



## local Bifurcation of the Dynamic Behavior of Predator-Prey System with Refuge for both Species

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### Abstract

The main purpose of this paper is to study a predator – prey dynamical system consisting of three species prey, specialized predator and generalist predator namely  $H(t)$ ,  $I(t)$  and  $J(t)$  respectively, with food web and refuge for the prey and specialized predator population. The considered system has five equilibrium points  $A_0 = (0, 0, 0)$ ,  $A_1 = (1, 0, 0)$ ,  $A_2 = (\bar{h}, \bar{i}, 0)$ ,  $A_3 = (\bar{h}, 0, \bar{j})$ , and the positive equilibrium point  $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$ . The stability and bifurcation of the equilibrium points was studied and the main influence was the qualitative behavior of the solution. It was found that  $A_0$  was unstable while the other equilibrium points are stable under condition so we study their bifurcation and show that  $A_1$ ,  $A_2$  and  $A_3$  are transcritical while  $A_4$  is saddle node bifurcation. Numerical simulations were used to illustrate the occurrence of local bifurcation of this model.

**Keywords:** Local bifurcation, predator–prey, stability analysis, Lyapunov's function, ecological, Refuge.

### 1. Introduction

The mathematical study of changes in a dynamical system's qualitative asymptotic structure is known as bifurcation theory (1, 2). As well as its attempts to explain various phenomena that have been described in the natural sciences over the centuries. Where performing bifurcation analysis is often a powerful way to analyze the properties of such systems. The prey and predator model is an important topic at present as it is used to solve many problems in the ecological nature and other sciences. The prey system includes several interactions, such as competition co-existence and stage-structured (3). The system is also impacted by a number of other factors, such as shelter, sickness, and others. A bifurcation, which is the primary qualitative shift in the behavior of a dynamic system as a result of changing one of its coefficients, can occasionally



emerge from variations in any parameter in the system, leading to complex behavior that leads to system instability. Local and global bifurcations were the two main classes that made up the bifurcation. Changes in the local stability parameters of equilibria or periodic orbits can be used to evaluate local bifurcation. Global bifurcation, on the other hand, happens when periodic orbits run into equilibrium. This leads to changes in the topology of the trajectories in phase space that cannot be contained within a limited region as is the case with local bifurcation. These bifurcations occur when a single parameter is changed (4-9). Perko (10), on the other hand, identified the prerequisites for local bifurcation, including saddle-node, transcritical, pitchfork, and period-doubling. Finally, Sotomayor's theorem(11) for local bifurcation was applied in this work to examine the occurrence of local bifurcation at equilibrium sites local bifurcation methods close to all equilibrium points, and a number of fundamental results (12-22).

## 2. Model formulation

An ecological model was suggested for investigation in this section. A prey was included in the model, and its overall population density at time  $t$  is represented by the symbol  $H(t)$ , engaging with a specialized predator  $I(t)$  is the population density at time  $t$  and generalist predator whose population density at time  $t$  is denoted by  $J(t)$ . It was assumed that the prey with refuge and specialist predator is with the prey refuge.

The following presumptions are now used to create the fundamental ecological model shown in **Table 1:**

$$\begin{aligned} \frac{dH}{dt} &= \rho H \left(1 - \frac{H}{K}\right) - \alpha (1-m_1) H I - \beta (1-m_2) H J \\ \frac{dI}{dt} &= \alpha_1 (1 - m_1) H I - \gamma (1-m_3) IJ - d_1 I \\ \frac{dJ}{dt} &= \beta_1 (1 - m_2) H J - \gamma_1 (1-m_3) IJ - d_2 J \end{aligned} \tag{1}$$

**Table 1.** The parameters of model (1)

parameters	Biological meaning
$\rho > 0$	intrinsic growth
$k > 0$	carrying capacity (in logistic growth)
$\alpha > 0$	maximum attack rate by specialist predator
$\beta > 0$	maximum attack rate by generalist predator
$d_1$ and $d_2$	natural death rate of specialist and generalist predator
$\alpha_1 > 0$	maximum predation rate of the specialist predator over the prey
$\beta_1 > 0$	maximum predation rate of the generalist predator over the prey
$\gamma > 0$	maximum attack rate by generalist predator on specialist predator
$\gamma_1 > 0$	maximum predation rate of the generalist predator over the specialist predator
$0 < m_1 < 1,$	The refuge rates constants of the prey from the specialist predator
$0 < m_2 < 1$	The refuge rates constants of the prey from the generalist predator
$0 < m_3 < 1$	The refuge rates constants of the specialist predator from the generalist predator

The above model has 13 parameters so, it is difficult to study all of them, there for reduce them a dimensionless variables and parameters are defined:

$$t^* = \rho t, \quad h = \frac{H}{k}, \quad i = \frac{I}{k}, \quad j = \frac{J}{k}, \quad r_1 = \frac{\alpha k(1-m_1)}{\rho}, \quad r_2 = \frac{\beta k(1-m_2)}{\rho}$$

$$r_3 = \frac{\alpha_1 k(1-m_1)}{\rho}, \quad r_4 = \frac{\gamma k(1-m_3)}{\rho}, \quad r_5 = \frac{d_1}{\rho}, \quad r_6 = \frac{\beta_1 k(1-m_2)}{\rho}, \quad r_7 = \frac{\gamma_1 k(1-m_1)}{\rho},$$

$$r_8 = \frac{d_2}{\rho}.$$

Now, for simplicity rename  $t^* = t$ .

So, the dimensional system (1) can be formulated as:

$$\left. \begin{aligned} \frac{dh}{dt} &= h[1 - h - r_1 i - r_2 j] = f_1(h, i, j) \\ \frac{di}{dt} &= i[r_3 h - r_4 j - r_5] = f_2(h, i, j) \\ \frac{dj}{dt} &= j[r_6 h + r_7 i - r_8] = f_3(h, i, j) \end{aligned} \right\} \quad (2)$$

With  $h(0) \geq 0, i(0) \geq 0$  and  $j(0) \geq 0$ . In system (2) there are 8 parameters. All of the functions on system (2) right side are  $C^2(\mathbb{R}^3, \mathbb{R}^+)$ .

$$R_+^3 = \{(h, i, j) \in \mathbb{R}^3 : h(0) \geq 0, \quad i(0) \geq 0, \quad j(0) \geq 0\}.$$

Therefore, these functions are Lipschitzian on  $R_+^3$ , and as a result, system (2) has a unique and existing solution.

**Theorem 1 [Uniformly Boundedness]:**

All the solutions of system (2) with nonnegative initial conditions are uniformly bounded.

**Proof:** Let the solution of (2) be  $[h(t), i(t), j(t)]$  the initial condition  $[h(0), i(0), j(0)] \in R_+^3$  are nonnegative.

Now, let  $\dot{H}(t) = h(t) + i(t) + j(t)$ ,

$$\frac{d\dot{H}}{dt} < 2h - (r_1 - r_3)hi - (r_2 - r_6)hj - (r_4 - r_7)ij - h - r_5i - r_8j.$$

Now, ecologically  $r_3 < r_1, r_6 < r_2$  and  $r_7 < r_4$ .

$$\frac{d\dot{H}}{dt} < 2 - \delta \dot{H}, \quad \text{where } \delta = \min \{1, r_5, r_8\}.$$

Now, by solving this differential inequality for the initial value  $H(0) = H_0$ , we get that:

$$\dot{H}(t) \leq \frac{2}{\delta} + \left( \dot{H}(0) - \frac{2}{\delta} \right) e^{-\delta t}$$

Thus  $0 \leq \dot{H}(t) \leq \frac{2}{\delta}$  as  $t \rightarrow \infty$ .

Hence system (2) has uniformly bounded.

**3. Equilibrium points are existence and stable:**

There are maximum of five equilibrium points in System (2), which are listed below:

- The point of equilibrium  $A_0 = (0, 0, 0)$ , it is always present it is referred to as the vanishing point, which is unstable and always existing.
- The axial equilibrium point  $A_1 = (1, 0, 0)$ , existence without conditions additionally.

Therefore, the characteristic equation of  $J(A_1)$  is as follows:

$$J_1 = J(A_1) = \begin{bmatrix} -1 & -r_1 & -r_2 \\ 0 & r_3 - r_5 & 0 \\ 0 & 0 & r_6 - r_8 \end{bmatrix}. \tag{1.a}$$

$$(-1 - \lambda) [(r_3 - r_5) - \lambda] [(r_6 - r_8) - \lambda] = 0.$$

Which gives the eigenvalues of  $J_1$  by:

$$\lambda_{1h} = -1 < 0, \quad \lambda_{1i} = r_3 - r_5 < 0 \quad \text{and} \quad \lambda_{1j} = (r_6 - r_8) < 0$$

The equilibrium point  $A_1$  then becomes asymptotically stable under the following conditions:

$$r_3 > r_5, \tag{3}$$

$$r_8 > r_6. \tag{4}$$

Otherwise,  $A_1$  is unstable. However, it is a saddle point.

● The equilibrium point  $A_2(\bar{h}, \bar{i}, 0)$  exists uniquely in  $Int. R_+^2$  (Interior of  $R_+^2$ ) of  $hi$  – plane provided that:

$$r_3 > r_5. \tag{5}$$

Where:

$$\bar{h} = \frac{r_5}{r_3} > 0. \tag{6}$$

$$\bar{i} = \frac{r_3 - r_5}{r_1 r_3}. \tag{7}$$

And the Jacobian matrix of system (2) at  $A_2$  can be written as

$$J_2 = J(A_2) = [\mu_{ij}]_{3 \times 3}, \tag{2.a}$$

where:

$$\mu_{11} = 1 - 2\bar{h} - r_1\bar{i}, \quad \mu_{12} = -r_1\bar{h} < 0, \quad \mu_{13} = -r_2\bar{h},$$

$$\mu_{21} = r_3\bar{i} > 0, \quad \mu_{22} = r_3\bar{h} - r_5, \quad \mu_{23} = -r_4\bar{i},$$

$$\mu_{31} = 0, \quad \mu_{32} = 0, \quad \mu_{33} = r_6\bar{h} + r_7\bar{i} - r_8, \quad ,$$

Consequently, the characteristic equation of  $J(A_2)$  is as follows:

$$(\mu_{33} - \lambda)[\lambda^2 - \hat{A}]\lambda + \det(\hat{A}) = 0$$

$$\text{where : } \hat{A} = \begin{bmatrix} 1 - 2\bar{h} - r_1\bar{i} & -r_1\bar{h} \\ r_3\bar{i} & r_3\bar{h} - r_5 \end{bmatrix},$$

$$\text{Then } [(r_6\bar{h} + r_7\bar{i} - r_8) - \lambda] [(1 - 2\bar{h} - r_1\bar{i} - \lambda)(r_3\bar{h} - r_5 - \lambda) + r_1r_3\bar{h}\bar{i}] = 0$$

$$\text{Either, } \lambda_{2h} = r_6\bar{h} + r_7\bar{i} - r_8$$

Which, as a result of the following condition, produces the first eigenvalues  $J_2$  with negative real parts:

$$r_8 > r_6\bar{h} + r_7\bar{i} \tag{8}$$

$$\text{or, } \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

Where:

$$\begin{aligned} \text{tr}(\hat{A}) &= \lambda_{2i} + \lambda_{2j} = 1 - 2\bar{h} - r_1\bar{i} + r_3\bar{h} - r_5 \\ &= (1 + r_3\bar{h}) - (2\bar{h} + r_1\bar{i} + r_5) > 0, \end{aligned}$$

$$\begin{aligned} \det(\hat{A}) &= \lambda_{2i}\lambda_{2j} \\ &= (r_3\bar{h}-r_5)\left(1-2\bar{h}-r_1\bar{i}\right)+r_1r_3\ 2r_5\ \bar{i} \\ &= (2r_5+r_3)\bar{h}+r_1r_5\ \bar{i}-(2r_3\ \bar{h}^2+r_5)>0. \end{aligned}$$

Which, as a result of the following criteria, produces the second two eigenvalues of  $J_2$  with negative real parts:

$$2\bar{h}+r_1\bar{i}+r_5<1+r_3\bar{h}, \tag{9}$$

$$2r_3\ \bar{h}^2+r_5<(2r_5+r_3)\bar{h}+r_1r_5\ \bar{i}. \tag{10}$$

Therefore,  $A_2$  is stable equilibrium point if conditions (8), (9) and (10) are satisfied. However otherwise, it is unstable.

⊙ The specialist predator free equilibrium point  $A_3=(\bar{\bar{h}}, 0, \bar{\bar{j}})$  exists if the solutions to the following set of equations are positive:

$$\bar{\bar{h}}=\frac{r_8}{r_6}>0 \tag{11}$$

$$\bar{\bar{j}}=\frac{r_6-r_8}{r_2\ r_6} \tag{12}$$

The equation (12) is positive, provided that:

$$r_6>r_8. \tag{13}$$

The Jacobian matrix of system ( 2 ) at  $A_2$  can be written as:

$$J_3=J(A_3)=[\eta_{ij}]_{3\times 3}, \tag{3.a}$$

where:  $\eta_{11}=1-2\bar{\bar{h}}-r_2\bar{\bar{j}}, \eta_{12}=-r_1\bar{\bar{h}}<0,$

$\eta_{13}=-r_2\bar{\bar{h}}<0, \eta_{21}=0, \eta_{22}=r_3\bar{\bar{h}}-r_4\bar{\bar{j}}-r_5,$

$\eta_{33}=r_6\bar{\bar{h}}-r_8, \eta_{23}=0, \eta_{31}=r_6\bar{\bar{j}}, \eta_{32}=r_7\bar{\bar{j}}$

Characteristic equation for  $J(A_3)$  is then provided by:

$$(\eta_{22}-\lambda)[\lambda^2-\text{tr}(\hat{A})\lambda+\det(\hat{A})]=0$$

where :  $\hat{A}=\begin{bmatrix} 1-2\bar{\bar{h}}-r_2\bar{\bar{j}} & -r_2\bar{\bar{h}} \\ r_6\bar{\bar{j}} & r_6\bar{\bar{h}}-r_8 \end{bmatrix}.$

Then  $[(r_3\bar{\bar{h}}-r_4\bar{\bar{j}}-r_5)-\lambda][(1-2\bar{\bar{h}}-r_2\bar{\bar{j}}-\lambda)](r_6\bar{\bar{h}}-r_8-\lambda)+r_2r_6\ \bar{\bar{h}}\bar{\bar{j}}=0$

Either,  $\lambda_{3h}=r_3\bar{\bar{h}}-r_4\bar{\bar{j}}-r_5$

Because of the following circumstance the first eigenvalues of  $J_3$  have negative real portions:

$$r_3\bar{\bar{h}}>r_4\bar{\bar{j}}+r_5 \tag{14}$$

or,  $\lambda^2-\text{tr}(\hat{A})\lambda+\det(\hat{A})=0$

Where,

$$\text{tr}(\hat{A})=\lambda_{3i}+\lambda_{3j}=1-2\bar{\bar{h}}-r_2\bar{\bar{j}}+r_6\bar{\bar{h}}-r_8>0$$

$$\begin{aligned} \det(\hat{A}) &= \lambda_{3i}\lambda_{3j} \\ &= (1-2\bar{\bar{h}}-r_2\bar{\bar{j}})(r_6\bar{\bar{h}}-r_8)+r_2r_6\ \bar{\bar{h}}\bar{\bar{j}} \end{aligned}$$

$$= (r_6 + 2 r_8)\bar{h} + r_2 r_8 \bar{j} - (2r_6 \bar{h}^2 + r_8) > 0.$$

Therefore, as a result of the following requirements, results in the second two eigenvalues of  $J_3$  having negative real portions:

$$2\bar{h} + r_2\bar{j} - r_6\bar{h} < 1, \tag{15}$$

$$2r_3 \bar{h}^2 + r_5 < (2r_5 + r_3)\bar{h} + r_1 r_5 \bar{i}. \tag{16}$$

Therefore,  $A_3$  is stable equilibrium point if conditions (14), (15) and (16) are satisfied. On the other hand, it is unstable.

● Finally, the positive (coexistence) equilibrium point  $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$  exists if the following system of equations has a positive solution:

$$\begin{aligned} \tilde{h} &= \frac{(r_4 + r_1 r_5) - r_1 r_4 r_8 / r_7}{(r_4 + r_1 r_3) - r_1 r_4 r_6 / r_7} \\ \tilde{i} &= \frac{r_8 - r_6 \tilde{h}}{r_7}, \quad \tilde{j} = \frac{r_3 \tilde{h} - r_5}{r_4}. \end{aligned}$$

Note that  $\tilde{h}$  is positive, provided that:

$$(r_4 + r_1 r_5) < r_1 r_4 r_8 / r_7 \quad \text{and} \quad (r_4 + r_1 r_3) < r_1 r_4 r_6 / r_7.$$

Or

$$(r_4 + r_1 r_5) > r_1 r_4 r_8 / r_7 \quad \text{and} \quad (r_4 + r_1 r_3) > r_1 r_4 r_6 / r_7$$

So,  $\tilde{i}$  and  $\tilde{j}$  are positive, provided that:

$$r_3 \tilde{h} > r_5 \quad \text{and} \quad r_8 > r_6 \tilde{h} \quad \text{respectively.}$$

For  $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$ , can be expressed as:

$$J_4 = J(A_4) = [\tilde{n}_{ij}]_{3 \times 3}, \tag{4.a}$$

Where:

$$\begin{aligned} \tilde{n}_{11} &= 1 - 2\tilde{h} - r_1\tilde{i} - r_2\tilde{j}, \quad \tilde{n}_{12} = -r_1\tilde{h} < 0, \quad \tilde{n}_{13} = -r_2\tilde{h} < 0, \quad \tilde{n}_{21} = r_3\tilde{i} > 0, \\ \tilde{n}_{22} &= r_3\tilde{h} - r_4\tilde{j} - r_5, \quad \tilde{n}_{23} = -r_4\tilde{i}, \quad \tilde{n}_{31} = r_6\tilde{j} > 0, \quad \tilde{n}_{32} = r_7\tilde{j} > 0, \quad \tilde{n}_{33} = r_6\tilde{h} + r_7\tilde{i} - r_8 \end{aligned}$$

A characteristic equation for  $J(A_4)$  is then provided by:

$$\lambda^3 + \check{R}_1 \lambda^2 + \check{R}_2 \lambda + \check{R}_3 = 0, \tag{4.b}$$

where:  $\check{R}_1 = -(\tilde{n}_{11} + \tilde{n}_{22} + \tilde{n}_{33}),$

$$\check{R}_2 = -[\tilde{n}_{23}\tilde{n}_{32} - \tilde{n}_{22}\tilde{n}_{33} - \tilde{n}_{11}(\tilde{n}_{22} + \tilde{n}_{33}) + \tilde{n}_{21}\tilde{n}_{12} + \tilde{n}_{13}\tilde{n}_{31}]$$

$$\check{R}_3 = -\tilde{n}_{11}(\tilde{n}_{22}\tilde{n}_{33} - \tilde{n}_{23}\tilde{n}_{32}) + \tilde{n}_{12}\tilde{n}_{21}\tilde{n}_{33} - \tilde{n}_{12}\tilde{n}_{31}\tilde{n}_{23} - \tilde{n}_{13}\tilde{n}_{21}\tilde{n}_{32} - \tilde{n}_{13}\tilde{n}_{31}\tilde{n}_{22}.$$

Now,  $\check{R}_1 > 0$  and  $\check{R}_2 > 0$  provided that:

$$1 < 2\tilde{h} + r_1\tilde{i} + r_2\tilde{j}, \tag{17}$$

$$\tilde{h} < r_4\tilde{j} + r_5, \tag{18}$$

$$r_6\tilde{h} + r_7\tilde{i} < r_8 \tag{19}$$

Also,  $\Delta = \check{R}_1 \check{R}_2 - \check{R}_3 > 0.$

By the following condition:

$$\tilde{h} < \frac{r_4}{r_1 r_3 + 2 r_4} (1 - r_1 \tilde{i} - r_2 \tilde{j}), \tag{20}$$

$$r_2 r_6 \tilde{h} \tilde{j} < w_1 w_2, \tag{21}$$

$$r_4 r_6 \tilde{i} \tilde{j} < r_1 \tilde{h} (r_6 \tilde{h} + r_7 \tilde{i} - r_8). \tag{22}$$

Where,  $w_1 = 2\tilde{h} + r_1\tilde{i} + r_2\tilde{j} - 1,$   $w_2 = r_6\tilde{h} + r_7\tilde{i} - r_8.$

Using the Routh-Hurwitz criterion, however, allows each of the additional eigenvalues of eq. (4. b), have negative real parts if and only if  $R_1 > 0, R_3 > 0$  and  $R_1R_2 - R_3 > 0$ .

Therefore, all of  $J(A_4)$  eigenvalues have a negative real portion if the additional criteria from (17) - (22) hence  $A_4$  is asymptotically stable locally. In contrast, it is unstable.

#### 4. Local Bifurcation Analysis

This section investigates the dynamical behavior of system (2) around each equilibrium point as a result of altering the parameter values. Remember that the existence of the system (2)'s non-hyperbolic equilibrium point is a required, but not sufficient, need for bifurcation. As a result, it is appropriate to apply the Sotomayor's Theorem for local bifurcation in the following theorems. Currently, in accordance with the Jacobian matrix of system (2).

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial h} & \frac{\partial f_1}{\partial i} & \frac{\partial f_1}{\partial j} \\ \frac{\partial f_2}{\partial h} & \frac{\partial f_2}{\partial i} & \frac{\partial f_2}{\partial j} \\ \frac{\partial f_3}{\partial h} & \frac{\partial f_3}{\partial i} & \frac{\partial f_3}{\partial j} \end{bmatrix} \tag{5. a}$$

where  $f_i ; i=1, 2, 3$  are displayed on the system's right side (2) and

$$\frac{\partial f_1}{\partial h} = 1 - 2h - r_1i - r_2, \quad \frac{\partial f_1}{\partial i} = -r_1h, \quad \frac{\partial f_1}{\partial j} = -r_2h, \quad \frac{\partial f_2}{\partial h} = r_3i, \\ \frac{\partial f_2}{\partial i} = r_3h - r_4j - r_5, \quad \frac{\partial f_2}{\partial j} = -r_4i, \quad \frac{\partial f_3}{\partial h} = r_6j, \quad \frac{\partial f_3}{\partial i} = r_7j, \quad \frac{\partial f_3}{\partial j} = r_6h + r_7j - r_8.$$

It is clear to verify that for any nonzero vector  $\dot{\rho} = (\dot{\rho}_1, \dot{\rho}_2, \dot{\rho}_3)^T$  we have:

$$J\dot{\rho} = [\tau_{ij}]_{3 \times 1}$$

Where:

$$\tau_{11} = (1 - 2h - r_1i - r_2j)\dot{\rho}_1 - r_1h\dot{\rho}_2 - r_2h\dot{\rho}_3, \\ \tau_{21} = r_3i\dot{\rho}_1 + (r_3h - r_4j - r_5)\dot{\rho}_2 - r_4i\dot{\rho}_3, \\ \tau_{31} = r_6j\dot{\rho}_1 + r_7j\dot{\rho}_2 + (r_6h + r_7i - r_8)\dot{\rho}_3. \\ D^{2\circ}F_{\mu}(\dot{Y}, \mu)(\dot{\rho}, \dot{\rho}) = [\ddot{\tau}_{ij}]_{3 \times 1}. \tag{23}$$

Where:

$$\ddot{\tau}_{11} = (-2\dot{\rho}_1 - r_1\dot{\rho}_2 - r_2\dot{\rho}_3)\dot{\rho}_1 - r_1\dot{\rho}_1\dot{\rho}_2 - r_2\dot{\rho}_1\dot{\rho}_3, \\ \ddot{\tau}_{21} = r_3\dot{\rho}_1\dot{\rho}_2 + (r_3\dot{\rho}_1 - r_4\dot{\rho}_3)\dot{\rho}_2 - r_4\dot{\rho}_2\dot{\rho}_3,$$

$$\ddot{\tau}_{31} = r_6\dot{\rho}_1\dot{\rho}_3 + r_7\dot{\rho}_2\dot{\rho}_3 + (r_6\dot{\rho}_1 + r_7\dot{\rho}_2)\dot{\rho}_3.$$

Where  $\dot{Y} = (h, i, j)^T$  and  $\mu$  is any bifurcation parameter.

Theorems in the following the local bifurcation conditions near equilibrium points are established.

#### 4.1 Local bifurcation analysis near $A_1$ :

**Theorem (2):** If the value of the parameter  $r_3$  passes through  $\check{r}_3 = r_3$  then, system (2) at the axial equilibrium point  $A_1 = (1, 0, 0)$  possesses:

- No saddle-node bifurcation .
- Transcritical bifurcation .

**Proof:** According to the Jacobian matrix  $J(A_1)$  given by eq.(1. a ): Zero eigenvalue exists for system (2) at equilibrium point  $A_1 = (1,0,0)$ . (say  $\lambda_{1i} = 0$ ) at  $r_3 = \check{r}_3$ , and the Jacobian matrix  $J_1$  with  $r_3 = \check{r}_3$  becomes:

$$\ddot{j}_1 = J_1(r_3 = \ddot{r}_3) = \begin{bmatrix} -1 & -r_1 & -r_2 \\ 0 & 0 & 0 \\ 0 & 0 & r_3 - r_5 \end{bmatrix},$$

Now, let  $\dot{\rho}^{[1]} = (\dot{\rho}_1^{[1]}, \dot{\rho}_2^{[1]}, \dot{\rho}_3^{[1]})^T$  be the eigenvector corresponding to the eigenvalue  $\lambda_{1i} = 0$ . Thus  $(\ddot{j}_1 - \lambda_{1i}I) \dot{\rho}^{[1]} = 0$ , which gives:

$$\rho^{[1]} = (-r_1 \dot{\rho}_2^{[1]}, \dot{\rho}_2^{[1]}, 0) \text{ where } \dot{\rho}_2^{[1]} \text{ and } \dot{\rho}_3^{[1]} \text{ are any nonzero real number.}$$

Let  $\zeta^{[1]} = (\zeta_1^{[1]}, \zeta_2^{[1]}, \zeta_3^{[1]})^T$  be the eigenvector associated with the eigenvalue  $\lambda_{1i} = 0$  of the matrix  $[\ddot{j}_1]^T$ . Then we have,  $(\ddot{j}_1^T - \lambda_{1i}I) \zeta^{[1]} = 0$ . By solving this equation  $\zeta^{[1]}$ ,

We obtain,  $\zeta^{[1]} = (0, \zeta_2^{[1]}, 0)^T$  where  $\zeta_2^{[1]}$  any nonzero real number.

Now,

$$\frac{\partial f}{\partial r_3} = f_{r_3}(A_1, r_3) = \left( \frac{\partial f_1}{\partial r_3}, \frac{\partial f_2}{\partial r_3}, \frac{\partial f_3}{\partial r_3} \right)^T = (0, h_i, 0)^T.$$

So,  $\frac{\partial f}{\partial r_3}(A_1, \ddot{r}_3) = (0, 0, 0)^T$  that's why  $(\zeta)^T \frac{\partial f}{\partial r_3}(A_1, \ddot{r}_3) = 0$ .

Therefore, according to Sotomayor's theorem the saddle - node bifurcation cannot occur.

While the first condition of transcritical bifurcation is satisfied . Now, since

$$Df_{r_3}(A_1, \ddot{r}_3) \dot{\rho}^{[1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -r_1 \dot{\rho}_2^{[1]} \\ \dot{\rho}_2^{[1]} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{\rho}_2^{[1]} \\ 0 \end{pmatrix},$$

$$(\zeta^{[1]})^T [Df_{r_3}(A_1, \ddot{r}_3) \dot{\rho}^{[1]}] = (0, \zeta_2^{[1]}, 0)^T \begin{pmatrix} 0 \\ \dot{\rho}_2^{[1]} \\ 0 \end{pmatrix} = \zeta_2^{[1]} \dot{\rho}_2^{[1]} \neq 0.$$

Now, by substituting  $\dot{\rho}^{[1]}$  in (23), we get:

$$D^2 f(A_1, \ddot{r}_3)(\dot{\rho}^{[1]}, \dot{\rho}^{[1]}) = \begin{pmatrix} 4r_1^2 [\dot{\rho}_2^{[1]}]^2 \\ -r_1 r_3 [\dot{\rho}_2^{[1]}]^2 \\ 0 \end{pmatrix}.$$

Hence, it is obtained that:

$$(\zeta^{[1]})^T D^2 f(A_1, \ddot{r}_3)(\dot{\rho}^{[1]}, \dot{\rho}^{[1]}) = (0, \zeta_2^{[1]}, 0)^T \begin{pmatrix} 4r_1^2 [\dot{\rho}_2^{[1]}]^2 \\ -r_1 r_3 [\dot{\rho}_2^{[1]}]^2 \\ 0 \end{pmatrix} = -r_1 r_3 [\dot{\rho}_2^{[1]}]^2 \zeta_2^{[1]} \neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation but not experience a saddle node bifurcation at  $A_1$  with the parameter  $r_3$ , where  $r_3 = \ddot{r}_3$ .

#### 4.2 Near by local bifurcation analysis $A_2(\bar{h}, \bar{i}, \mathbf{0})$ :

**Theorem (3):** Assume that the following conditions are met:

$$\epsilon_2 \neq 0, \tag{24}$$

$$\epsilon_4 \neq 0, \tag{25}$$

$$\vartheta = -2\epsilon_1 \epsilon_3 (\epsilon_1 + r_1 + r_2 \epsilon_2) + r_3 \epsilon_1 - r_4 \epsilon_2 + \epsilon_2 \epsilon_4 (r_6 \epsilon_1 + r_7) \neq 0 \tag{26}$$



$$\epsilon_1 = \frac{\bar{h} [r_1 r_4 \bar{h} \bar{i} - r_2 (r_3 \bar{h} - r_3)]}{i [r_4 (1 - r_1 \bar{i}) - (2r_4 + r_2 r_3) \bar{h}]}, \quad \epsilon_2 = \frac{(1 - 2 \bar{h} - r_1 \bar{i}) \bar{i} - r_1 \bar{h}}{r_2 \bar{h}}$$

$$\epsilon_3 = \frac{r_3 \bar{h} - r_5}{r_1 \bar{h}}, \quad \epsilon_4 = \frac{r_2 \bar{h} \epsilon_3 + r_4 \bar{i}}{r_6 \bar{h} + r_7 \bar{i} - r_8}$$

Then system (2) at the equilibrium point  $A_2 = (\bar{h}, \bar{i}, 0)$  with the parameter

$$\bar{r}_8 = r_6 \bar{h} + r_7 \bar{i}$$

possesses:

- No saddle- node bifurcation.
- Transcritical bifurcation.

**Proof:** According to the Jacobian matrix  $J(A_2)$  given by eq.(2.a) of system (2) at the equilibrium point  $A_2 = (\bar{h}, \bar{i}, 0)$  has zero eigenvalue (say  $\lambda_{2j} = 0$ ) at  $r_8 = \bar{r}_8$ , and the

Jacobian matrix  $J_2$  with  $\bar{r}_8 = r_6 \bar{h} + r_7 \bar{i}$  becomes:

$$\dot{J}_2 = J_2(r_8 = \bar{r}_8) = [\mu_{ij}]_{3 \times 3},$$

where,  $\mu_{ij} = \mu_{ij}$  for all  $i, j = 1, 2, 3$  except  $\mu_{33} = r_6 \bar{h} + r_7 \bar{i} - r_8 = 0$ .

Let  $\dot{\rho}^{[2]} = (\dot{\rho}_1^{[2]}, \dot{\rho}_2^{[2]}, \dot{\rho}_3^{[2]})^T$  be the eigenvector corresponding to the eigenvalue  $\lambda_{2j} = 0$ .

Thus  $(\dot{J}_2 - \lambda_{2j} I) \dot{\rho}^{[2]} = 0$ , which gives :

$$\dot{\rho}_1^{[2]} = \epsilon_1 \dot{\rho}_2^{[2]} \text{ and } \dot{\rho}_3^{[2]} = \epsilon_2 \dot{\rho}_2^{[2]},$$

where  $\dot{\rho}_2^{[2]}$  any nonzero, real number with  $\epsilon_1, \epsilon_2$  and which and which are mentioned in the state of the theorem.

Let  $\zeta^{[2]} = (\zeta_1^{[2]}, \zeta_2^{[2]}, \zeta_3^{[2]})^T$  be the eigenvector associated with the eigenvalue  $\lambda_{2w} = 0$  of the matrix  $\ddot{J}_2$ .

Then we have  $(\ddot{J}_2 - \lambda_{2w} I) \zeta^{[2]} = 0$ . By solving this equation for  $\zeta^{[2]}$ , we obtain  $\zeta^{[2]} = (\epsilon_3 \zeta_2^{[2]}, \zeta_2^{[2]}, \epsilon_4 \zeta_2^{[2]})^T$ , where  $\zeta_2^{[2]}$  any real numbers that are not zero, with  $\epsilon_3, \epsilon_4$  which are mentioned in the state of the theorem.

Now, consider:

$$\frac{\partial f}{\partial r_8}(\bar{Y}, r_8) = f_{r_8}(\bar{Y}, r_8) = \left( \frac{\partial f_1}{\partial r_8}, \frac{\partial f_2}{\partial r_8}, \frac{\partial f_3}{\partial r_8} \right)^T = (0, 0, j)^T$$

$$\text{So, } \frac{\partial f}{\partial r_8}(A_2, r_8) = (0, 0, 0)^T.$$

That's why  $(\zeta^{[2]})^T f_{r_8}(A_2, \bar{r}_8) = 0$ .

Therefore, according to Sotomayor's theorem there can be no saddle-node bifurcation. Although the first need for transcritical bifurcation has been satisfied. Now, since

$$Df_{r_8}(A_2, \ddot{r}_8)\dot{\rho}^{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \dot{\rho}_2^{[2]} \\ \dot{\rho}_2^{[2]} \\ \epsilon_2 \dot{\rho}_2^{[2]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\epsilon_2 \dot{\rho}_2^{[2]} \end{pmatrix}$$

$$(\zeta^{[2]})^T [Df_{r_8}(A_2, \ddot{r}_8)\dot{\rho}^{[2]}] = (\epsilon_3 \zeta_2^{[2]}, \zeta_2^{[2]}, \epsilon_4 \zeta_2^{[2]}) \begin{pmatrix} 0 \\ 0 \\ -\epsilon_2 \dot{\rho}_2^{[2]} \end{pmatrix} = -\epsilon_2 \epsilon_4 \zeta_2^{[2]} \dot{\rho}_2^{[2]} \neq 0.$$

Moreover, by substituting  $\dot{\rho}^{[2]}$  in (23), we get:

$$D^2 f(A_2, \ddot{r}_8)(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}) = \begin{bmatrix} -2\epsilon_1(\dot{\rho}_2^{[2]})^2(\epsilon_1 + r_1 + r_2\epsilon_2) \\ 2(\dot{\rho}_2^{[2]})^2(r_3\epsilon_1 - r_4\epsilon_2) \\ 2\epsilon_2(\dot{\rho}_2^{[2]})^2(r_6\epsilon_1 + r_7) \end{bmatrix}.$$

Hence, it is obtained that:  $(\epsilon_3 \zeta_2^{[2]}, \zeta_2^{[2]}, \epsilon_4 \zeta_2^{[2]})$

$$(\zeta^{[2]})^T D^2 f(A_2, \ddot{r}_8)(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}) = 2\vartheta(\dot{\rho}_2^{[2]})^2 \zeta_2^{[2]}$$

Where,  $\epsilon_1, \epsilon_3$  and  $\epsilon_4$  are mentioned in the theorem's state.

As a result of condition (26), we get that:

$$(\zeta^{[2]})^T D^2 f(A_2, \ddot{r}_8)(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}) \neq 0.$$

Thus, according to Sotomayor's theorem system (2) has a transcritical bifurcation at the equilibrium point  $A_2 = (\bar{h}, \bar{i}, 0)$  with the parameter  $\ddot{r}_8 = r_6\bar{h} + r_7\bar{i}$ .

### 4.3 Local bifurcation analysis near $A_3 = (\bar{\bar{h}}, 0, \bar{\bar{j}})$ :

**Theorem (4):** Assume that the following criteria are fulfilled:

$$r_4 > \frac{r_3 \bar{\bar{h}} - r_5}{\bar{\bar{j}}}, \tag{27}$$

$$\dot{\epsilon}_2 \neq 0, \tag{28}$$

$$\mathbb{R} \neq 0 \tag{29}$$

where:

$$\dot{\epsilon}_1 = \frac{r_1 r_6 \bar{\bar{h}} + r_7 \bar{\bar{j}}(1 - 2\bar{\bar{h}} - r_2)}{\bar{\bar{h}}[r_2 + r_1(r_6 \bar{\bar{h}} - r_8)]}, \quad \dot{\epsilon}_2 = \frac{1 - r_2 \bar{\bar{j}} - (2 - r_2 \dot{\epsilon}_1) \bar{\bar{h}}}{r_1 \bar{\bar{h}}}, \quad \dot{\epsilon}_3 = \frac{r_7 \bar{\bar{j}}}{r_1 \bar{\bar{h}}},$$

$$\mathbb{R} = \zeta_3^{[3]}[\epsilon_2(r_6\epsilon_1 + r_7) - \epsilon_1\epsilon_3(\epsilon_1 + r_1 + r_2\epsilon_2)] + \zeta_2^{[3]}[r_3\epsilon_1 - r_4\epsilon_2].$$

Then system (2) at the equilibrium point  $A_3 = (\bar{\bar{h}}, 0, \bar{\bar{j}})$  with the parameter value:

$\ddot{r}_4 = r_3\bar{\bar{h}} - r_4\bar{\bar{j}}$  has a transcritical bifurcation, but a saddle – node cannot occur at  $A_3$ .

**Proof:** The characteristic equation represented by eq. (3.a) of system (2) at the equilibrium point  $A_3$  has zeroeigenvalue (say  $\lambda_{3i} = 0$ ) at  $r_4 = \ddot{r}_4$  and the Jacobian matrix  $J_3$  with 4parameter  $r_4 = \ddot{r}_4$  becomes:

$$\ddot{J}_3 = J_3(r_4 = \dot{r}_4) = [\tilde{\eta}_{ij}]_{3 \times 3},$$

Where:  $\tilde{\eta}_{ij} = \eta_{ij}$  for all  $i, j = 1, 2, 3$  except  $\tilde{\eta}_{ij} = r_3 \bar{h} - \dot{r}_4 \bar{j} - r_4 = 0$ .

Let  $\dot{\rho}^{[3]} = (\dot{\rho}_2^{[3]}, \dot{\rho}_2^{[3]}, \dot{\rho}_3^{[3]})^T$  be the eigenvector which follows the eigenvalue be the eigenvector which follows the eigenvalue.  $\lambda_{3s} = 0$ . Thus  $(\ddot{J}_3 - \lambda_{3s}I) \dot{\rho}^{[3]} = 0$ , which gives:

$\dot{\rho}^{[3]} = (\dot{\rho}_1^{[3]}, \dot{\epsilon}_2 \dot{\rho}_1^{[3]}, \dot{\epsilon}_1 \dot{\rho}_1^{[3]})^T$ , where  $\dot{\rho}_1^{[3]}$  any nonzero number with,  $\dot{\epsilon}_1$  which are mentioned in the state of the theorem.

Let  $\zeta^{[3]} = (\zeta_1^{[3]}, \zeta_2^{[3]}, \zeta_3^{[3]})^T$  become the eigenvector linked to the eigenvalue  $\lambda_{3s} = 0$  of the matrix  $\ddot{J}_3$ . Then we have  $(\ddot{J}_3^T - \lambda_{3i}I) \zeta^{[3]} = 0$ .

By solving this equation for  $\zeta^{[3]}$ , we obtain:

$\zeta^{[3]} = (\dot{\epsilon}_3 \zeta_3^{[3]}, \zeta_2^{[3]}, \zeta_3^{[3]})^T$  where  $\zeta_3^{[3]}$  any nonzero number with  $\dot{\epsilon}_3$  those are referred to in the theorem's state. Now, consider:

$$\frac{\partial f}{\partial r_4}(\dot{Y}, r_4) = f_{r_4}(\dot{Y}, r_4) = \left( \frac{\partial f_1}{\partial r_4}, \frac{\partial f_2}{\partial r_4}, \frac{\partial f_3}{\partial r_4} \right)^T = (0, -ij, 0)^T$$

So, 
$$\frac{\partial f}{\partial r_4}(A_3, r_4) = (0, 0, 0)^T.$$

And hence 
$$(\zeta^{[3]})^T f_{r_4}(A_3, \dot{r}_4) = 0.$$

Therefore, the saddle-node bifurcation is ruled out by Sotomayor's theorem. While the first transcritical bifurcation condition has been satisfied. Now, since  $\dot{\rho}^{[3]} = [\dot{\rho}_1^{[3]}, \dot{\epsilon}_2 \dot{\rho}_1^{[3]}, \dot{\epsilon}_1 \dot{\rho}_1^{[3]}]$

$$Df_{r_4}(A_3, \dot{r}_4) \dot{\rho}^{[3]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -j & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\rho}_1^{[3]} \\ \dot{\epsilon}_2 \dot{\rho}_1^{[3]} \\ \dot{\epsilon}_1 \dot{\rho}_1^{[3]} \end{pmatrix} = \begin{pmatrix} 0 \\ -j \dot{\epsilon}_2 \dot{\rho}_1^{[3]} \\ 0 \end{pmatrix}$$

$$(\zeta^{[2]})^T [Df_{r_4}(A_2, \dot{r}_4) \dot{\rho}^{[2]}] = (\epsilon_3 \zeta_3^{[3]}, \zeta_2^{[3]}, \zeta_3^{[3]}) \begin{pmatrix} 0 \\ -j \dot{\epsilon}_2 \dot{\rho}_1^{[3]} \\ 0 \end{pmatrix} = -\dot{\epsilon}_2 \zeta_2^{[3]} \dot{\rho}_1^{[3]}.$$

So, by condition (28), we obtain that:

$$(\zeta^{[2]})^T [Df_{r_4}(A_2, \dot{r}_4) \dot{\rho}^{[2]}] \neq 0$$

Moreover, by substituting  $\dot{\rho}^{[3]}$  in (23), we get:

$$D^2 f(A_3, \dot{r}_4) (\dot{\rho}^{[3]}, \dot{\rho}^{[3]}) = \begin{bmatrix} -2\epsilon_1 (\dot{\rho}_2^{[2]})^2 (\epsilon_1 + r_1 + r_2 \epsilon_2) \\ 2(\dot{\rho}_2^{[2]})^2 (r_3 \epsilon_1 - r_4 \epsilon_2) \\ 2\epsilon_2 (\dot{\rho}_2^{[2]})^2 (r_6 \epsilon_1 + r_7) \end{bmatrix}.$$

Hence, it is obtained that:  $(\epsilon_3 \zeta_3^{[3]}, \zeta_2^{[3]}, \zeta_3^{[3]})$

$$(\zeta^{[2]})^T D^2 f(A_3, \dot{r}_4)(\dot{\rho}^{[3]}, \dot{\rho}^{[3]}) = 2(\dot{\rho}_2^{[2]})^2 \mathbb{R}$$

Where,  $\epsilon_1, \epsilon_2, \epsilon_3$  and  $\mathbb{R}$  are mentioned in the state of the theorem. As a result of condition (29), we get that:  $(\zeta^{[2]})^T D^2 f(A_2, \dot{r}_8)(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}) \neq 0$ .

Thus, according to Sotomayor's theorem system (2) has a transcritical bifurcation at the equilibrium point  $A_3 = (\bar{h}, 0, \bar{j})$  with the parameter  $\dot{r}_4 = r_3 \bar{h} - r_4 \bar{j}$ .

#### 4.4 Local bifurcation analysis near $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$ :

**Theorem (5):** Assume that the following criteria are fulfilled:

$$\epsilon_1 = \frac{-r_3}{r_6}, \quad \epsilon_2 = \frac{r_8 - r_6 \tilde{h} - (r_7 \tilde{i} + \tilde{j} \epsilon_1)}{r_6 \tilde{j}}, \quad \epsilon_3 = \frac{-r_3 \tilde{i}}{r_6 \tilde{j}},$$

$$\epsilon_4 = \frac{(r_6 \tilde{h} + r_7 \tilde{i} - r_8) \epsilon_3 - r_4 \tilde{j}}{r_2 \tilde{h}} \neq 0, \tag{30}$$

$$1 = 2 \tilde{h} + r_1 \tilde{i} + r_2 \tilde{j}, \tag{31}$$

$$\epsilon_2 \epsilon_4 [\epsilon_2 + r_1 \epsilon_1 + r_2] \neq \epsilon_1 [r_3 \epsilon_2 + r_4] + \epsilon_4 [r_6 \epsilon_2 + r_7 \epsilon_1]. \tag{32}$$

Then system (2) at the equilibrium point  $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$  with the parameter value:

$$\dot{r}_1 = \frac{1 - 2 \tilde{h} - r_2 \tilde{j}}{\tilde{i}},$$

has a saddle – node bifurcation, but neither a transcritical nor a pitchfork bifurcation at  $A_4$ .

**Proof:** The characteristic equation given by eq. (4.a) if  $H_4 = 0$  and  $A_4$  becomes a non-hyperbolic equilibrium point, of system (2) having zero eigenvalue (say  $\lambda_{4h} = 0$ ).

The Jacobian matrix for system (2) at equilibrium point  $E_4$  with parameter ( $r_1 = \dot{r}_1$ )

clearly becomes:  $\check{J}_4 = J_4(r_1 = \dot{r}_1) = [\check{n}_{ij}]_{3 \times 3}$ ,

for all  $i, j = 1, 2, 3$  except  $\check{n}_{11}$  which is given by: where,  $\check{n}_{ij} = \check{n}_{11}$

$$\check{n}_{11} = 1 - 2 \tilde{h} - \dot{r}_1 \tilde{i} - r_2 \tilde{j}$$

Let  $\dot{\rho}^{[4]} = (\dot{\rho}_1^{[4]}, \dot{\rho}_2^{[4]}, \dot{\rho}_3^{[4]})^T$  be the eigenvector that follows the eigenvalue  $\lambda_{4h} = 0$ .

Thus  $(\check{J}_4 - \lambda_{4h} I) \dot{\rho}^{[4]} = 0$ , which gives:

any non-zero value that has the  $\epsilon_1$  and  $\epsilon_2$   $\dot{\rho}^{[4]} = (\epsilon_2 \dot{\rho}_3^{[4]}, \epsilon_1 \dot{\rho}_3^{[4]}, \dot{\rho}_3^{[4]})^T$ , where  $\dot{\rho}_3^{[4]}$  conditions given in the theorem.

Let  $\zeta^{[4]} = (\zeta_1^{[4]}, \zeta_2^{[4]}, \zeta_3^{[4]})^T$  be the eigenvector associated with the eigenvalue  $\lambda_{4i} = 0$  of the matrix  $\check{J}_4$ . Next, we have  $(\check{J}_4^T - \lambda_{4i} I) \zeta^{[4]} = 0$ . By solving this equation for  $\zeta^{[4]}$ , we obtain:

any nonzero number with  $\epsilon_3$  and  $\epsilon_4$  which are  $\zeta^{[4]} = (\epsilon_4 \zeta_2^{[4]}, \zeta_2^{[4]}, \epsilon_3 \zeta_2^{[4]})^T$  where  $\zeta_2^{[4]}$  mentioned in the state of the theorem.

Now,

$$\frac{\partial f}{\partial r_1} = f_{r_1}(\dot{Y}, r_1) = \left( \frac{\partial f_1}{\partial r_1}, \frac{\partial f_2}{\partial r_1}, \frac{\partial f_3}{\partial r_1}, \right)^T = (-hi, 0, 0)^T,$$

So,  $f_{a_1}(A_4, \dot{r}_1) = (-\tilde{h} \tilde{i}, 0, 0)^T$ , and hence, it is obtained that:

$$(\zeta^{[4]})^T f_{r_1}(A_4, \dot{r}_1) = -\tilde{h} \tilde{i} \epsilon_4 \zeta_2^{[4]}.$$

In the context of condition (30), we thus have that:  $(\zeta^{[4]})^T f_{r_1}(A_4, \dot{r}_1) \neq 0$ .

$$D^2 f(A_4, \dot{r}_1)(\dot{\rho}^{[4]}, \dot{\rho}^{[4]}) = \begin{bmatrix} -2 \epsilon_2 (\dot{\rho}_3^{[4]})^2 [\epsilon_2 + r_1 \epsilon_1 + r_2] \\ 2 \epsilon_1 (\dot{\rho}_3^{[4]})^2 [r_3 \epsilon_2 + r_4] \\ 2 (\dot{\rho}_3^{[4]})^2 [r_6 \epsilon_2 + r_7 \epsilon_1] \end{bmatrix}.$$

Hence, it is obtained that:

$$\begin{aligned} & (\zeta^{[4]})^T D^2 f(A_4, \dot{r}_1)(\dot{\rho}^{[4]}, \dot{\rho}^{[4]}) \\ &= -2 (\dot{\rho}_3^{[4]})^2 \zeta_2^{[4]} (\epsilon_2 \epsilon_4 [\epsilon_2 + r_1 \epsilon_1 + r_2] - \epsilon_1 [r_3 \epsilon_2 + r_4] - \epsilon_4 [r_6 \epsilon_2 + r_7 \epsilon_1]). \end{aligned}$$

Therefore, in accordance with condition (32) we get that:

$$(\zeta^{[4]})^T D^2 f(A_4, \dot{r}_1)(\dot{\rho}^{[4]}, \dot{\rho}^{[4]}) \neq 0$$

Sotomayor's theorem is used to show that system (2) has a saddle-node bifurcation  $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$  at  $\dot{r}_1$ .

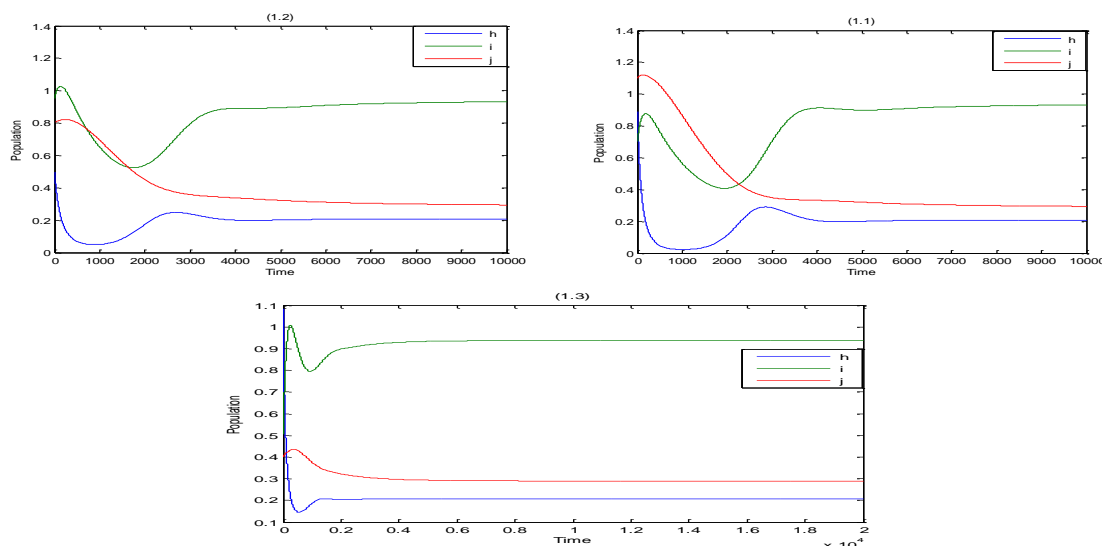
### 5. Numerical Simulation

This section numerically analyzes System (2)'s dynamical behavior for a given set of parameters and various initial point sets. These are the study's objectives, including:

1. Analyze the impact of changing the value of each parameter on the system's dynamic behavior (2).
2. Verify the analytical results that were found.

The following hypothetical collection of parameters is found to satisfy the stability requirements of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in **Figure 5.1 (a, b, c)**,

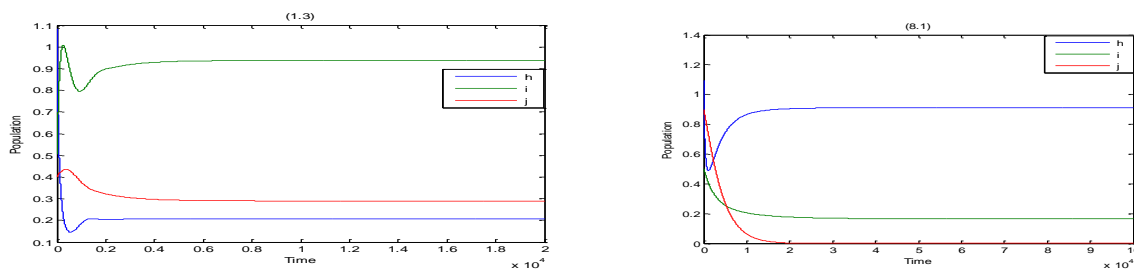
$\begin{aligned} r_1 = 0.05, & \quad r_2 = 0.1, & \quad r_3 = 0.03, & \quad r_4 = 0.1, \\ r_6 = 0.06, & \quad r_7 = 0.07, & \quad r_8 = 0.1 & \quad r_5 = 0.01, \end{aligned}$	(5.1)
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**Figure1:** The time series of the solution of system (2) started from the three different initial point (0.8, 0.7, 1.64), (1.2, 0.7, 1.64) and (0.2, 0.4, 0.64) for the data given by (5.1), (a) the trajectories of **h** as a function of time, (b) the trajectories of **i** as a function of time, (c) the trajectories of **j** as a function of time.

As the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_4 = (0.4544, 0.6567, 2.1722)$  beginning with three distinct starting points, **Figure1**. clearly demonstrates that system (2) has a globally asymptotically stable, and this is supporting our obtained analytical results. We will now talk about how system's (2) parameter settings affect the system's dynamical behavior. The system is numerically solved for the data in (5.1) by changing one parameter at a time, sometimes even two, and the results are shown below.

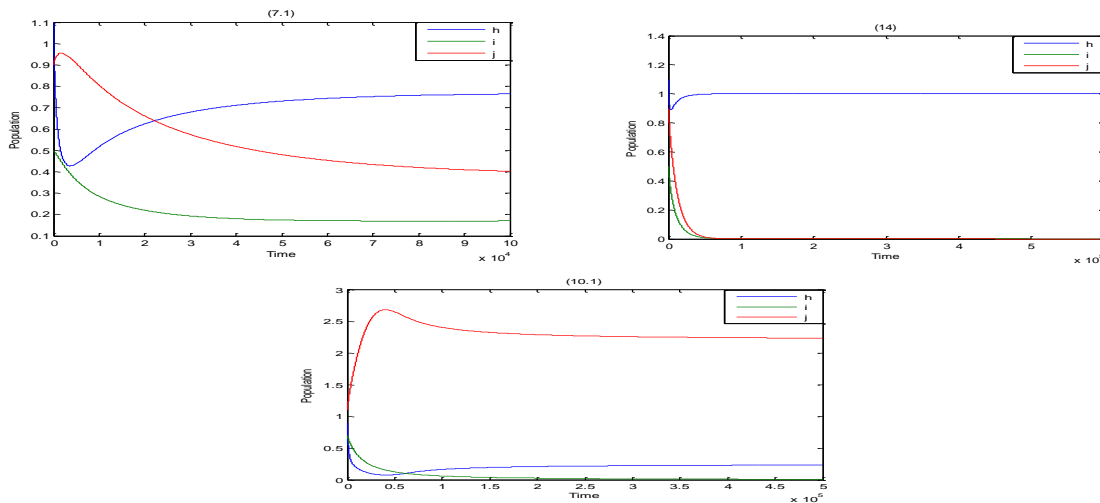
The system will get closer to the point of positive equilibrium  $A_4$  in the interior of the positive quadrant of the hij-plane as shown in **Figure2.a1** when the predation rate on a prey is varied from the specialist predator in the range  $0.0001 < r_1 \leq 0.595$  while maintaining other parameters as data given in (5.1),  $r_1$  has a usual value of 0.15. As shown in **Figure2.a2** for average value  $r_1 = 0.9$ , it is seen that the solution of system (2) approaches asymptotically to the equilibrium point  $A_2$  in the range  $0.595 < r_1 < 2$



**Figure2.a1:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_4 = (0.8182, 0.7275, 0.7271)$  in the interior of  $R_+^3$ . For the data in (5.1) with  $r_1 = 0.15$ . **Figure2.a2:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_2 = (0.3333, 0.7407, 0)$  in the interior of  $R_+^3$ . For the data in (5.1) with  $r_1 = 0.9$ .

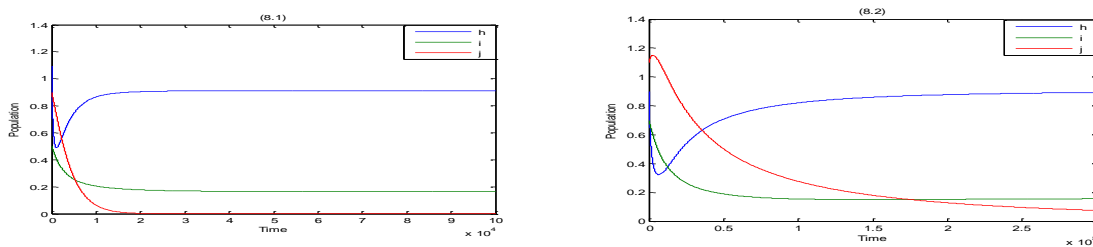
Additionally, changing the specialist predator's mortality death rate between 0.0001 and 0.03 while maintaining the other parameters according to the data in (5.1) results in the specialized predator going extinct, and **Figure3.e1** illustrates how system (2) approaches asymptotically to the positive equilibrium point  $A_4$  with a typical value of  $r_5 = 0.01$ , however, for a typical value of

$r_5=0.1$ , as shown in **Figure3.e2**, at  $0.03 < r_5 < 0.15$  approaches asymptotically to the axial equilibrium point  $A_1$ , additionally, for a typical value of  $r_5=0.5$ , **Figure3.e3** shows that for  $0.15 \leq r_5 < 0.6$  approaches asymptotically to the equilibrium point  $A_3$ .



**Figure3.e1:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_4=(0.99043, 0.6535, 0.6304)$  in the interior of  $R_+^3$ . For the data in (5.1) with  $r_5 = 0.01$ . **Figure3.e2:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_1=(1, 0, 0)$  in the interior of  $R_+^3$ . For the data in (5.1) with  $r_5 = 0.1$ . **Figure3.e3** time series of the solution of system (2) approaches asymptotically to point  $A_3=(0.6667, 0, 3.3333)$ .

The system (2) still approaches asymptotically to the equilibrium point  $A_2$  despite changing the parameter  $r_6$  which represents the conversion rate from the prey to the generalist predator in the range  $0.0001 \leq r_6 < 0.015$  this causes extinction in the prey, however in additional for  $0.015 \leq r_6 < 0.1$  approaches asymptotically to the positive equilibrium point  $A_4$ , as shown in **Figure4.g2**, for typical value  $r_6=0.08$ .



**Figure4.g1:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_2=(0.3392, 13.2187, 0)$  in the interior of  $R_+^3$ . For the data in (5.1) with  $r_6 = 0.013$ . **Figure4.g2:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point  $A_4=(0.7311, 4.1682, 0.6044)$  in the interior of  $R_+^3$ . For the data in (5.1) with  $r_6 = 0.08$

### 6. Discussion

Starting with the hypothetical set of data provided by eq. (5.1), system (2) has been numerically solved for several sets of initial points and various sets of parameters, and the following observations are obtained:

1. When approaching globally stable locations via  $\text{Int. } R_+^3$  techniques system (2) only has two types of attractors. The system (2) approaches asymptotically to the globally stable positive point  $A_4 = (0.4544, 0.6567, 2.1722)$  for the set parameter value specified in (5.1).
2. The positive equilibrium point  $A_4$  being approached by the solution of system (2) as the assault rate on a victim from the specialized predator  $r_1$  increases to 0.15 while maintaining the

other parameters as in eq. (5.1), but if  $r_1 = 0.59$  it can be seen that the solution to system (2) approaches the equilibrium point  $A_2$  asymptotically;  $r_1 = 0.595$  is a bifurcation point.

3. The trajectory changed from the axial point  $A_1$  to the equilibrium point  $A_2$  in the range  $0.0001 \leq r_3 < 0.006$ , and from the equilibrium point  $A_2$  to the equilibrium point  $A_4$  in the range  $0.006 \leq r_3 < 0.0125$  consequently the bifurcation points for the parameter  $r_3$  are at  $r_3 = 0.006$  and  $r_3 = 0.01$  respectively.

4. As the attack rate of generalist predator on specialist predator  $r_4$  keeping the rest of parameters as in eq.(5.1), the solution of system (2) approaches to the positive point  $A_4$ , if  $r_4 = 1.5$ , it is shown that the solution of system (2) approaches asymptotically to the equilibrium point  $A_3$ , indicating that  $r_4 = 1.5$  is a bifurcation point.

5. The natural death rate of specialist predator  $r_5$  the solution of system (2) advances from the positive equilibrium point  $A_4$  to the axial equilibrium point  $A_1$  with  $0.03 < r_5 < 0.15$  keeping the other parameters as in eq.(5.1), and from the axial equilibrium point  $A_1$  to the equilibrium point  $A_3$  in the range  $0.15 \leq r_5 < 0.6$ , the parameter  $r_5 = 0.045$  is bifurcation point.

6. As the conversion rate of prey to the generalist predator  $r_6$  decreasing, and keeping the rest parameters values as in eq. (5.1) the solution of system (2) approaches the equilibrium point  $A_4$ , while for  $0.0001 \leq r_6 < 0.015$ , the generalist predator population revives and then the trajectory changed from the point  $A_2$  to the positive equilibrium point  $A_4$ , while for  $0.015 \leq r_6 < 0.1$  thus, the parameter  $r_6 = 0.015$  is a bifurcation point.

7. In light of the conversion rate of specialist predator to the generalist predator  $r_7$  decreasing to 0.0002 and keeping the rest parameters values as in eq. (5.1) the solution of system (2) approaches the equilibrium point  $A_2$ , while for  $0.00001 \leq r_7 < 0.005$  the generalist predator population revives and when  $0.005 \leq r_7 < 0.2$ , the trajectory changed from the point  $A_2$  to the positive equilibrium point  $A_4$ , the parameter  $r_7 = 0.005$  is a bifurcation point.

## 7. Conclusion

This study used an ecological mathematical model that includes a predator-prey model and a food web, as well as a population of prey and a population of specialized predators as refuges. Additionally, this model includes linear types of functional reactions for the predation of creatures that were not protected.

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## Conflict of Interest

There are no conflicts of interest.

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