Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2024, 37(4)

Analysing the Result Involution Graph of the Group J_3

Ahmed Arkan Meteab[1] **and Ali Abd Aubad**[2,*]
[1,2]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
*Corresponding Author.

Received: 12 April 2023 Accepted: 18 May 2023 Published: 20 October 2024
doi.org/10.30526/37.4.3405

Abstract

Assume that G be a finite group and let I(G) be the set of the involution elements in G. The result involution graph denoted by , Γ_G^{RI}, is an undirected simple graph having the elements of G as a vertex set. Moreover, two vertices in Γ_G^{RI} are connected by an edge if they are distinct and their product belong to I(G). The objective of this work is to investigate the result involution graph for the Janko group J_3. In this paper we compute different result involution graph features, such as the radius, the diameter, the clique number, and the girth. Furthermore, the connectedness of the result involution graph is determined. All of the steps needed for analyzing the result involution graph were carried out using the computational technique along with theoretical support.

Keywords: Janko Groups, Result Involution Graph, Connectedness, Girth.

1. Introduction

One of the most important techniques for examining group structure is possibly the action of a group on a graph. The numerous and diverse graphs that have been analyzed for various groups are highlighted by the substantial citations in [1-14]. If an element of a group has order 2, it is said to be involution. Involutions of groups significantly affect the way it is structured. For instance, the categorization of finite simple groups heavily depends on the study of involutions. An S3-involution graph for a group G was first introduced by Devillers and Giudici in [15] as a graph with a G-classes of involution as a vertex set, where two vertices are adjacent if they generate an S_3-subgroup in a certain G-class. They looked at the structure of S_3-involution graph and general characteristics for the group PSL (2,q) when q is greater than 3. Define I(G) as a set of the involution elements in a finite group G. The result involution graph is known as the simple undirected graph with vertex set being the elements of G with two vertices x,y G are adjacent if $x \neq y$ and $xy \in$ I(G), and is denoted by Γ_G^{RI}. Jund and Salih reported the result involution graph and its attributes for the first time in [16]. In their analysis, they demonstrated that the graphs Γ_{Sn}^{RI} and Γ_{An}^{RI} are connected graphs for n>4 and have a maximum diameter, radius, and girth of 3 and 3. Also, they provide some really helpful qualities for the result involution graphs of the quaternion group and the dihedral group. Nevertheless, Aubad and Salih looked at the structure of the involution graphs that were produced for the whole list of the Mathieu sporadic simple groups as mentioned in [17]. Furthermore, the result involution graphs for the Conway group Co_3 are analyzed with full information in [18].

Examining the result involution graph for the Janko group J_3, is the aim of this article. Together with the connectedness of the result involution graph, we show many graph properties including the radius, diameter, clique number, and girth. The paper is set up like follows: We offer some observations and graph notations for the result involution graph in section 2. A number of results about Γ_{J3}^{RI} are provided in section 3. We draw our concluding conclusions and make suggestions for more study in section 4.

2. Preliminary

We begin the section by defining a few concepts and outlining several facts that will be important to the thesis later on. First let G be a finite group moving forward. Let $\mathbf{sI}_G=|I(G)|$ be the size of the set I(G), where I(G) be the set of the involution elements in G as mentioned in the introduction. We begin with the following formula to determine how many edges there are in the result involution graph:

Proposition 2.1 [17]: Let G be a finite group, and let F represent the number of elements of order 4 in G. Then the number of edges in the result involution graph is calculated using the following formula ½ $(\boldsymbol{s}\mathbf{I}_{\boldsymbol{G}}\ |G| - F)$.

We should notice that if G is a finite group with large number of elements, then the Γ_G^{RI} has a huge vertex set. The result involution graph of G is then exceedingly difficult to handle. To resolve this issue, we use the resize graph idea, which define as follows:

Definition 2.2 [17]: Assume that G be a finite group, the resize graph of G, Γ_G^{RS}, has a vertex set the whole G-conjugacy classes with two vertices (G-conjugacy classes) X,Y $\subseteq$ G, are adjacent if X≠Y and there are $v_1 \in$X and $v_2 \in$ Y such that v_1, v_2 are adjacent vertices in Γ_G^{RI}. The following results displays a relationship between a result involution graph and a resize graph:

Proposition 2.3 [17]: Let G be a finite group. Then the connectivity of the graphs Γ_G^{RI} and Γ_G^{RS} are isomorphic graphs.

These findings will be illustrated by the example that follows. We will mostly employ GAP [19] and YAGS [20] as a computational approach in the following example.

Example 2.4: Let G be an alternating group of degree 4. Then the result involution graph Γ_{A4}^{RI} contains the following vertices:{e,(2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4), (1,3,2), (1,3,4), (1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3)}.Then we can conclude the figure below that the graph Γ_{A4}^{RI} is disconnected with two components. One of these components has four vertices labels by the numbers {1,9,4,12} and the subgraph induced with this vertex is the compete graph K_4. On the other hand, the subgraph induced by the vertices labels by {2,3,5,6,7,8,10,11}. This graph is 3-regular graph with dimeter and radius of size 3. Moreover, clique number of size 2 and girth 4. This information can be seen the following **Figure**:

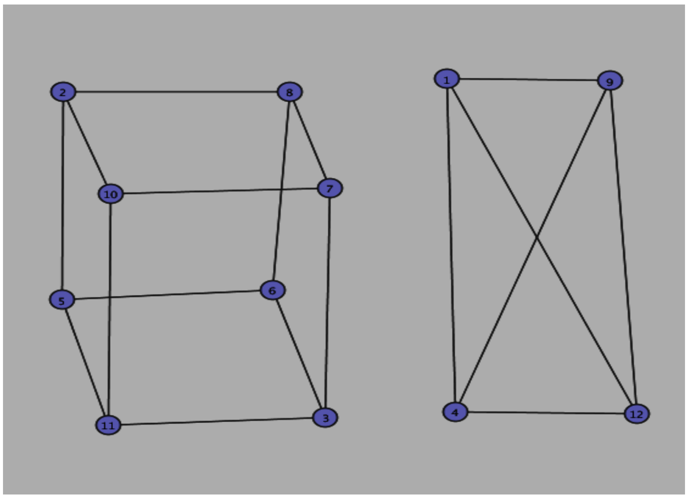

Figure 1. The Structure of the Result Involution Graph Γ_G^{RI}, G≅A$_4$.

On the other hand, the resize graph of G, Γ_{A4}^{RS} has vertex set present as {1A,2A,3A,3B}. In the next figure we can see that resize graph of A$_4$ is (1,1)-biregular disconnected graph has vertex set can be split into two disjoint vertices can be labeled by X={1,2} and Y={3,4}:

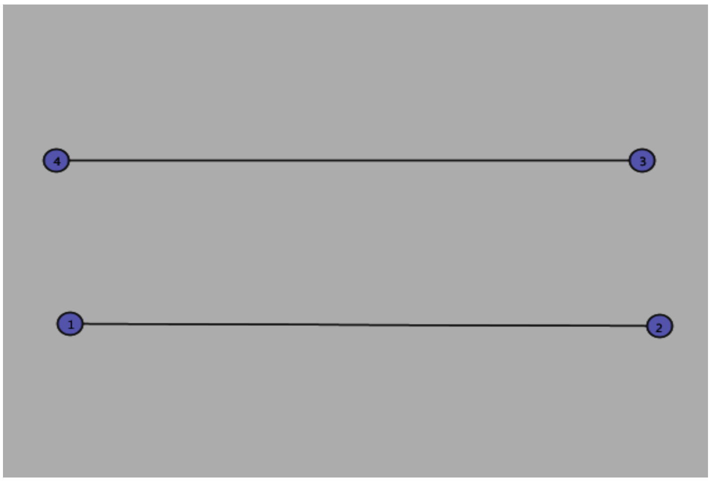

Figure 2. The Structure of the Resize Graph Γ_G^{RS}, G≅A$_4$.

3. Results

In this section we will look at the structure of the result involution graph for the Janko group J$_3$. To achieve the goal of the study, we will use GAP and the Online Atlas [21]. We should note that the group J$_3$ has only one class of involution namely 2A with size equal to 26163. There are several involutions to take into account as a consequence. Hence, in order to arrive at our observations, we will investigate the resizing graph Γ_{J3}^{RS}.

3.1 The Structures of Γ_{J3}^{RS}

The Online Atlas reveals that J$_3$ has different 21 conjugacy classes. Therefore , the resizing graph Γ_{J3}^{RS} has order 42. Furthermore, the vertex set of Γ_{J3}^{RS} could now be stated in the following manner:{1A , 2A , 3A , 3B , 4A , 5A , 5B , 6A , 8A , 9A , 9B , 9C , 10A , 10B , 12A , 15A , 15B , 17A , 17B ,19A , 19B}.

In addition, the vertex degree sequence is shown below:

{1, 18, 14, 17, 18, 19, 19, 18, 19, 18, 18, 18, 19, 19, 19, 19,19, 19, 19, 18, 18}

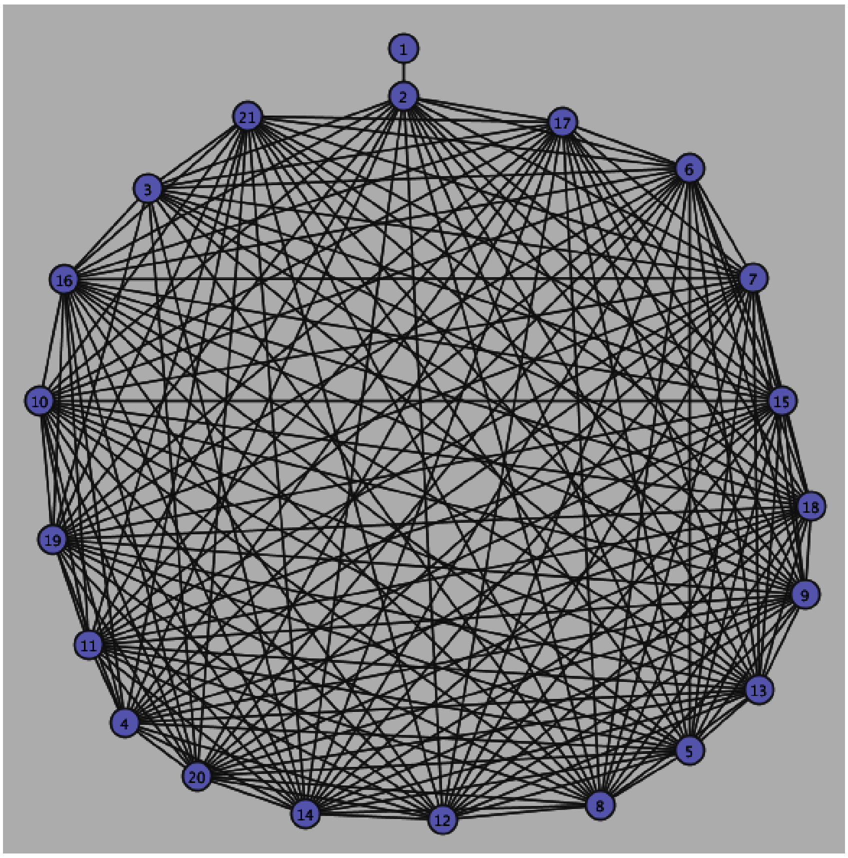

Figure 3. The Structure of the Resize Graph Γ_G^{RS}, G≅J$_3$.

According to the data in **Figure 3**, Γ_{J3}^{RS} is connected and has the following characteristics:

Table 1. Properties of the Resize Graph Γ_{J3}^{RS}.

Γ_G^{RS}	$E(\Gamma_G^{RS})$	Girth	Clique Number	Radius	Diameter
G≅J$_3$	183	3	17	2	3

3.2 The Edges Set of Result Involution Graphs

In this section, we provide comprehensive information about the edges set of the result involution graph Γ_{J3}^{RI} for the Janko group J$_3$. The data concern the number of edges connecting any two G-Conjugacy elements. The findings will be crucial in assessing the construct of the result involution graph of J$_3$. We will use GAP and the Online Atlas to get the intended result. We should be informed of that, in the elements of the class 5A there are an edges with the remaining J$_3$-conjugacy classes except the identity element. This can be seen in the following set:
{2A(33488640),3A(25116480),3B(167443200),4A(418608000),5A(598609440),5B(1573966080),6A(1783270080),8A(5425159680),9A(1557221760),9B(1557221760),9C(1557221760),10A(4651572096),10B(4470733440),12A(3717239040),15A(3265142400),15B(2662346880),17A(2561880960),17B(2561880960),19A(2310716160),19B(2310716160)}.
In addition, we can find 26163 edges between the elements of the class 2A.

3.3 The Structures of Γ_{J3}^{RI}

For the Janko group J$_3$, the structures of the result involution graph are fully described in the following theorem:

Theorem 3.1: The result involution graph for the Janko group J$_3$ is connected with diameter 3, radius 2, and girth 3.

Proof: The resizing graph of J_3 is connected using **Table 1**. As a result, by utilizing **Proposition 2.3**, we are able to determine how Γ_{J3}^{RI} is connected. In addition, between the elements of the class 2A there are 26163 edges and these elements adjacent with the identity element of J_3. Hence, we have a cycle with a length of 3 in the graph Γ_{J3}^{RI}. Therefore, the result involution graph has girth 3 in this case. Furthermore, The class 5A vertices and the identity element are not adjacent. But, during the classes 2A, these vertices were connected to the identity. As a result, in this scenario, the radius is 2. Moreover, the diameter is 3 due to the other vertices maximum distance being 3.

We provide the following significant corollaries in light of the preceding finding:

Corollary 3.2: Suppose that G be the Janko groups J_3. Then for any distinct elements $x,y \in$ G, one of the following are holds:

- i- Their product is an involution element.
- ii- There exist an certain $z \in$ G, satisfy its product with x and y produce an involution elements.
- iii- There are an elements $z,w \in$ G, satisfy (xz), (zw) and (wy) are involution elements.
- iv- J_3 has three distinct components, any one of them has an involution product with the others.

Proof: By applying the outcomes of **Theorem 3.1** we have the diameter 2 and the radius 2 for the result involution graph of G . Therefore, when x and y have distance one then their product is an involution element so (**i**) is hold. If the distance between x and y equal 2, then one can find z in G satisfying (**ii**). Finally, to prove (**iii**) the distance between x and y equal to 3, implies there are such $z,w \in$ G satisfy the conditions of (**iii**). Finally, the result involution graph of G with girth 3. As a result, the characterization of the result involution graph normally leads to the proof of **iv**.

4. Conclusion

This study analyses the result involution graph for the Janko group J_3. the computational method used to determine certain graph properties. Detailed information about the resize graph, for example, as well as the radius, diameter and circumference. The results of this study can be applied to the study of more complicated simple groups, including monster groups, pariah groups and Lie-type exceptional groups.

Acknowledgment

The authors would like to thank the University of Baghdad (https://uobaghdad.edu.iq/), Department of Mathematics in the College of Sciences for their motivation and support.

Conflict of Interest

The authors declare that they have no conflicts of interest.

Funding

No funding.

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