112}	d_{113}	d_{114}	d_{115}	d_{116}	d_{117}	d_{118}	d_{119}	d_{120}	d_{121}	d_{122}
B_9	$\langle 21,3,2,1\rangle$	1																							
	$\langle 21,3,2,1\rangle'$		1																						
	$\langle 16,8,2,1\rangle$	1		1																					
	$\langle 16,8,2,1\rangle'$		1		1																				
	$\langle 15,8,3,1\rangle$			1		1																			
	$\langle 15,8,3,1\rangle'$				1		1																		
	$\langle 14,8,3,2\rangle$					1		1																	
	$\langle 14,8,3,2\rangle'$						1		1																
	$\langle 13,8,3,2,1\rangle^*$							1	1	1	1														
	$\langle 9,8,4,3,2,1\rangle$									1		1													
	$\langle 9,8,4,3,2,1\rangle'$										1		1												
	$\langle 8,7,6,3,2,1\rangle$											1													
	$\langle 8,7,6,3,2,1\rangle'$												1												
B_{10}	$\langle 20,5,2\rangle^*$													1											
	$\langle 18,7,2\rangle^*$													1	1										
	$\langle 15,7,5\rangle^*$														1	1									
	$\langle 13,7,5,2\rangle$															1	1								
	$\langle 12,7,5,2,1\rangle^*$																1	1							
	$\langle 10,7,5,3,2\rangle^*$																	1	1						
	$\langle 9,7,5,4,2\rangle^*$																		1						
B_{11}	$\langle 20,4,3\rangle^*$																			1					
	$\langle 17,7,3\rangle^*$																			1	1				
	$\langle 16,7,4\rangle^*$																				1	1			
	$\langle 13,7,4,3\rangle$																					1	1		
	$\langle 12,7,4,3,1\rangle^*$																						1	1	
	$\langle 11,7,4,3,2\rangle^*$																							1	1
	$\langle 8,7,5,4,3\rangle^*$																								1

(Decomposition matrix)

Proof: To find $\boldsymbol{B_9}$ using $(\mathrm{r},\bar{r})$-inducing of p.i.s. $D_{63}, D_{64}, D_{65}, D_{66}, D_{101}, D_{69}, D_{71}, D_{72}, D_{73}$, D_{74} of S_{26} to S_{27}we get on $d_{99}, d_{100}, d_{101}, d_{102}, k_1, k_2, d_{107}$,$d_{108}$,$d_{109}$,$d_{110}$ respectively. Since $\langle 14,8,3,2\rangle \neq \langle 14,8,3,2\rangle'$ then k_1 must split to d_{103}, d_{104}, also since $\boldsymbol{B_9}$ of defect one then k_2 must split to d_{105}, d_{106},then we get block $\boldsymbol{B_9}$.To find $\boldsymbol{B_{10}}$ and $\boldsymbol{B_{11}}$ since

- degree $\{\langle 20,5,2\rangle^*, \langle 15,7,5\rangle^*, \langle 12,7,5,2,1\rangle^*, \langle 9,7,5,4,2\rangle^*\} \equiv 143 \bmod 13^2$
- degree $\{\langle 18,7,2\rangle^*, \langle 13,7,5,2\rangle + \langle 13,7,5,2\rangle', \langle 10,7,5,3,2\rangle^*\} \equiv -143 \bmod 13^2$,
- degree $\{\langle 17,7,3\rangle^*, \langle 13,7,4,3\rangle + \langle 13,7,4,3\rangle', \langle 11,7,4,3,2\rangle^*\} \equiv 143 \bmod 13^2$
- degree $\{\langle 20,4,3\rangle^*, \langle 16,7,4\rangle^*, \langle 12,7,4,3,1\rangle^*, \langle 8,7,5,4,3\rangle^*\} \equiv -143 \bmod 13^2$,

by inducing of p.i.s. $D_{75}, D_{77}, \ldots, D_{97}$ of S_{26} to S_{27}, and

1. $\langle 13,7,5,2\rangle = \langle 13,7,5,2\rangle'$
2. $\langle 12,7,5,2,1\rangle^* = \langle 10,7,5,3,2\rangle^* - \langle 9,7,5,4,2\rangle^* + \langle 13,7,5,2\rangle - \langle 15,7,5\rangle^* + \langle 18,7,2\rangle^* - \langle 20,5,2\rangle^*$
3. $\langle 13,7,4,3\rangle = \langle 13,7,4,3\rangle'$
4. $\langle 12,7,4,3,1\rangle^* = \langle 11,7,4,3,2\rangle^* - \langle 8,7,5,4,3\rangle^* + \langle 13,7,4,3\rangle - \langle 16,7,4\rangle^* + \langle 17,7,3\rangle^* - \langle 20,4,3\rangle^*$

then ech blocks $\boldsymbol{B_{10}}, \boldsymbol{B_{11}}$ contains at most 6 columns, so we get **Table 5**.

Lemma 3.6. Block $\boldsymbol{B_{12}}$ of type associateand and $\boldsymbol{B_{13}}$ of type double as given in the **Table 6.**

Table 6. Blocks B_{12}, B_{13}

Block	Spin characters																		Decomposition matrix
B_{12}	$\langle 20,4,2,1\rangle$	1																	
	$\langle 20,4,2,1\rangle'$		1																
	$\langle 17,7,2,1\rangle$	1		1															
	$\langle 17,7,2,1\rangle'$		1		1														
	$\langle 15,7,4,1\rangle$			1		1													
	$\langle 15,7,4,1\rangle'$				1		1												
	$\langle 14,7,4,2\rangle$					1		1											
	$\langle 14,7,4,2\rangle'$						1		1										
	$\langle 13,7,4,2,1\rangle^*$							1	1	1	1								
	$\langle 10,7,4,3,2,1\rangle$									1		1							
	$\langle 10,7,4,3,2,1\rangle'$										1		1						
	$\langle 8,7,5,4,2,1\rangle$											1							
	$\langle 8,7,5,4,2,1\rangle'$												1						
B_{13}	$\langle 19,5,3\rangle^*$													1					
	$\langle 18,6,3\rangle^*$													1	1				
	$\langle 16,6,5\rangle^*$														1	1			
	$\langle 13,6,5,3\rangle$															1	1		
	$\langle 12,6,5,3,1\rangle^*$																1	1	
	$\langle 11,6,5,3,2\rangle^*$																	1	1
	$\langle 9,6,5,4,3\rangle^*$																		1
		d_{123}	d_{124}	d_{125}	d_{126}	d_{127}	d_{128}	d_{129}	d_{130}	d_{131}	d_{132}	d_{133}	d_{134}	d_{135}	d_{136}	d_{137}	d_{138}	d_{139}	d_{140}

Proof:. By using inducing of p.i.s. $D_{87}, D_{88}, D_{89}, D_{90}, D_{101}, D_{93}, D_{95}, D_{96}, D_{97}, D_{98}$ of S_{26} to S_{27}we get on $d_{123}, d_{124}, d_{125}, d_{126}, k_1, k_2, d_{131}$,$d_{132}$,$d_{133}$,$d_{134}$ respectively. Since $\langle 14,7,4,2\rangle \neq \langle 14,7,4,2\rangle' k_1$ must split to d_{127}, d_{128}, also since $\boldsymbol{B_{12}}$ of defect one then k_2 must split to d_{129}, d_{130}.To find block $\boldsymbol{B_{13}}$, Since

- degree $\{\langle 18,6,3\rangle^*, \langle 13,6,5,3\rangle + \langle 13,6,5,3\rangle', \langle 11,6,5,3,2\rangle^*\} \equiv 156 \bmod 13^2$
- degree $\{\langle 19,5,3\rangle^*, \langle 16,6,5\rangle^*, \langle 12,6,5,3,1\rangle^*, \langle 9,6,5,4,3\rangle^*\} \equiv -156 \bmod 13^2$,

by inducing of p.i.s.$D_{105}, D_{107}, D_{109}, D_{111}, D_{113}, D_{115}$ of S_{26} to S_{27}, and

1. $\langle 13,6,5,3\rangle = \langle 13,6,5,3\rangle'$
2. $\langle 12,6,5,3,1\rangle^* = \langle 11,6,5,3,2\rangle^* - \langle 9,6,5,4,3\rangle^* + \langle 13,6,5,3\rangle - \langle 16,6,5\rangle^* + \langle 18,6,3\rangle^* - \langle 19,5,3\rangle^*$

then the block $\boldsymbol{B_{13}}$ contains at most 6 columns , then we get **Table 6.**

Lemma 3.7. Blocks $\boldsymbol{B_{14}}$, $\boldsymbol{B_{15}}$ of type associate as shown in the **Table 7.**

Table 7. Blocks B_{14}, B_{15}

Block	Spin Character																Block
B_{14}	$\langle 19,5,2,1\rangle$	1															
	$\langle 19,5,2,1\rangle'$		1														
	$\langle 18,6,2,1\rangle$	1		1													
	$\langle 18,6,2,1\rangle'$		1		1												
	$\langle 15,6,5,1\rangle$			1		1											
	$\langle 15,6,5,1\rangle'$				1		1										
	$\langle 14,6,5,2\rangle$					1		1									
	$\langle 14,6,5,2\rangle'$						1		1								
	$\langle 13,6,5,2,1\rangle^*$							1	1	1	1						
	$\langle 10,6,5,3,2,1\rangle$									1		1					
	$\langle 10,6,5,3,2,1\rangle'$										1		1				
	$\langle 9,6,5,4,2,1\rangle$											1					
	$\langle 9,6,5,4,2,1\rangle'$												1				
B_{15}	$\langle 19,4,3,1\rangle$													1			
	$\langle 19,4,3,1\rangle'$														1		
	$\langle 17,6,3,1\rangle$													1		1	
	$\langle 17,6,3,1\rangle'$														1		1

	d_{141}	d_{142}	d_{143}	d_{144}	d_{145}	d_{146}	d_{147}	d_{148}	d_{149}	d_{150}	d_{151}	d_{152}	d_{153}	d_{154}	d_{155}	d_{156}	d_{157}	d_{158}	d_{159}	d_{160}	d_{161}	d_{162}	d_{163}	d_{164}
$\langle 16,6,4,1\rangle$															1		1							
$\langle 16,6,4,1\rangle'$																1		1						
$\langle 14,6,4,3\rangle$																	1		1					
$\langle 14,6,4,3\rangle'$																		1		1				
$\langle 13,6,4,3,1\rangle^*$																			1	1	1	1		
$\langle 11,6,4,3,2,1\rangle$																					1		1	
$\langle 11,6,4,3,2,1\rangle'$																						1		1
$\langle 8,6,5,4,3,1\rangle$																							1	
$\langle 8,6,5,4,3,1\rangle'$																								1

Proof: Using inducing of p.i.s. $D_{105}, D_{106}, D_{107}, D_{108}, D_{131}, D_{111}, D_{113}, D_{114}, D_{115}, D_{116}, D_{117}, D_{118}, D_{119}, D_{120}, D_{131}, D_{123}, D_{125}, D_{126}, D_{127}, D_{128}$ of S_{26} to S_{27}we get on d_{141}, d_{142}, d_{143}, d_{144}, k_1, k_2, d_{149},d_{150},d_{151},d_{152},d_{153}, d_{154}, d_{155}, d_{156}, k_3, k_4, d_{161}, d_{162}, d_{163}, d_{164}. Since $\langle 14,6,5,2\rangle \neq \langle 14,6,5,2\rangle'$ then k_1 split to d_{145}, d_{146}, also $\boldsymbol{B_{14}}$ of defect one k_2 must split to d_{147}, d_{148}. For the block $\boldsymbol{B_{15}}$, $\langle 14,6,4,3\rangle \neq \langle 14,6,4,3\rangle'$ then k_3 must split to d_{155}, d_{156}, also since $\boldsymbol{B_{15}}$ of defect one then k_4 must split to d_{157}, d_{158}, then we get **Table 7.**

Lemma 3.8. The block B_{16} of type associate as shown in the **Table 8.**

Table 8. Block B_{16}

spin characters	decomposition matrix											
$\langle 18,4,3,2\rangle$	1											
$\langle 18,4,3,2\rangle'$		1										
$\langle 17,5,3,2\rangle$	1		1									
$\langle 17,5,3,2\rangle'$		1		1								
$\langle 16,5,4,2\rangle$			1		1							
$\langle 16,5,4,2\rangle'$				1		1						
$\langle 15,5,4,3\rangle$					1		1					
$\langle 15,5,4,3\rangle'$						1		1				
$\langle 13,5,4,3,2\rangle^*$							1	1	1	1		
$\langle 12,5,4,3,2,1\rangle$									1		1	
$\langle 12,5,4,3,2,1\rangle'$										1		1
$\langle 7,6,5,4,3,2\rangle$											1	
$\langle 7,6,5,4,3,2\rangle'$												1
	d_{165}	d_{166}	d_{167}	d_{168}	d_{169}	d_{170}	d_{171}	d_{172}	d_{173}	d_{174}	d_{175}	d_{176}

Proof: We find the required matrix by using inducing of p.i.s. D_{135}, D_{136}, D_{137}, D_{138}, D_{180}, D_{181}, D_{139}, D_{140} get on k_1, k_2, k_3, k_4, d_{173}, d_{174}, k_5. Since $\langle 18,4,3,2\rangle \neq \langle 18,4,3,2\rangle'$ then k_1 split to d_{165}, d_{166}. Since $\langle 16,5,4,2\rangle \neq \langle 16,5,4,2\rangle'$ so k_2 or k_3 is split. If k_2 is split to d_{167}, d_{168}, but $\langle 15,5,4,3\rangle \neq \langle 15,5,4,3\rangle'$ then k_3 split to, d_{169}, d_{170}. If k_3 is split, and

$$\langle 16,5,4\rangle + \langle 13,5,4,3,2\rangle^* - \langle 15,5,4,3\rangle \neq \langle 16,5,4\rangle' + \langle 13,5,4,3,2\rangle^* - \langle 15,5,4,3\rangle' \qquad (6)$$

then k_2 split, so in both cases we get k_2 and k_3 are splits, also $\boldsymbol{B_{16}}$ of defect one then k_4 split to d_{171}, d_{172}. Finally. $\langle 7,6,5,4,3,2\rangle \neq \langle 7,6,5,4,3,2\rangle'$ then k_5 split to d_{175}, d_{176}, we get **Table 8.**

3.Decomposition matrix for $\bar{S}_{28}$

Decomposition matrix for $\bar{S}_{28}$ of degree (334,295), and it is decomposed in to blocks of character it consists of 69 blocks which $B_1$of defect two, B_2 , $B_3, \ldots, B_{21}$ are defect one, and the remaining blocks are defects zero, decomposition matrix is equal to $B_1 \oplus B_2 \oplus \ldots \oplus B_{69}$

Lemma 4.1. The blocks $\boldsymbol{B_2}$**,** $\boldsymbol{B_3}$, $\boldsymbol{B_4}$ of type double as shown in **Table 9.**

Table 9. Blocks B_2, B_3,B_4

Block	spin characters	decomposition matrix																	
B_2	$\langle 27,1\rangle^*$	1																	
	$\langle 14,13,1\rangle$	1	1																
	$\langle 14,11,2,1\rangle^*$		1	1															
	$\langle 14,10,3,1\rangle^*$			1	1														
	$\langle 14,9,4,1\rangle^*$				1	1													
	$\langle 14,8,5,1\rangle^*$					1	1												
	$\langle 14,7,6,1\rangle^*$						1												
B_3	$\langle 25,3\rangle^*$							1											
	$\langle 16,12\rangle^*$							1	1										
	$\langle 13,12,3\rangle$								1	1									
	$\langle 12,11,3,2\rangle^*$									1	1								
	$\langle 12,9,4,3\rangle^*$										1	1							
	$\langle 12,8,5,3\rangle^*$											1	1						
	$\langle 12,7,6,3\rangle^*$												1						
B_4	$\langle 24,4\rangle^*$													1					
	$\langle 17,11\rangle^*$													1	1				
	$\langle 13,11,4\rangle$														1	1			
	$\langle 12,11,4,1\rangle^*$															1	1		
	$\langle 11,10,4,3\rangle^*$																1	1	
	$\langle 11,8,5,4\rangle^*$																	1	1
	$\langle 11,7,6,4\rangle^*$																		1
		d_{55}	d_{56}	d_{57}	d_{58}	d_{59}	d_{60}	d_{61}	d_{62}	d_{63}	d_{64}	d_{65}	d_{66}	d_{67}	d_{68}	d_{69}	d_{70}	d_{71}	d_{72}

Proof: Since

- degree $\{\langle 14,13,1\rangle + \langle 14,13,1\rangle', \langle 14,10,3,1\rangle^*, \langle 14,8,5,1\rangle^*\} \equiv 117 \bmod 13^2$,
- degree $\{\langle 27,1\rangle^*, \langle 14,11,2,1\rangle^*, \langle 14,9,4,1\rangle^*, \langle 14,7,6,1\rangle^*\} \equiv -117 \bmod 13^2$,
- degree $\{\langle 16,12\rangle^*, \langle 12,11,3,2\rangle^*, \langle 12,8,5,3\rangle^*\} \equiv 91 \bmod 13^2$,
- degree $\{\langle 25,3\rangle^*, \langle 13,12,3\rangle + \langle 13,12,3\rangle', \langle 12,9,4,3\rangle^*, \langle 12,7,6,3\rangle^*\} \equiv -91 \bmod 13^2$,
- degree $\{\langle 17,11\rangle^*, \langle 12,11,4,1\rangle^*, \langle 11,8,5,4\rangle^*\} \equiv 156 \bmod 13^2$,
- degree $\{\langle 24,4\rangle^*, \langle 13,11,4\rangle + \langle 13,11,4\rangle', \langle 11,10,4,3\rangle^*, \langle 11,7,6,4\rangle^*\} \equiv -156 \bmod 13^2$,

used inducing of p.i.s.D_2, D_{11}, D_{10}, D_9, D_8, D_7, D_{27}, D_{29}, D_{31}, D_{33}, D_{35}, D_{37}, D_{39}, D_{41}, D_{43}, D_{45}, D_{47}, D_{49} of S_{27} to S_{28}, and

1. $\langle 14,13,1\rangle = \langle 14,13,1\rangle'$
2. $\langle 14,11,2,1\rangle^* = \langle 14,10,3,1\rangle^* - \langle 14,9,4,1\rangle^* + \langle 14,13,1\rangle - \langle 27,1\rangle^* + \langle 14,8,5,1\rangle^* - \langle 14,7,6,1\rangle^*$
3. $\langle 13,12,3\rangle = \langle 13,12,3\rangle'$
4. $\langle 12,11,3,2\rangle^* = \langle 12,9,4,3\rangle^* - \langle 12,8,5,3\rangle^* + \langle 13,12,3\rangle - \langle 16,12\rangle^* + \langle 25,3\rangle^* + \langle 12,7,6,3\rangle^*$
5. $\langle 13,11,4\rangle = \langle 13,11,4\rangle'$
6. $\langle 12,11,4,1\rangle^* = \langle 11,10,4,3\rangle^* - \langle 11,8,5,4\rangle^* + \langle 13,11,4\rangle - \langle 17,11\rangle^* + \langle 24,4\rangle^* + \langle 11,7,6,4\rangle^*$

so each of these blocks contains 6 columns, so we get **Table 9**

Lemma 4.2. The block $\boldsymbol{B_5}$ of type associate and $\boldsymbol{B_6}$ of type doubleas shown in the **Tables 10**.

Table 10. Blocks B_5, B_6

Block	Spin characters								Decomposition matrix
B_5	$\langle 24,3,1\rangle$	1							
	$\langle 24,3,1\rangle'$		1						
	$\langle 16,11,1\rangle$	1		1					
	$\langle 16,11,1\rangle'$		1		1				
	$\langle 14,11,3\rangle$			1		1			
	$\langle 14,11,3\rangle'$				1		1		
	$\langle 13,11,3,1\rangle^*$					1	1	1	1

	$\langle 11,9,4,3,1\rangle$							1		1									
	$\langle 11,9,4,3,1\rangle'$								1		1								
	$\langle 11,8,5,3,1\rangle$									1		1							
	$\langle 11,8,5,3,1\rangle'$										1		1						
	$\langle 11,7,6,3,1\rangle$											1							
	$\langle 11,7,6,3,1\rangle'$												1						
	$\langle 23,5\rangle^*$													1					
	$\langle 18,10\rangle^*$													1	1				
	$\langle 13,10,5\rangle$														1	1			
B_6	$\langle 12,10,5,1\rangle^*$															1	1		
	$\langle 11,10,5,2\rangle^*$																1	1	
	$\langle 10,9,5,4\rangle^*$																	1	1
	$\langle 10,7,6,5\rangle^*$																		1
		d_{73}	d_{74}	d_{75}	d_{76}	d_{77}	d_{78}	d_{79}	d_{80}	d_{81}	d_{82}	d_{83}	d_{84}	d_{85}	d_{86}	d_{87}	d_{88}	d_{89}	d_{90}

Proof: By using inducing ofp.i.s. D_{39}, D_{40}, D_{11}, D_{12}, D_{45}, $D_{46}, \ldots, D_{51}$, D_{53}, D_{55}, D_{57}, D_{59}, D_{61} of S_{27} to S_{28}we get on $d_{73}, d_{74}, k_1, k_2, d_{79}, d_{80}, \ldots, d_{90}$. Since $\langle 14,11,3\rangle \neq \langle 14,11,3\rangle'$ then k_1 split to d_{75}, d_{76}, also since $\boldsymbol{B_5}$ of defect one then k_2 split to d_{77}, d_{78}. To find block $\boldsymbol{B_6}$, since

- degree $\{\langle 23,5\rangle^*, \langle 13,10,5\rangle + \langle 13,10,5\rangle', \langle 11,10,5,2\rangle^*, \langle 10,7,6,5\rangle^*\} \equiv 117 \bmod 13^2$
- degree $\{\langle 18,10\rangle^*, \langle 12,10,5,1\rangle^*, \langle 10,9,5,4\rangle^*\} \equiv -117 \bmod 13^2$,

and on $(13, \alpha)$-regular classes we have:

1. $\langle 13,10,5\rangle = \langle 13,10,5\rangle'$
2. $\langle 12,10,5,1\rangle^* = \langle 11,10,5,2\rangle^* + \langle 13,10,5\rangle - \langle 18,10\rangle^* + \langle 23,5\rangle^* - \langle 10,9,5,4\rangle^* + \langle 10,7,6,5\rangle^*$

then the block contains at most 6 columns , so we get **Table 10**

Lemma 4.3. The block $\boldsymbol{B_7}$ of type associate and $\boldsymbol{B_8}$ of type double as shown in the **Table 11**.

Table 11. Blocks B_7, B_8

Block	Spin characters	Decomposition matrix																	
	$\langle 23,4,1\rangle$	1																	
	$\langle 23,4,1\rangle'$		1																
	$\langle 17,10,1\rangle$	1		1															
	$\langle 17,10,1\rangle'$		1		1														
	$\langle 14,10,4\rangle$			1		1													
	$\langle 14,10,4\rangle'$				1		1												
B_7	$\langle 13,10,4,1\rangle^*$					1	1	1	1										
	$\langle 11,10,4,2,1\rangle$							1		1									
	$\langle 11,10,4,2,1\rangle'$								1		1								
	$\langle 10,8,5,4,1\rangle$									1		1							
	$\langle 10,8,5,4,1\rangle'$										1		1						
	$\langle 10,7,6,4,1\rangle$											1							
	$\langle 10,7,6,4,1\rangle'$												1						
	$\langle 22,6\rangle^*$													1					
	$\langle 19,9\rangle^*$													1	1				
	$\langle 13,9,6\rangle$														1	1			
B_8	$\langle 12,9,6,1\rangle^*$															1	1		
	$\langle 11,9,6,2\rangle^*$																1	1	
	$\langle 10,9,6,3\rangle^*$																	1	1
	$\langle 9,8,6,5\rangle^*$																		1
		d_{91}	d_{92}	d_{93}	d_{94}	d_{95}	d_{96}	d_{97}	d_{98}	d_{99}	d_{100}	d_{101}	d_{102}	d_{103}	d_{104}	d_{105}	d_{106}	d_{107}	d_{108}

Proof: By inducing of p.i.s. D_{51}, D_{52}, D_8, D_{14}, D_{57}, D_{58}, ..., D_{62}, D_{63}, D_{65}, D_{67}, D_{69}, D_{71}, D_{73} of S_{27} to S_{28}we get on d_{91}, d_{92}, k_1, k_2, d_{97}, d_{98}, ..., d_{108}. Since $\langle 14,10,4\rangle \neq \langle 14,10,4\rangle'$ then k_1 split to d_{93}, d_{94}, also $\boldsymbol{B_7}$ of defect one then k_2 split to d_{95}, d_{96}.To find the $\boldsymbol{B_8}$ since

- degree $\{\langle 22,6\rangle^*, \langle 13,9,6\rangle + \langle 13,9,6\rangle', \langle 11,9,6,2\rangle^*, \langle 9,8,6,5\rangle^*\} \equiv 117 \text{ mod } 13^2$,
- degree $\{\langle 19,9\rangle^*, \langle 12,9,6,1\rangle^*, \langle 10,9,6,3\rangle^*\} \equiv -117 \text{ mod } 13^2$,

and on $(13, \alpha)$-regular classes we have:

1. $\langle 13,9,6\rangle = \langle 13,9,6\rangle'$
2. $\langle 12,9,6,1\rangle^* = \langle 11,9,6,2\rangle^* + \langle 13,9,6\rangle - \langle 19,9\rangle^* + \langle 22,6\rangle^* - \langle 10,9,6,3\rangle^* + \langle 9,8,6,5\rangle^*$

then the block $\boldsymbol{B_8}$ contains at most 6 columns, so we get **Table 11**

Lemma 4.4. Block $\boldsymbol{B_9}$ is associate and $\boldsymbol{B_{10}}$, $\boldsymbol{B_{11}}$ are double as shown in the **Table 12**.

Table 12. Blocks B_9, B_{10}, B_{11}

Block	Spin characters	Decomposition matrix																							
	$\langle 22,5,1\rangle$	1																							
	$\langle 22,5,1\rangle'$		1																						
	$\langle 18,9,1\rangle$	1		1																					
	$\langle 18,9,1\rangle'$		1		1																				
	$\langle 14,9,5\rangle$			1		1																			
	$\langle 14,9,5\rangle'$				1		1																		
B_9	$\langle 13,9,5,1\rangle^*$					1	1	1	1																
	$\langle 11,9,5,2,1\rangle$							1		1															
	$\langle 11,9,5,2,1\rangle'$								1		1														
	$\langle 10,9,5,3,1\rangle$									1		1													
	$\langle 10,9,5,3,1\rangle'$										1		1												
	$\langle 9,7,6,5,1\rangle$											1													
	$\langle 9,7,6,5,1\rangle'$												1												
	$\langle 22,3,2,1\rangle^*$													1											
	$\langle 16,9,2,1\rangle^*$													1	1										
	$\langle 15,9,3,1\rangle^*$														1	1									
B_{10}	$\langle 14,9,3,2\rangle^*$															1	1								
	$\langle 13,9,3,2,1\rangle$																1	1							
	$\langle 9,8,5,3,2,1\rangle^*$																	1	1						
	$\langle 9,7,6,3,2,1\rangle^*$																		1						
	$\langle 21,7\rangle^*$																			1					
	$\langle 20,8\rangle^*$																			1	1				
	$\langle 13,8,7\rangle$																				1	1			
B_{11}	$\langle 12,8,7,1\rangle^*$																					1	1		
	$\langle 11,8,7,2\rangle^*$																						1	1	
	$\langle 10,8,7,3\rangle^*$																							1	1
	$\langle 9,8,7,4\rangle^*$																								1
		d_{109}	d_{110}	d_{111}	d_{112}	d_{113}	d_{114}	d_{115}	d_{116}	d_{117}	d_{118}	d_{119}	d_{120}	d_{121}	d_{122}	d_{123}	d_{124}	d_{125}	d_{126}	d_{127}	d_{128}	d_{129}	d_{130}	d_{131}	d_{132}

Proof: Using inducing of p.i.s. $D_{63}, D_{64}, D_7, D_{15}$, D_{69}, D_{70}, ..., D_{74}, D_{99}, D_{101}, D_{103}, D_{105}, D_{107}, D_{109}, D_{81}, D_{83}, $D_{85}, D_{87}, D_{89}, D_{91}$ of S_{27} to S_{28}we get on d_{109}, d_{110}, k_1, k_2, d_{115}, d_{116} , ..., d_{132}. Since $\langle 14,9,5\rangle \neq \langle 14,9,5\rangle'$ then k_1 split to d_{111}, d_{112}, also $\boldsymbol{B_9}$ of defect k_2 split to d_{113}, d_{114}. To find blocks $\boldsymbol{B_{10}}$, $\boldsymbol{B_{11}}$, since

- degree $\{\langle 16,9,2,1\rangle^*, \langle 14,9,3,2\rangle^*, \langle 9,8,5,3,2,1\rangle^*\} \equiv 143 \text{ mod } 13^2$,
- degree$\{\langle 22,3,2,1\rangle^*, \langle 15,9,3,1\rangle^*, \langle 13,9,3,2,1\rangle + \langle 13,9,3,2,1\rangle', \langle 9,7,6,3,2,1\rangle^*\} \equiv -143 \text{ mod } 13^2$,
- degree $\{\langle 20,8\rangle^*, \langle 12,8,7,1\rangle^*, \langle 10,8,7,3\rangle^*\} \equiv 143 \text{ mod } 13^2$,
- degree $\{\langle 21,7\rangle^*, \langle 13,8,7\rangle + \langle 13,8,7\rangle', \langle 11,8,7,2\rangle^*, \langle 9,8,7,4\rangle^*\} \equiv -143 \text{ mod } 13^2$,

and on $(13, \alpha)$-regular classes we have:

1. $\langle 13,9,3,2,1\rangle = \langle 13,9,3,2,1\rangle'$

2. $\langle 9,8,5,3,2,1\rangle^* = \langle 9,7,6,3,2,1\rangle^* + \langle 13,9,3,2,1\rangle - \langle 14,9,3,2\rangle^* + \langle 15,9,3,1\rangle^* - \langle 16,9,2,1\rangle^* + \langle 22,3,2,1\rangle^*$
3. $\langle 13,8,7\rangle = \langle 13,8,7\rangle'$
4. $\langle 12,8,7,1\rangle^* = \langle 11,8,7,2\rangle^* + \langle 13,8,7\rangle - \langle 20,8\rangle^* + \langle 21,7\rangle^* - \langle 10,8,7,3\rangle^* + \langle 9,8,7,4\rangle^*$

so each of these blocks contains 6 columns, thenwe get **Table 12**

Lemma 4.5. The blocks $\boldsymbol{B_{12}}$, $\boldsymbol{B_{13}}$ are associate as shown in the **Table 13**.

Table 13. Blocks B_{12}, B_{13}

Decomposition matrix

Block	Spin characters	d_{133}	d_{134}	d_{135}	d_{136}	d_{137}	d_{138}	d_{139}	d_{140}	d_{141}	d_{142}	d_{143}	d_{144}	d_{145}	d_{146}	d_{147}	d_{148}	d_{149}	d_{150}	d_{151}	d_{152}	d_{153}	d_{154}	d_{155}	d_{156}
	$\langle 21,6,1\rangle$	1																							
	$\langle 21,6,1\rangle'$		1																						
	$\langle 19,8,1\rangle$	1		1																					
	$\langle 19,8,1\rangle'$		1		1																				
	$\langle 14,8,6\rangle$			1		1																			
	$\langle 14,8,6\rangle'$				1		1																		
B_{12}	$\langle 13,8,6,1\rangle^*$					1	1	1	1																
	$\langle 11,8,6,2,1\rangle$							1		1															
	$\langle 11,8,6,2,1\rangle'$								1		1														
	$\langle 10,8,6,3,1\rangle$									1		1													
	$\langle 10,8,6,3,1\rangle'$										1		1												
	$\langle 9,8,6,4,1\rangle$											1													
	$\langle 9,8,6,4,1\rangle'$												1												
	$\langle 21,4,3\rangle$													1											
	$\langle 21,4,3\rangle'$														1										
	$\langle 17,8,3\rangle$													1		1									
	$\langle 17,8,3\rangle'$														1		1								
	$\langle 16,8,4\rangle$															1		1							
	$\langle 16,8,4\rangle'$																1		1						
B_{13}	$\langle 13,8,4,3\rangle^*$																	1	1	1	1				
	$\langle 12,8,4,3,1\rangle$																			1		1			
	$\langle 12,8,4,3,1\rangle'$																				1		1		
	$\langle 11,8,4,3,2\rangle$																					1		1	
	$\langle 11,8,4,3,2\rangle'$																						1		1
	$\langle 8,7,6,4,3\rangle$																							1	
	$\langle 8,7,6,4,3\rangle'$																								1

Proof: By inducing of p.i.s. $D_{81}, D_{82}, D_8, D_{16}, D_{87}, D_{88}, \dots, D_{92}, D_{117}, D_{118}, D_{119}, D_{217}, D_{218}, D_{221}, D_{222}$ of S_{27} to S_{28}we get on d_{133}, d_{134}, k_1, k_2, d_{139}, d_{140}, ..., d_{144}, k_3, k_4, k_5, d_{151}, d_{152}, k_6, k_7. Since $\langle 14,8,6\rangle \neq \langle 14,8,6\rangle'$ then k_1 split to d_{135}, d_{136}, also $\boldsymbol{B_{12}}$ of defect one then k_2 split to d_{137}, d_{138}. To find block $\boldsymbol{B_{12}}$, $\langle 17,8,3\rangle \neq \langle 17,8,3\rangle'$ so k_3 or k_4 is split. If k_3 is split to d_{145}, d_{146}, but $\langle 16,8,4\rangle \neq \langle 16,8,4\rangle'$ then k_4 split to, d_{147}, d_{148}. If k_4 is split and

$$\langle 16,8,4\rangle + \langle 17,8,3\rangle - \langle 21,4,3\rangle \neq \langle 16,8,4\rangle' + \langle 17,8,3\rangle' - \langle 21,4,3\rangle' \quad (7)$$

then k_3 split, so, in both cases we get k_3 and k_4 are splits, also $\boldsymbol{B_{13}}$ of defect one then k_5 split to d_{149}, d_{150}. Since $\langle 11,8,4,3,2\rangle \neq \langle 11,8,4,3,2\rangle'$ so k_6 or k_7 is split. If k_6 is split to d_{153}, d_{154}, but $\langle 8,7,6,4,3\rangle \neq \langle 8\ 7\ 6\ 4\ 3\rangle'$ then k_7 split to, d_{155}, d_{156}. If k_7 is split, and

$$\langle 11,8,4,3,2\rangle - \langle 8,7,6,4,3\rangle \neq \langle 11,8,4,3,2\rangle' - \langle 8,7,6,4,3\rangle' \quad (8)$$

then k_6 split, so in both cases we get k_6 and k_7 are splits, then we get Table 13.

Lemma 4.6. Blocks $\boldsymbol{B_{14}, B_{16}}$ of type double and $\boldsymbol{B_{15}}$ is associate as shown in Table 14.

Table 14. Blocks B_{14}, B_{15},B_{16}

Block	Spin characters	Decomposition matrix																							
	$\langle 21,4,2,1\rangle^*$	1																							
	$\langle 17,8,2,1\rangle^*$	1	1																						
	$\langle 15,8,4,1\rangle^*$		1	1																					
B_{14}	$\langle 14,8,4,2\rangle^*$			1	1																				
	$\langle 13,8,4,2,1\rangle$				1	1																			
	$\langle 10,8,4,3,2,1\rangle^*$					1	1																		
	$\langle 8,7,6,4,2,1\rangle^*$						1																		
	$\langle 20,5,3\rangle$							1																	
	$\langle 20,5,3\rangle'$								1																
	$\langle 18,7,3\rangle$							1		1															
	$\langle 18,7,3\rangle'$								1		1														
	$\langle 16,7,5\rangle$									1		1													
	$\langle 16,7,5\rangle'$										1		1												
B_{15}	$\langle 13,7,5,3\rangle^*$											1	1	1	1										
	$\langle 12,7,5,3,1\rangle$													1		1									
	$\langle 12,7,5,3,1\rangle'$														1		1								
	$\langle 11,7,5,3,2\rangle$															1		1							
	$\langle 11,7,5,3,2\rangle'$																1		1						
	$\langle 9,7,5,4,3\rangle$																	1							
	$\langle 9,7,5,4,3\rangle'$																		1						
	$\langle 20,5,2,1\rangle^*$																			1					
	$\langle 18,7,2,1\rangle^*$																			1	1				
	$\langle 15,7,5,1\rangle^*$																				1	1			
B_{16}	$\langle 14,7,5,2\rangle^*$																					1	1		
	$\langle 13,7,5,2,1\rangle$																						1	1	
	$\langle 10,7,5,3,2,1\rangle^*$																							1	1
	$\langle 9,7,5,4,2,1\rangle^*$																								1
		d_{157}	d_{158}	d_{159}	d_{160}	d_{161}	d_{162}	d_{163}	d_{164}	d_{165}	d_{166}	d_{167}	d_{168}	d_{169}	d_{170}	d_{171}	d_{172}	d_{173}	d_{174}	d_{175}	d_{176}	d_{177}	d_{178}	d_{179}	d_{180}

Proof: By inducing of p.i.s.D_{99}, D_{101}, D_{103}, D_{105}, D_{107}, D_{109}, D_{111}, D_{112}, D_{113}, D_{219}, D_{220}, D_{115}, D_{116}, D_{123}, D_{125}, D_{127}, D_{129}, D_{131}, D_{133} of S_{27} to S_{28}we get on d_{157},d_{158},…,d_{162}, k_1, k_2, k_3,d_{169}, d_{170}, k_4, k_5, d_{175}, d_{176},…,d_{180}. To find blocks $\boldsymbol{B_{14}}$, $\boldsymbol{B_{16}}$, since

- degree $\{\langle 21,4,2,1\rangle^*, \langle 15,8,4,1\rangle^*, \langle 13,8,4,2,1\rangle + \langle 13,8,4,2,1\rangle', \langle 8,7,6,4,2,1\rangle^*\} \equiv 156 \bmod 13^2$,
- degree $\{\langle 17,8,2,1\rangle^*, \langle 14,8,4,2\rangle^*, \langle 10,8,4,3,2,1\rangle^*\} \equiv -156 \bmod 13^2$,
- degree $\{\langle 20,5,2,1\rangle^*, \langle 15,7,5,1\rangle^*, \langle 13,7,5,2,1\rangle + \langle 13,7,5,2,1\rangle', \langle 9,7,5,4,2,1\rangle^*\} \equiv 104 \bmod 13^2$,
- degree $\{\langle 18,7,2,1\rangle^*, \langle 14,7,5,2\rangle^*, \langle 10,7,5,3,2,1\rangle^*\} \equiv -104 \bmod 13^2$,

and on $(13,\alpha)$-regular classes:

1. $\langle 13,8,4,2,1\rangle = \langle 13,8,4,2,1\rangle'$
2. $\langle 14,8,4,2\rangle^* = \langle 15,8,4,1\rangle^* + \langle 13,8,4,2,1\rangle - \langle 17,8,2,1\rangle^* + \langle 21,4,2,1\rangle^* - \langle 10,8,4,3,2,1\rangle^* + \langle 8,7,6,4,2,1\rangle^*$
3. $\langle 13,7,5,2,1\rangle = \langle 13,7,5,2,1\rangle'$
4. $\langle 14,7,5,2\rangle^* = \langle 15,7,5,1\rangle^* + \langle 13,7,5,2,1\rangle - \langle 10,7,5,3,2,1\rangle^* + \langle 9,7,5,4,2,1\rangle^* - \langle 18,7,2,1\rangle^* + \langle 20,5,2,1\rangle^*$

so each blocks contains 6 columns. In $\boldsymbol{B_{15}}$, $\langle 18,7,3\rangle \neq \langle 18,7,3\rangle'$ so k_1 or k_2 is split. If k_1 is split to d_{163}, d_{164}, but $\langle 16,7,5\rangle \neq \langle 16,7,5\rangle'$ then k_2 split to, d_{165}, d_{166}. If k_2 is split and

$$\langle 16,7,5\rangle - \langle 18,7,3\rangle + \langle 20,5,3\rangle \neq \langle 16,7,5\rangle' - \langle 18,7,3\rangle' + \langle 20,5,3\rangle' \qquad (9)$$

then k_1 split, so, in both cases we get k_1 and k_2 are splits, also $\boldsymbol{B_{15}}$ of defect one then k_3 split to d_{167}, d_{168}. Since $\langle 11,8,4,3,2\rangle \neq \langle 11,8,4,3,2\rangle'$ so k_4 or k_5 is split. If k_4 is split to d_{171},d_{172}, but $\langle 9,7,5,4,3\rangle \neq \langle 9,7,5,4,3\rangle'$ then k_5 split to, d_{173}, d_{174}. If k_5 is split, and

$\langle 11,7,5,3,2\rangle - \langle 9,7,5,4,3\rangle \neq \langle 11,7,5,3,2\rangle' - \langle 9,7,5,4,3\rangle'$ (10)

then k_4 split to d_{171}, d_{172}, so, in both cases we get k_4 , k_5 are splits, then we get Table 14.

Lemma 4.7 Blocks $\boldsymbol{B_{17}}$ **,** $\boldsymbol{B_{19}}$ are double and $\boldsymbol{B_{18}}$ of type associate as shown in Table 15.

Table 15. Blocks B_{17}, B_{18}, B_{19}

Block	Spin characters	Decomposition matrix																							
	$\langle 20,4,3,1\rangle^*$	1																							
	$\langle 17,7,3,1\rangle^*$	1	1																						
	$\langle 16,7,4,1\rangle^*$		1	1																					
B_{17}	$\langle 14,7,4,3\rangle^*$			1	1																				
	$\langle 13,7,4,3,1\rangle$				1	1																			
	$\langle 11,7,4,3,2,1\rangle^*$					1	1																		
	$\langle 8,7,5,4,3,1\rangle^*$						1																		
	$\langle 19,5,4\rangle$							1																	
	$\langle 19,5,4\rangle'$								1																
	$\langle 18,6,4\rangle$							1		1															
	$\langle 18,6,4\rangle'$								1		1														
	$\langle 17,6,5\rangle$									1		1													
	$\langle 17,6,5\rangle'$										1		1												
B_{18}	$\langle 13,6,5,4\rangle^*$											1	1	1	1										
	$\langle 12,6,5,4,1\rangle$													1		1									
	$\langle 12,6,5,4,1\rangle'$														1		1								
	$\langle 11,6,5,4,2\rangle$															1		1							
	$\langle 11,6,5,4,2\rangle'$																1		1						
	$\langle 10,6,5,4,3\rangle$																	1							
	$\langle 10,6,5,4,3\rangle'$																		1						
	$\langle 19,5,3,1\rangle^*$																			1					
	$\langle 18,6,3,1\rangle^*$																			1	1				
	$\langle 16,6,5,1\rangle^*$																				1	1			
B_{19}	$\langle 14,6,5,3\rangle^*$																					1	1		
	$\langle 13,6,5,3,1\rangle$																						1	1	
	$\langle 11,6,5,3,2,1\rangle^*$																							1	1
	$\langle 9,6,5,4,3,1\rangle^*$																								1
		d_{181}	d_{182}	d_{183}	d_{184}	d_{185}	d_{186}	d_{187}	d_{188}	d_{189}	d_{190}	d_{191}	d_{192}	d_{193}	d_{194}	d_{195}	d_{196}	d_{197}	d_{198}	d_{199}	d_{200}	d_{201}	d_{202}	d_{203}	d_{204}

Proof: By inducing of p.i.s.D_{153}, D_{155}, $D_{157}, D_{159}, D_{161}, D_{163}$,$D_{135}, D_{136}$, D_{137}, D_{221}, D_{222}, D_{139}, D_{140}, D_{141}, D_{143}, D_{145}, D_{147}, D_{149}, D_{151} of S_{27} to S_{28}we get ond_{181}, d_{182},…,d_{186},k_1, k_2, k_3 ,d_{193}, d_{194}, k_4,k_5, d_{199}, d_{200},…,d_{204}. To find blocks $\boldsymbol{B_{17}}$, $\boldsymbol{B_{19}}$ since

- degree $\{\langle 20,4,3,1\rangle^*, \langle 16,7,4,1\rangle^*, \langle 13,7,4,3,1\rangle + \langle 13,7,4,3,1\rangle', \langle 8,7,5,4,3,1\rangle^*\} \equiv 130 \bmod 13^2$,
- degree $\{\langle 17,7,3,1\rangle^*, \langle 14,7,4,3\rangle^*, \langle 11,7,4,3,2,1\rangle^*\} \equiv -130 \bmod 13^2$,
- degree $\{\langle 19,5,3,1\rangle^*, \langle 16,6,5,1\rangle^*, \langle 13,6,5,3,1\rangle + \langle 13,6,5,3,1\rangle', \langle 9,6,5,4,3,1\rangle^*\} \equiv 143 \bmod 13^2$,
- degree $\{\langle 18,6,3,1\rangle^*, \langle 14,6,5,3\rangle^*, \langle 11,6,5,3,2,1\rangle^*\} \equiv -143 \bmod 13^2$,

and on $(13, \alpha)$-regular classes:

1. $\langle 13,7,4,3,1\rangle = \langle 13,7,4,3,1\rangle'$
2. $\langle 14,7,4,3\rangle^* = \langle 16,7,4,1\rangle^* + \langle 13,7,4,3,1\rangle - \langle 11,7,4,3,2,1\rangle^* + \langle 8,7,5,4,3,1\rangle^* - \langle 17,7,3,1\rangle^* + \langle 20,4,3,1\rangle^*$
3. $\langle 13,6,5,3,1\rangle = \langle 13,6,5,3,1\rangle'$
4. $\langle 14,6,5,3\rangle^* = \langle 16,6,5,1\rangle^* + \langle 13,6,5,3,1\rangle - \langle 11,6,5,3,2,1\rangle^* + \langle 9,6,5,4,3,1\rangle^* - \langle 18,6,3,1\rangle^* + \langle 19,5,3,1\rangle^*$

so each blocks contains 6 columns.To find blocks $\boldsymbol{B_{18}}$**,** $\langle 18,6,4\rangle \neq \langle 18,6,4\rangle'$ so k_1 or k_2 is split. If k_1 is split to d_{187}, d_{188}, but $\langle 17,6,5\rangle \neq \langle 1,6,5\rangle'$ then k_2 split to, d_{189}, d_{190}. If k_2 is split, and

$$\langle 17,6,5\rangle - \langle 18,6,4\rangle + \langle 19,5,4\rangle \neq \langle 17,6,5\rangle' - \langle 18,6,4\rangle' + \langle 19,5,4\rangle' \quad (11)$$

then k_1 split, so, in both cases we get k_1 and k_2 are splits, also $\boldsymbol{B_{18}}$ of defect one then k_3 split to d_{191}, d_{192}. Since $\langle 11,6,5,4,2\rangle \neq \langle 11,6,5,4,2\rangle'$ so k_4 or k_5 is split. If k_4 is split to d_{195},d_{196}, but $\langle 10,6,5,4,3\rangle \neq \langle 10,6,5,4,3\rangle'$ then k_5 split to, d_{197}, d_{198}. If k_5 is split and

$$\langle 11,6,5,4,2\rangle - \langle 10,6,5,4,3\rangle \neq \langle 11,6,5,4,2\rangle' - \langle 10,6,5,4,3\rangle' \quad (12)$$

then k_4 split, so, in both cases we get k_4 and k_5 are splits, then we get Table 15.

Lemma 4.8. Block $\boldsymbol{B_{20}}$is a double and $\boldsymbol{B_{21}}$is associateas given in Table 16.

Table 16. Blocks B_{20}, B_{21}

Block	Spin characters	Decomposition matrix																	
B_{20}	$\langle 19,4,3,2\rangle^*$	1																	
	$\langle 17,6,3,2\rangle^*$	1	1																
	$\langle 16,6,4,2\rangle^*$		1	1															
	$\langle 15,6,4,3\rangle^*$			1	1														
	$\langle 13,6,4,3,2\rangle$				1	1													
	$\langle 12,6,4,3,2,1\rangle^*$					1	1												
	$\langle 8,6,5,4,3,2\rangle^*$						1												
B_{21}	$\langle 18,4,3,2,1\rangle$							1											
	$\langle 18,4,3,2,1\rangle'$								1										
	$\langle 17,5,3,2,1\rangle$							1		1									
	$\langle 17,5,3,2,1\rangle'$								1		1								
	$\langle 16,5,4,2,1\rangle$									1		1							
	$\langle 16,5,4,2,1\rangle'$										1		1						
	$\langle 15,5,4,3,1\rangle$											1		1					
	$\langle 15,5,4,3,1\rangle'$												1		1				
	$\langle 14,5,4,3,2\rangle$													1		1			
	$\langle 14,5,4,3,2\rangle'$														1		1		
	$\langle 13,5,4,3,2,1\rangle^*$															1	1	1	1
	$\langle 7,6,5,4,3,2,1\rangle$																	1	
	$\langle 7,6,5,4,3,2,1\rangle'$																		1
		d_{205}	d_{206}	d_{207}	d_{208}	d_{209}	d_{210}	d_{211}	d_{212}	d_{213}	d_{214}	d_{215}	d_{216}	d_{217}	d_{218}	d_{219}	d_{220}	d_{221}	d_{222}

Proof: By inducing of p.i.s. D_{153}, D_{155}, D_{157}, D_{159}, D_{161}, D_{163}, D_{165}, $D_{166}, \ldots$, D_{170}, D_{212}, $D_{173}, D_{175}, D_{176}$, we get on d_{211}, d_{212}, d_{213}, d_{214}, d_{215}, d_{216}, k_1, k_2, d_{221}, d_{222}. Since

- degree $\{\langle 19,4,3,2\rangle^*, \langle 16,6,4,2\rangle^*, \langle 13,6,4,3,2\rangle + \langle 13,6,4,3,2\rangle', \langle 8,6,5,4,3,2\rangle^*\} \equiv 117 \bmod 13^2$
- degree $\{\langle 17,6,3,2\rangle^*, \langle 15,6,4,3\rangle^*, \langle 12,6,4,3,2,1\rangle^*\} \equiv -117 \bmod 13^2$,

and on $(13, \alpha)$-regular classes we:

1. $\langle 13,6,4,3,2\rangle = \langle 13,6,4,3,2\rangle'$
2. $\langle 15,6,4,3\rangle^* = \langle 16,6,4,2\rangle^* + \langle 13,6,4,3,2\rangle - \langle 12,6,4,3,2,1\rangle^* + \langle 8,6,5,4,3,2\rangle^* - \langle 17,6,3,2\rangle^* + \langle 19,4,3,2\rangle^*$

then the block $\boldsymbol{B_{20}}$ contains at most 6 columns. To find the $\boldsymbol{B_{21}}$**,** $\langle 14,5,4,3,2\rangle \neq \langle 14,5,4,3,2\rangle'$, so k_1 divided to d_{217}, d_{218} or there are two columns:
$\varphi_1 = a_1\langle 15,5,4,3,1\rangle + a_2\langle 14,5,4,3,2\rangle + a_3\langle 13,5,4,3,2,1\rangle^* + a_4\langle 7,6,5,4,3,2,1\rangle$,
$\varphi_2 = a_1\langle 15,5,4,3,1\rangle' + a_2\langle 14,5,4,3,2\rangle' + a_3\langle 13,5,4,3,2,1\rangle^* + a_4\langle 7,6,5,4,3,2,1\rangle'$,
to describe columns, since $\boldsymbol{B_{21}}$ of defect one then $a_1, a_2, a_3, a_4 \in \{0,1\}$, but $\langle 7,6,5,4,3,2,1\rangle \downarrow S_{27}$ has only one of i.m.s. and from table has only one of i.m.s. then $a_4 = 0$, so that degree $\varphi_1, \varphi_2 \equiv 0 \bmod 7^3$ **(theorem 2.3)** only when $\varphi_1 + \varphi_2 = d_{217} + d_{218}$, then $k_1 = d_{217} + d_{218}$, also $\boldsymbol{B_{21}}$ of defect one then k_2 split to d_{219}, d_{220}, from abve we get Table 16.

Lemma 4.9. Decomposition matrix for the block B_1 of type double as shown in the Tables 17.

Table 17. Block B_1

spin characters	spin characters																																																					
⟨28⟩	1																																																					
⟨28⟩′		1																																																				
⟨26,2⟩*	1	1	1	1																																																		
⟨25,2,1⟩			1		1																																																	
⟨25,2,1⟩′				1		1																																																
⟨23,3,2⟩					1		1																																															
⟨23,3,2⟩′						1		1																																														
⟨22,4,2⟩							1		1																																													
⟨22,4,2⟩′								1		1																																												
⟨21,5,2⟩									1		1																																											
⟨21,5,2⟩′										1		1																																										
⟨20,6,2⟩											1		1																																									
⟨20,6,2⟩′												1		1																																								
⟨19,7,2⟩											1		1		1																																							
⟨19,7,2⟩′												1		1		1																																						
⟨18,8,2⟩									1		1				1		1																																					
⟨18,8,2⟩′										1		1				1		1																																				
⟨17,9,2⟩							1		1								1		1																																			
⟨17,9,2⟩′								1		1								1		1																																		
⟨16,10,2⟩					1		1												1		1																																	
⟨16,10,2⟩						1		1												1		1																																
⟨15,13⟩*	1	1	1	1																			1	1																														
⟨15,12,1⟩	1		1		1																		1		1																													
⟨15,12,1⟩		1		1		1																		1		1																												
⟨15,11,2⟩					1																1				1		1																											
⟨15,11,2⟩						1																1				1		1																										
⟨15,10,3⟩																			1		1						1		1																									
⟨15,10,3⟩																				1		1						1		1																								
⟨15,9,4⟩																	1		1										1		1																							
⟨15,9,4⟩′																		1		1										1		1																						
⟨15,8,5⟩															1		1														1		1																					
⟨15,8,5⟩′																1		1														1		1																				
⟨15,7,6⟩															1																		1																					
⟨15,7,6⟩′																1																		1																				
⟨14,12,2⟩	1	1																					1	1	1		1											1	1																		
⟨14,12,2⟩	1	1																					1	1		1		1										1	1																		
⟨13,12,2,																						1	1			1	1										1	1	1	1																
⟨13,10,3,																										1	1	1	1								1	1	1	1	1	1														
⟨13,9,4,2																												1	1	1	1								1	1	1	1												
⟨13,8,5,2																														1	1	1	1								1	1	1	1										
⟨13,7,6,2																																1	1										1	1										
⟨12,10,3,																																		1				1						1									
⟨12,10,3,																																			1				1						1								
⟨12,9,4,2																																				1		1		1				1		1							
⟨12,9,4,2																																					1		1		1				1		1						
⟨12,8,5,2																																								1		1				1		1					
⟨12,8,5,2																																									1		1				1		1				
⟨12,7,6,2																																										1						1					
⟨12,7,6,2																																											1						1				
⟨10,9,4,3																																					1										1				1			
⟨10,9,4,3																																						1										1				1		
⟨10,8,5,3																																													1		1		1		1		1	
⟨10,8,5,3																																														1		1		1		1		1
⟨10,7,6,3																																																	1				1	
⟨10,7,6,3																																																		1				1
⟨9,8,5,4,2																																													1								1	

⟨9,8,5,4,2																																													1								1
⟨9,7,6,4,2																																																		1		1	
⟨9,7,6,4,2																																																			1		1
⟨8,7,6,5,2																																																		1			
⟨8,7,6,5,2																																																			1		
	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d

Proof: By inducing of p.i.s. $D_1, D_{27}, D_{28}, D_3, D_4, D_5, D_6, D_{179}, D_7, D_8, D_9, D_{193}, D_{29}, D_{30}, D_{11}, D_{14}, D_{15}, D_{16}, D_{17}, D_{31}, D_{32}, D_{18}, D_{33}, D_{34}, \ldots, D_{38}, D_{22}, D_{23}, \ldots, D_{26}$ of S_{27} to S_{28}. All i.m.s. are associated in block B_1, since $\langle 28\rangle \neq \langle 28\rangle'$, according to **(theorem 2.4)** $\langle 28\rangle, \langle 28\rangle'$ have the same multiplicity, hence $k_1 = d_1 + d_2$. Since$\langle 23,3,2\rangle \neq \langle 23,3,2\rangle'$ so k_2 or k_3 is split. If k_2 is split to d_5, d_6, but $\langle 22,4,2\rangle \neq \langle 22,4,2\rangle'$ then k_3 split to, d_7, d_8. If k_3 is split, and

$$\langle 23,3,2\rangle + \langle 21,5,2\rangle - \langle 22,4,2\rangle \neq \langle 23,3,2\rangle' + \langle 21,5,2\rangle' - \langle 22,4,2\rangle' \quad (13)$$

then k_2 , so in both cases we get k_2 , k_3 splits. Since$\langle 21,5,2\rangle \neq \langle 21,5,2\rangle'$ so k_4 or k_5 is split. If k_4 is split to d_9, d_{10}, but $\langle 20,6,2\rangle \neq \langle 20,6,2\rangle'$ then k_5 split to, d_{11}, d_{12}. If k_5 is split, and

$$21{,}5{,}2 + 19{,}7{,}2 - \langle 20,6,2\rangle \neq \langle 21,5,2\rangle' + \langle 19,7,2\rangle' - \langle 20,6,2\rangle' \quad (14)$$

then k_4 is split, so we get k_4 , k_5 splits. Since$\langle 19,7,2\rangle \neq \langle 19,7,2\rangle'$ so k_6 or k_7 is split. If k_7 is split to d_{15}, d_{16}, but $\langle 20,6,2\rangle \neq \langle 20,6,2\rangle'$ then k_6 split to, d_{13}, d_{14}. If k_6 is split, and

$$\langle 19,7,2\rangle - \langle 20,6,2\rangle \neq \langle 19,7,2\rangle' - \langle 20,6,2\rangle' \quad (15)$$

then k_7 is split, so we get k_6 and k_7 are splits.Since$\langle 17,9,2\rangle \neq \langle 17,9,2\rangle'$ so k_8 or k_9 is split. If k_8 is split to d_{17}, d_{18}, but $\langle 16,10,2\rangle \neq \langle 16,10,2\rangle'$ then k_9 split to, d_{19}, d_{20}. If k_9 is split, and

$$\langle 17,9,2\rangle - \langle 16,10,2\rangle \neq \langle 17,9,2\rangle' - \langle 16,10,2\rangle' \quad (16)$$

then k_8 is split, then k_8 , k_9 splits. Since $\langle 16,10,2\rangle \neq \langle 16,10,2\rangle'$. then $k_{10} = d_{21} + d_{22}$ has been divided or has two columns φ_1, φ_2, to explain these columns, since $\langle 16,10,2\rangle \downarrow S_{27} = \langle 15,10,2\rangle^{*1} + \langle 16,9,2\rangle^{*2} + \langle 16,10,1\rangle^{*4}$ has 7 of i.m.s. we have $a_1 \in \{0,1,2,3\}$. In the same way we $a_{24} \in \{0,1\}, a_2, a_5, a_8, a_{10}, a_{19} \in \{0,1,2\}, a_{14}, a_{18}, a_{21}, a_{22}, a_{23} \in \{0,1,2,3\}, a_6, a_7, a_{11}, a_{15} \in \{0,1,\ldots,4\}$ $a_{12}, a_{13}, a_{16}, a_{17}, a_{20} \in \{0,1,\ldots,6\}$, $a_4 \in \{0,1,\ldots,7\}, a_3 \in \{0,1,\ldots,8\}$, $a_9 \in \{0,1,\ldots,10\}$. Let $a_1 \in \{1,2,3\}$ (if$a_1 = 0$ contradiction). Since $\langle 16,10,2\rangle \downarrow S_{27} \cap \langle 15,13\rangle^* \downarrow S_{23}$ has no i.m.s so $a_2 = 0$, the same way we get $a_7, a_8, a_{10}, a_{11}, \ldots, a_{24}$ are equal to zero, and since inducing m.s. is m.s. then we get:

$$(\langle 17,8,2\rangle^* - \langle 15,8,4\rangle^* + \langle 13,8,4,2\rangle) \uparrow^{(5,9)} S_{28}\ hence\ a_6 = 0 \quad (17)$$

$$(\langle 13,12,2\rangle^* - \langle 15,12\rangle + \langle 25,2\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_3 = 0 \quad (18)$$

Therefore, we only obtain degree $\varphi_1, \varphi_2 \equiv 0\ mod 13^2$ when $\varphi_1 + \varphi_2 = m(d_{21} + d_{22}), m \in \{1,2\}$, which is basically the partition of k_{10} into d_{21}, d_{22}. $\langle 15,10,3\rangle \neq \langle 15,10,3\rangle'$ so k_{12} or k_{13} is split. If k_{12} split to d_{27}, d_{28}, but $\langle 15,9,4\rangle \neq \langle 15,9,4\rangle'$ then k_{13} split to, d_{29}, d_{30}. If k_{13} split, and

$$\langle 15,8,5\rangle + \langle 15,10,3\rangle + \langle 17,9,2\rangle + \langle 21,5,2\rangle + \langle 23,3,2\rangle - \langle 15,9,4\rangle - \langle 16,10,2\rangle - \langle 18,8,2\rangle - \langle 22,4,2\rangle \neq \langle 15,8,5\rangle' + \langle 15,10,3\rangle' + \langle 17,9,2\rangle' + \langle 21,5,2\rangle' + \langle 23,3,2\rangle' - \langle 15,9,4\rangle' - \langle 16,10,2\rangle' - \langle 18,8,2\rangle' - \langle 22,4,2\rangle' \quad (19)$$

so that k_{12} is split, then k_{12} , k_{13} splits. Since$\langle 15,8,5\rangle \neq \langle 15,8,5\rangle'$ so k_{14} or k_{15} is split. If k_{14} is split to d_{31}, d_{32}, but $\langle 15,7,6\rangle \neq \langle 15,7,6\rangle'$ then k_{15} split to, d_{33}, d_{34}. If k_{15} is split, and

$$\langle 15,8,5\rangle - \langle 18,8,2\rangle - \langle 20,6,2\rangle - \langle 15,7,6\rangle + \langle 19,7,2\rangle + \langle 21,5,2\rangle \neq \langle 15,8,5\rangle' - \langle 18,8,2\rangle' - \langle 20,6,2\rangle' - \langle 15,7,6\rangle' + \langle 19,7,2\rangle' + \langle 21,5,2\rangle' \quad (20)$$

then k_{14} split, so k_{14},k_{15} splits. Since$\langle 10,9,4,3,2\rangle \neq \langle 10,9,4,3,2\rangle'$ so k_{18} or k_{20} is split. If k_{18} is split to d_{47}, d_{48}, but $\langle 8,7,6,5,2\rangle \neq \langle 8,7,6,5,2\rangle'$ then k_{20} split to, d_{51}, d_{52}. If k_{20} is split, and

$$\langle 10,9,4,3,2\rangle - \langle 8,7,6,5,2\rangle \neq \langle 10,9,4,3,2\rangle' - \langle 8,7,6,5,2\rangle' \quad (21)$$

then k_{18} is split, so k_{18}, k_{20} splits. Since$\langle 10,7,6,3,2\rangle \neq \langle 10,7,6,3,2\rangle'$ so k_{19} or k_{21} is split. If k_{19} is split d_{49}, d_{50}, but $\langle 9,8,5,4,2\rangle \neq \langle 9,8,5,4,2\rangle'$ then k_{21} split, d_{53}, d_{54}. If k_{21} , is split and

$$\langle 10,8,5,3,2\rangle - \langle 10,7,6,3,2\rangle \neq \langle 10,8,5,3,2\rangle' - \langle 8,7,6,5,2\rangle' \quad (22)$$

then k_{19} is split, so k_{19}, k_{21} splits. Since$\langle 12,9,4,2,1\rangle \neq \langle 12,9,4,2,1\rangle'$ so k_{16} or k_{17} is split. If k_{16} split d_{37}, d_{38}, but $\langle 10,8,5,3\rangle \neq \langle 10,8,5,3\rangle'$ then k_{17} split d_{45}, d_{46}. If k_{17} split, and

$$\langle 10,9,4,3,2\rangle + \langle 10,7,6,3,2\rangle - \langle 9,7,6,4,2\rangle - \langle 10,8,5,3,2\rangle + \langle 9,8,5,4,2\rangle + \langle 8,7,6,5,2\rangle \neq \langle 10,9,4,3,2\rangle' + \langle 10,7,6,3,2\rangle' - \langle 9,7,6,4,2\rangle' - \langle 10,8,5,3,2\rangle' - \langle 9,8,5,4,2\rangle' + \langle 8,7,6,5,2\rangle' \quad (23)$$

then k_{19} is split, so k_{19}, k_{21} splits. Since $\langle 15,11,2\rangle \neq \langle 15,11,2\rangle'$ on $(13, \alpha)$-regular classes and we have **294** columns in the decomposition matrix, then k_{11} must be split to d_{25}, d_{26}.

Conclusions

There is no prescribed method to find irreducible modular spin properties when the field property is primary, especially when the investigation concerns the same field with a group change. As a result, we had to conduct a series of studies to collect enough information to find new properties and theorems, including decomposition matrices that establish a connection between irreducible spin characteristics and irreducible modular spin characteristics. This opens the way for a comprehensive investigation that first examines the properties of irreducible modular spins before classifying entities.

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Conflict of Interest

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