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local Bifurcation of the Dynamic Behavior of Predator-Prey System with Refuge for both Species

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Abstract

The main purpose of this paper is to study a predator – prey dynamical system consisting of three species prey, specialized predator and generalist predator namely H (t), I (t) and J (t) respectively, w food web and refuge for the prey and specialized predator population. The consider system has five equilibrium points $A_0 = (0,0,0)$, $A_1 = (1,0,\ 0)$, $A_2 = (\bar{h}, \bar{i}, 0)$, $A_3 = (\bar{\bar{h}}, 0,\ \bar{\bar{j}})$, and the positive equilibrium point $A_4 = (\tilde{h}, \tilde{i}, \tilde{j})$.The stability and bifurcation of the equilibrium points was studied and the main influence was the qualitative behavior of the solution. It was found that A_0 was unstable while the other equilibrium points are stable under condition so we study their bifurcation and show that A_1, A_2 and A_3 are transcritical while A_4 is saddle node bifurcation. Numerical simulations were used to illustrate the occurrence of local bifurcation of this model.

Keywords: Local bifurcation, predator–prey, stability analysis, Lyapunov's function, ecological, Refuge.

1. Introduction

The mathematical study of changes in a dynamical system's qualitative asymptotic structure is known as bifurcation theory (1, 2). As well as its attempts to explain various phenomena that have been described in the natural sciences over the centuries. Where performing bifurcation analysis is often a powerful way to analyze the properties of such systems. The prey and predator model is an important topic at present as it is used to solve many problems in the ecological nature and other sciences. The prey system includes several interactions, such as competition co-existence and stage-structured (3). The system is also impacted by a number of other factors, such as shelter, sickness, and others. A bifurcation, which is the primary qualitative shift in the behavior of a dynamic system as a result of changing one of its coefficients, can occasionally emerge from variations in any parameter in the system, leading to complex behavior that leads to system instability. Local and global bifurcations were the two main classes that made up the bifurcation. Changes in the local stability parameters of equilibria or periodic orbits can be used to evaluate local bifurcation. Global bifurcation, on the other hand, happens when periodic orbits run into equilibrium. This leads to changes in the topology of the trajectories in phase space that cannot be contained within a limited region as is the case with local bifurcation. These bifurcations occur when a single parameter is changed (4-9). Perko (10), on the other hand, identified the prerequisites for local bifurcation, including saddle-node, transcritical, pitchfork, and period-doubling. Finally, Sotomayor's theorem(11) for local bifurcation was applied in this work to examine the occurrence of local bifurcation at equilibrium sites local bifurcation methods close to all equilibrium points, and a number of fundamental results (12-22).

2. Model formulation

An ecological model was suggested for investigation in this section. A prey was included in the model, and its overall population density at time t is represented by the symbol H(t), engaging with a specialized predator I(t) is the population density at time t and generalist predator whose population density at time t is denoted by J(t). It was assumed that the prey with refuge and specialist predator is with the prey refuge.

The following presumptions are now used to create the fundamental ecological model shown in **Table 1:**

$$\begin{aligned}
\frac{dH}{dT} &= \rho H \left(1 - \frac{H}{K}\right) - \alpha\,(1-m_1)\,H\,I - \beta\,(1-m_2)\,H\,J \\
\frac{dI}{dT} &= \alpha_1\,(1 - m_1)\,H\,I - \gamma\,(1-m_3)\,IJ - d_1\,I \\
\frac{dJ}{dT} &= \beta_1\,(1 - m_2)\,H\,J - \gamma_1\,(1-m_3)\,IJ - d_2\,J
\end{aligned} \tag{1}$$

Table 1. The parameters of model (1)

parameters	Biological meaning
$\rho > 0$	intrinsic growth
$k > 0$	carrying capacity (in logistic growth)
$\alpha > 0$	maximum attack rate by specialist predator
$\beta > 0$	maximum attack rate by generalist predator
$d_1 and\ d_2$	natural death rate of specialist and generalist predator
$\alpha_1 > 0$	maximum predation rate of the specialist predator over the prey
$\beta_1 > 0$	maximum predation rate of the generalist predator over the prey
$\gamma > 0$	maximum attack rate by generalist predator on specialist predator
$\gamma_1 > 0$	maximum predation rate of the generalist predator over the specialist predator
$0< m_1 < 1$,	The refuge rates constants of the prey from the specialist predator
$0< m_2 < 1$	The refuge rates constants of the prey from the generalist predator
$0< m_3 < 1$	The refuge rates constants of the specialist predator from the generalist predator

The above model has 13 parameters so, it is difficult to study all of them, there for reduce them a dimensionless variables and parameters are defined:

$t^* = \rho\, t$, $\mathrm{h} = \frac{\mathrm{H}}{\mathrm{k}}$, $\mathrm{i} = \frac{\mathrm{I}}{\mathrm{k}}$, $\mathrm{j} = \frac{\mathrm{J}}{\mathrm{k}}$, $\mathrm{r}_1 = \frac{\alpha \mathrm{k}(1 - m_1)}{\rho}$, $\mathrm{r}_2 = \frac{\beta \mathrm{k}(1 - m_2)}{\rho}$

$\mathrm{r}_3 = \frac{\alpha_1 \mathrm{k}(1 - m_1)}{\rho}$, $\mathrm{r}_4 = \frac{\gamma \mathrm{k}(1 - m_3)}{\rho}$, , $r_5 = \frac{\mathrm{d}_1}{\rho}$, $\mathrm{r}_6 = \frac{\beta_1 \mathrm{k}(1 - m_2)}{\rho}$, $r_7 = \frac{\gamma_1 k(1 - m_1)}{\rho}$,

$r_8 = \frac{\mathrm{d}_2}{\rho}$.

Now, for simplicity rename $t^* = t$.

So, the dimensional system (1) can be formulated as:

$$\left.\begin{aligned} \frac{dh}{dt} &= h[1 - h - \mathrm{r}_1 i - \mathrm{r}_2 \mathrm{j}] = f_1(h, i, j) \\ \frac{di}{dt} &= i[\mathrm{r}_3 h - \mathrm{r}_4 j - r_5] = f_2(h, i, j) \\ \frac{dj}{dt} &= j[\mathrm{r}_6 h + \mathrm{r}_7 i - \mathrm{r}_8] = f_3(h, i, j) \end{aligned}\right\} \quad (2)$$

With $h(0) \geq 0$, $i(0) \geq 0$ *and* $j(0) \geq 0$. In system (2) there are 8 parameters. All of the functions on system (2) right side are C^2 $(\mathbb{R}^3, \mathbb{R}^+)$.

$$R_+^3 = \{(h, i, j) \in R^3 : h(0) \geq 0, \quad i(0) \geq 0, \ j(0) \geq 0\}.$$

Therefore, these functions are Lipschitzian on R_+^3 , and as a result, system (2) has a unique and existing solution.

Theorem 1 [**Uniformly Boundedness**]:

All the solutions of system (2) with nonnegative initial conditions are uniformly bounded.

Proof: Let the solution of (2) be $[h(t), i(t), j(t)]$ the initial condition $[h(0), i(0), j(0)] \in R_+^3$ are nonnegative.

Now, let $\dot{\mathrm{H}}(t) = h(t) + i(t) + j(t)$,

$$\frac{d\dot{\mathrm{H}}}{dt} < 2h - (r_1 - r_3)hi - (r_2 - r_6)\, hj - (r_4 - r_7)\, ij - h - r_5 i - r_8\, j.$$

Now, ecologically $\mathrm{r}_3 < \mathrm{r}_1, \mathrm{r}_6 < \mathrm{r}_2$ and $\mathrm{r}_7 < \mathrm{r}_4$.

$$\frac{d\dot{\mathrm{H}}}{dt} < 2 - \delta\, \dot{\mathrm{H}} \ , \quad where \ \ \delta = min\ \{1, \mathrm{r}_5, \mathrm{r}_8\}.$$

Now, by solving this differential inequality for the initial value H (0) = H_0 , we get that:

$$\dot{\mathrm{H}}(\mathrm{t}) \leq \frac{2}{\delta} + \left(\dot{\mathrm{H}}(0) - \frac{2}{\delta}\right) \mathrm{e}^{-\delta\, \mathrm{t}}$$

Thus $\quad 0 \leq \dot{\mathrm{H}}(t) \leq \frac{2}{\delta} \qquad$ as $t \to \infty$.

Hence system (2) has uniformly bounded.

3. Equilibrium points are existence and stable:

There are maximum of five equilibrium points in System (2), which are listed below:

- The point of equilibrium $A_0 = (0, 0, 0)$, it is always present it is referred to as the vanishing point, which is unstable and always existing.
- The axial equilibrium point $A_1 = (1, 0, 0)$, existence without conditions additionally.

Therefore, the characteristic equation of J (A_1) is as follows:

$$J_1 = J(A_1) = \begin{bmatrix} -1 & -r_1 & -r_2 \\ 0 & r_3 - r_5 & 0 \\ 0 & 0 & r_6 - r_8 \end{bmatrix}. \qquad (1.a)$$

$$(-1-\lambda)\,[(r_3 - r_5) - \lambda]\,[(r_6 - r_8) - \lambda] = 0.$$

Which gives the eigenvalues of J_1 *by*:

$\lambda_{1h} = -1 < 0,\quad \lambda_{1i} = r_3 - r_5 < 0$ and $\lambda_{1j} = (r_6 - r_8) < 0$

The equilibrium point A_1 then becomes asymptotically stable under the following conditions:

$r_3 > r_5$, (3)

$r_8 > r_6$. (4)

Otherwise, A_1 is unstable. However, it is a saddle point.

◉ The equilibrium point $A_2\left(\bar{h}, \bar{i}, 0\right)$ exists uniquely in $Int.R_+^2$ (Interior of R_+^2) of hi − plane provided that:

$r_3 > r_5$. (5)

Where:

$$\bar{h} = \frac{r_5}{r_3} > 0. \qquad (6)$$

$$\bar{i} = \frac{r_3 - r_5}{r_1\, r_3}. \qquad (7)$$

And the Jacobian matrix of system (2) at A_2 can be written as

$$J_2 = J(A_2) = [\mu_{ij}]_{3\times 3}, \qquad (2.a)$$

where:

$\mu_{11} = 1 - 2\bar{h} - r_1\bar{i},\ \mu_{12} = -r_1\bar{h} < 0,\ \mu_{13} = -r_2\bar{h},$

$\mu_{21} = r_3\bar{i} > 0,\ \mu_{22} = r_3\bar{h} - r_5,\ \mu_{23} = -r_4\bar{i},$

$\mu_{31} = 0,\ \mu_{32} = 0,\ \mu_{33} = r_6\bar{h} + r_7\bar{i} - r_8.$,

Consequently, the characteristic equation of J (A_2) is as follows:

$$(\mu_{33} - \lambda)[\lambda^2 - \hat{A})\lambda + \det(\hat{A})] = 0$$

where : $\hat{A} = \begin{bmatrix} 1 - 2\bar{h} - r_1\bar{i} & -r_1\bar{h} \\ r_3\bar{i} & r_3\bar{h} - r_5 \end{bmatrix},$

Then $\left[(r_6\bar{h} + r_7\bar{i} - r_8) - \lambda\right]\left[(1 - 2\bar{h} - r_1\bar{i} - \lambda)(r_3\bar{h} - r_5 - \lambda) + r_1 r_3\,\bar{h}\,\bar{i}\right] = 0$

Either, $\lambda_{2h} = r_6\bar{h} + r_7\bar{i} - r_8$

Which, as a result of the following condition, produces the first eigenvalues J_2 with negative real parts: $r_8 > r_6\bar{h} + r_7$ (8)

or, $\lambda^2 - \mathrm{tr}(\hat{A})\lambda + \det(\hat{A}) = 0$

Where:

$$\mathrm{tr}\left(\hat{A}\right) = \lambda_{2i} + \lambda_{2j} = 1 - 2\bar{h} - r_1\bar{i} + r_3\bar{h} - r_5$$
$$= (1 + r_3\bar{h}) - (2\bar{h} + r_1\bar{i} + r_5) > 0,$$

$\det(\hat{A}) = \lambda_{2i}\lambda_{2j}$

$= \left(r_3\bar{h} - r_5\right)\left(1 - 2\bar{h} - r_1\bar{i}\right) + r_1 r_3\, 2r_5\, \bar{i}$

$= (2r_5 + r_3)\bar{h} + r_1 r_5\, \bar{i} - (2r_3\, \bar{h}^2 + r_5) > 0.$

Which, as a result of the following criteria, produces the second two eigenvalues of J_2 with negative real parts:

$$2\bar{h} + r_1\bar{i} + r_5 < 1 + r_3\bar{h}, \tag{9}$$

$$2r_3\, \bar{h}^2 + r_5 < (2r_5 + r_3)\bar{h} + r_1 r_5\, \bar{i}. \tag{10}$$

Therefore, A_2 is stable equilibrium point if conditions (8), (9) and (10) are satisfied. However otherwise, it is unstable.

◉ The specialist predator free equilibrium point $A_3 = (\bar{\bar{h}}, 0, \bar{\bar{j}})$ exists if the solutions to the following set of equations are positive:

$$\bar{\bar{h}} = \frac{r_8}{r_6} > 0 \tag{11}$$

$$\bar{\bar{j}} = \frac{r_6 - r_8}{r_2\, r_6} \tag{12}$$

The equation (12) is positive, provided that:

$$r_6 > r_8\,. \tag{13}$$

The Jacobian matrix of system (2) at A_2 can be written as:

$$J_3 = J(A_3) = \left[\eta_{ij}\right]_{3\times 3}, \qquad (3.a)$$

where: $\eta_{11} = 1 - 2\bar{\bar{h}} - r_2\bar{\bar{j}}, \eta_{12} = -r_1\bar{\bar{h}} < 0$,

$\eta_{13} = -r_2\bar{\bar{h}} < 0\ \eta_{21} = 0$, $\eta_{22} = r_3\bar{\bar{h}} - r_4\,\bar{\bar{j}} - r_5$,

$\eta_{33} = r_6\bar{\bar{h}} - r_8. \eta_{23} = 0$, $\eta_{31} = r_6\,\bar{\bar{j}}$, $\eta_{32} = r_7\,\bar{\bar{j}}$

Characteristic equation for J (A_3) is then provided by:

$$(\eta_{22} - \lambda)[\lambda^2 - \mathrm{tr}(\mathring{A})\lambda + \det(\mathring{A})] = 0$$

where : $\mathring{A} = \begin{bmatrix} 1 - 2\bar{\bar{h}} - r_2\bar{\bar{j}} & -r_2\bar{\bar{h}} \\ r_6\,\bar{\bar{j}} & r_6\bar{\bar{h}} - r_8 \end{bmatrix}.$

Then $\left[(r_3\bar{\bar{h}} - r_4\,\bar{\bar{j}} - r_5) - \lambda\right]\left[\left(1 - 2\bar{\bar{h}} - r_2\bar{\bar{j}} - \lambda\right)\right]\left(r_6\bar{\bar{h}} - r_8 - \lambda\right) + r_2 r_6\, \bar{\bar{h}}\,\bar{\bar{j}} = 0$

Either, $\lambda_{3h} = r_3\bar{\bar{h}} - r_4\,\bar{\bar{j}} - r_5$

Because of the following circumstance the first eigenvalues of J_3 have negative real portions:

$$r_3\bar{\bar{h}} > r_4\,\bar{\bar{j}} + r_5 \tag{14}$$

or, $\lambda^2 - \mathrm{tr}(\mathring{A})\lambda + \det(\mathring{A}) = 0$

Where,

$\mathrm{tr}\left(\mathring{A}\right) = \lambda_{3i} + \lambda_{3j} = 1 - 2\bar{\bar{h}} - r_2\bar{\bar{j}} + r_6\bar{\bar{h}} - r_8 > 0$

$\det(\mathring{A}) = \lambda_{3i}\lambda_{3j}$

$= \left(1 - 2\bar{\bar{h}} - r_2\bar{\bar{j}}\right)\left(r_6\bar{\bar{h}} - r_8\right) + r_2 r_6\, \bar{\bar{h}}\,\bar{\bar{j}}$

$= (r_6 + 2r_8)\bar{\bar{h}} + r_2 r_8\ \bar{\bar{j}} - (2r_6\ \bar{\bar{h}}^2 + r_8) > 0.$

Therefore, as a result of the following requirements, results in the second two eigenvalues of J_3 having negative real portions:

$$2\bar{\bar{h}} + r_2\bar{\bar{j}} - r_6\bar{\bar{h}} < 1, \qquad (15)$$

$$2r_3\ \bar{h}^2 + r_5 < (2r_5 + r_3)\bar{h} + r_1 r_5\ \bar{i}. \qquad (16)$$

Therefore, A_3 is stable equilibrium point if conditions (14), (15) and (16) are satisfied. On the other hand, it is unstable**.**

◉ Finally, the positive (coexistence) equilibrium point $A_4 = (\tilde{h}, \tilde{\imath}, \tilde{j})$ exists if the following system of equations has a positive solution:

$$\tilde{h} = \frac{(r_4 + r_1 r_5) - r_1 r_4 r_8 / r_7}{(r_4 + r_1 r_3) - r_1 r_4 r_6 / r_7}$$

$$\tilde{\imath} = \frac{r_8 - r_6\tilde{h}}{r_7}, \quad \tilde{j} = \frac{r_3\tilde{h} - r_5}{r_4}.$$

Note that $\bar{\bar{h}}$ is positive, provided that:

$(r_4 + r_1 r_5) < r_1 r_4 r_8 / r_7$ and $(r_4 + r_1 r_3) < r_1 r_4 r_6 / r_7$.

Or

$(r_4 + r_1 r_5) > r_1 r_4 r_8 / r_7$ *and* $(r_4 + r_1 r_3) > r_1 r_4 r_6 / r_7$

So, $\tilde{\imath}$ and $\tilde{j}$ are positive , provided that:

$r_3\ \tilde{h} > r_5$ and $r_8 > r_6\ \tilde{h}$ respectively.

For $A_4 = (\tilde{h}, \tilde{\imath}, \tilde{j})$, can be expressed as:

$$J_4 = J(A_4) = [\,\tilde{n}_{ij}\,]_{3\times 3}, \qquad (4.a)$$

Where:

$\tilde{n}_{11} = 1 - 2\tilde{h} - r_1\tilde{\imath} - r_2\tilde{j}$, $\tilde{n}_{12} = -r_1\tilde{h} < 0$, $\tilde{n}_{13} = -r_2\tilde{h} < 0$, $\tilde{n}_{21} = r_3\tilde{\imath} > 0$,
$n_{22} = r_3\tilde{h} - r_4\tilde{j} - r_5$, $\tilde{n}_{23} = -r_4\tilde{\imath}$**,** $\tilde{n}_{31} = r_6\tilde{j} > 0$, $\tilde{n}_{32} = r_7\tilde{j} > 0$, $\tilde{n}_{33} = r_6\tilde{h} + r_7\tilde{\imath} - r_8$

A characteristic equation for J (A_4) is then provided by:

$$\lambda^3 + \check{R}_1\lambda^2 + \check{R}_2\lambda + \check{R}_3 = 0, \qquad (4.b)$$

where: $\check{R}_1 = -(\tilde{n}_{11} + \tilde{n}_{22} + \tilde{n}_{33})$,

$$\check{R}_2 = -[\tilde{n}_{23}\tilde{n}_{32} - \tilde{n}_{22}\tilde{n}_{33} - \tilde{n}_{11}(\tilde{n}_{22} + \tilde{n}_{33}) + \tilde{n}_{21}\tilde{n}_{12} + \tilde{n}_{13}\tilde{n}_{31}]$$

$$\check{R}_3 = -\tilde{n}_{11}(\tilde{n}_{22}\tilde{n}_{33} - \tilde{n}_{23}\tilde{n}_{32}) + \tilde{n}_{12}\tilde{n}_{21}\tilde{n}_{33} - \tilde{n}_{12}\tilde{n}_{31}\tilde{n}_{23} - \tilde{n}_{13}\tilde{n}_{21}\tilde{n}_{32} - \tilde{n}_{13}\tilde{n}_{31}\tilde{n}_{22}.$$

Now, $\check{R}_1 > 0$ and $\check{R}_2 > 0$ provided that:

$$1 < 2\tilde{h} + r_1\tilde{\imath} + r_2\tilde{j}, \qquad (17)$$

$$\tilde{h} < r_4\tilde{j} + r_5, \qquad (18)$$

$$r_6\tilde{h} + r_7\tilde{\imath} < r_8 \qquad (19)$$

Also, $\Delta = \check{R}_1\ \check{R}_2 - \check{R}_3 > 0$.

By the following condition:

$$\tilde{h} < \frac{r_4}{r_1 r_3 + 2r_4}(1 - r_1\tilde{\imath} - r_2\tilde{j}), \qquad (20)$$

$$r_2 r_6\ \tilde{h}\,\tilde{j} < w_1\ w_2, \qquad (21)$$

$$r_4 r_6\ \tilde{\imath}\ \tilde{j} < r_1\ \tilde{h}(r_6\ \tilde{h} + r_7\ \tilde{\imath} - r_8). \qquad (22)$$

Where, $w_1 = 2\tilde{h} + r_1\tilde{\imath} + r_2\tilde{j} - 1$, $w_2 = r_6\tilde{h} + r_7\tilde{\imath} - r_8$.

Using the Routh-Hurwitz criterion, however, allows each of the additional eigenvalues of eq. $(4.b)$, have negative real parts if and only if $R_1 > 0, R_3 > 0$ and $R_1R_2 - R_3 > 0$.

Therefore, all of $J(A_4)$ eigenvalues have a negative real portion if the additional criteria from (17) - (22) hence A_4 is asymptotically stable locally. In contrast, it is unstable.

4. Local Bifurcation Analysis

This section investigates the dynamical behavior of system (2) around each equilibrium point as a result of altering the parameter values. Remember that the existence of the system (2)'s non-hyperbolic equilibrium point is a required, but not sufficient, need for bifurcation. As a result, it is appropriate to apply the Sotomayor's Theorem for local bifurcation in the following theorems. Currently, in accordance with the Jacobian matrix of system (2).

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial h} & \frac{\partial f_1}{\partial i} & \frac{\partial f_1}{\partial j} \\ \frac{\partial f_2}{\partial h} & \frac{\partial f_2}{\partial i} & \frac{\partial f_2}{\partial j} \\ \frac{\partial f_3}{\partial h} & \frac{\partial f_3}{\partial i} & \frac{\partial f_3}{\partial j} \end{bmatrix} \qquad (5.a)$$

where f_i ; i =1, 2, 3 are displayed on the system's right side (2) and

$\frac{\partial f_1}{\partial h} = 1 - 2h - r_1 i - r_2$, $\frac{\partial f_1}{\partial i} = -r_1 h$, $\frac{\partial f_1}{\partial j} = -r_2 h$, $\frac{\partial f_2}{\partial h} = r_3 i$,

$\frac{\partial f_2}{\partial i} = r_3 h - r_4 j - r_5$, $\frac{\partial f_2}{\partial j} = - r_4 i$, $\frac{\partial f_3}{\partial h} = r_6 j$, $\frac{\partial f_3}{\partial i} = r_7 j$, $\frac{\partial f_3}{\partial j} = r_6 h + r_7 j - r_8$.

It is clear to verify that for any nonzero vector $\dot{\rho} = (\dot{\rho}_1, \dot{\rho}_2, \dot{\rho}_3)^T$ we have:

$J\dot{\rho} = [\, \tau_{ij} \,]_{3\times1}$

Where:

$\tau_{11} = (1 - 2h - r_1 i - r_2 j)\dot{\rho}_1 - r_1 h\dot{\rho}_2 - r_2 h\dot{\rho}_3,$

$\tau_{21} = r_3 i\dot{\rho}_1 + (r_3 h - r_4 j - r_5)\dot{\rho}_2 - r_4 i\dot{\rho}_3,$

$\tau_{31} = r_6 j\dot{\rho}_1 + r_7 j\dot{\rho}_2 + (r_6 h + r_7 i - r_8)\dot{\rho}_3.$

$$D^2{}^\circ F_\mu(\acute{Y}, \mu)(\dot{\rho}, \dot{\rho}) = [\, \ddot{\tau}_{ij} \,]_{3\times1}. \qquad (23)$$

Where:

$\ddot{\tau}_{11} = (-2\dot{\rho}_1 - r_1\dot{\rho}_2 - r_2\dot{\rho}_3)\dot{\rho}_1 - r_1\dot{\rho}_1\dot{\rho}_2 - r_2\dot{\rho}_1\dot{\rho}_3,$

$\ddot{\tau}_{21} = r_3\dot{\rho}_1\dot{\rho}_2 + (r_3\dot{\rho}_1 - r_4\dot{\rho}_3)\dot{\rho}_2 - r_4\dot{\rho}_2\dot{\rho}_3,$

$\ddot{\tau}_{31} = r_6\dot{\rho}_1\dot{\rho}_3 + r_7\dot{\rho}_2\dot{\rho}_3 + (r_6\dot{\rho}_1 + r_7\dot{\rho}_2)\dot{\rho}_3.$

Where $\acute{Y} = (h, i, j)^T$ and μ is any bifurcation parameter.

Theorems in the following the local bifurcation conditions near equilibrium points are established.

4.1 Local bifurcation analysis near A_1:

Theorem (2): If the value of the parameter r_3 passes through $\ddot{r}_3 = r_3$ then, system (2) at the axial equilibrium point A_1 = (1, 0, 0) possesses:

- No saddle-node bifurcation .
- Transcritical bifurcation .

Proof: According to the Jacobian matrix $J(A_1)$ given by eq.$(1.a)$: Zero eigenvalue exists for system (2) at equilibrium point A_1= (1,0,0). (say $\lambda_{1i} = 0$) at $r_3 = \ddot{r}_3$, and the Jacobian matrix J_1 with $r_3 = \ddot{r}_3$ becomes:

$$\ddot{J}_1 = J_1(r_3 = \ddot{r}_3) = \begin{bmatrix} -1 & -r_1 & -r_2 \\ 0 & 0 & 0 \\ 0 & 0 & r_3 - r_5 \end{bmatrix},$$

Now, let $\dot{\rho}^{[1]} = \left(\dot{\rho}_1^{[1]}, \dot{\rho}_2^{[1]}, \dot{\rho}_3^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1i} = 0$. Thus $\left(\ddot{J}_1 - \lambda_{1i} I\right) \dot{\rho}^{[1]} = 0$, which gives:

$\rho^{[1]} = \left(-r_1 \dot{\rho}_2^{[1]}, \dot{\rho}_2^{[1]}, 0\right)$ where $\dot{\rho}_2^{[1]}$ and $\dot{\rho}_2^{[1]}$ are any nonzero real number.

Let $Ç^{[1]} = \left(Ç_1^{[1]}, Ç_2^{[1]}, Ç_3^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1i} = 0$ of the matrix $[\ddot{J}_1]^T$**.** Then we have, $\left(\ddot{J}_1^{\,T} - \lambda_{1i} I\right) Ç^{[1]} = 0$. By solving this equation $Ç^{[1]}$,

We obtain, $Ç^{[1]} = \left(0, Ç_2^{[1]}, 0\right)^T$ where $Ç_2^{[1]}$ any nonzero real number.

Now,

$$\frac{\partial f}{\partial r_3} = f_{r_3}(A_1, r_3) = \left(\frac{\partial f_1}{\partial r_3}, \frac{\partial f_2}{\partial r_3}, \frac{\partial f_3}{\partial r_3}\right)^{\mathrm{T}} = (0, \mathrm{h}i, 0)^{\mathrm{T}}.$$

So, $\frac{\partial f}{\partial r_3}(A_1, \ddot{r}_3) = (0,0,0)^T$ that's why $(Ç)^T \frac{\partial f}{\partial r_3}(A_1, \ddot{r}_3) = 0$.

Therefore, according to Sotomayor's theorem the saddle - node bifurcation cannot occur**.** While the first condition of transcritical bifurcation is satisfied . Now, since

$$Df_{r_3}(A_1, \ddot{r}_3)\dot{\rho}^{[1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -r_1 \dot{\rho}_2^{[1]} \\ \dot{\rho}_2^{[1]} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{\rho}_2^{[1]} \\ 0 \end{pmatrix},$$

$$\left(Ç^{[1]}\right)^T \left[Df_{r_3}(A_1, \ddot{r}_3)\dot{H}^{[1]}\right] = \left(0, Ç_2^{[1]}, 0\right)^T \begin{pmatrix} 0 \\ \dot{\rho}_2^{[1]} \\ 0 \end{pmatrix} = \dot{Ç}_2^{[1]} \rho_2^{[1]} \neq 0.$$

Now, by substituting $\dot{\rho}^{[1]}$ in (23)**,** we get**:**

$$D^2 f(A_1, \ddot{r}_3)\left(\dot{\rho}^{[1]}, \dot{\rho}^{[1]}\right) = \begin{pmatrix} 4{r_1}^2 [\dot{\rho}_2^{[1]}]^2 \\ -r_1 r_3 [\dot{\rho}_2^{[1]}]^2 \\ 0 \end{pmatrix}.$$

Hence, it is obtained that:

$$\left(Ç^{[1]}\right)^T D^2 f(A_1, \ddot{r}_3)\left(\dot{\rho}^{[1]}, \dot{\rho}^{[1]}\right) = \left(0, Ç_2^{[1]}, 0\right)^T \begin{pmatrix} 4{r_1}^2 [\dot{\rho}_2^{[1]}]^2 \\ -r_1 r_3 [\dot{\rho}_2^{[1]}]^2 \\ 0 \end{pmatrix} = -r_1 r_3 [\dot{\rho}_2^{[1]}]^2 Ç_2^{[1]} \neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcrirtical bifurcation but not experience a saddl node bifurcation at A_1 with the parameter r_3, where $r_3 = \ddot{r}_3$.

4.2 Near by local bifurcation analysis $A_2\left(\bar{\mathrm{h}}, \bar{i}, 0\right)$:

Theorem (3): Assume that the following conditions are met:

$$€_2 \neq 0, \tag{24}$$

$$€_4 \neq 0, \tag{25}$$

$$\vartheta = -2€_1€_3(€_1 + r_1 + r_2€_2) + r_3€_1 - r_4€_2 + €_2€_4(r_6€_1 + r_7) \neq 0 \tag{26}$$

$$€_1 = \frac{\bar{\mathrm{h}}\,[r_1 r_4\,\bar{\mathrm{h}}\,\bar{i}\ - r_2(r_3\bar{h} - r_3)]}{i\left[r_4\left(1 - r_1\bar{i}\right) - (2r_4 + r_2 r_3)\bar{\mathrm{h}}\right]}, \quad €_2 = \frac{(1 - 2\,\bar{\mathrm{h}}\ - r_1\bar{i})\bar{i} - r_1\bar{h}}{r_2\bar{h}}$$

$$€_3 = \frac{r_3\bar{h} - r_5}{r_1\bar{h}}, \quad €_4 = \frac{r_2\bar{h}\,€_3 + r_4\,\bar{i}}{r_6\bar{h} + r_7\,\bar{i} - r_8}.$$

Then system (2) at the equilibrium point $A_2 = \left(\bar{\mathrm{h}}\,,\bar{i}\,,0\right)$ with the parameter

$\ddot{r}_8\ = r_6\bar{h} + r_7\bar{i}$

possesses**:**

• No saddle- node bifurcation. • Transcritical bifurcation.

Proof: According to the Jacobian matrix $J(A_2)$ given by eq.$(2.a)$ of system (2) at the equilibrium point $A_2 = \left(\bar{\mathrm{h}}\,,\bar{i}\,,0\right)$ has zero eigenvalue (say $\lambda_{2j} = 0$) at $r_8 = \ddot{r}_8$ **,** and the Jacobian matrix J_2 with $\ddot{r}_8 = r_6\bar{h} + r_7\bar{i}$ becomes:

$\ddot{J_2} = J_2(r_8 = \ddot{r}_8) = \left[\mu_{ij_{ij}}\right]_{3x3}$,

where, $\dot{\mu_{\iota J}}_{\mathrm{ij}} = \mu_{ij}$ for all i, j =1,2,3 except $\dot{\mu_{\iota J}}_{33} = r_6\bar{h} + r_7\bar{i} - r_8$=0.

Let $\dot{\rho}^{[2]} = \left(\dot{\rho}_1^{[2]}, \dot{\rho}_2^{[2]}, \dot{\rho}_3^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2j} = 0$. Thus $\left(\ddot{J_2} - \lambda_{2j}I\right)\dot{\rho}^{[2]} = 0$, which gives :

$\dot{\rho}_1^{[2]} = €_1\dot{\rho}_2^{[2]}$ and $\dot{\rho}_3^{[2]} = €_2\dot{\rho}_2^{[2]}$,

where $\dot{\rho}_2^{[2]}$ any nonzero, real number with $€_1, €_2$ and which and which are mentioned in the state of the theorem.

Let $Ç^{[2]} = \left(Ç_1^{[2]}, Ç_2^{[2]}, Ç_3^{[2]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{2w} = 0$ of the matrix $\ddot{\jmath}_2$**.**

Then we have $(\ddot{\jmath}_2 - \lambda_{2w}I)\,Ç^{[2]} = 0$. By solving this equation for $Ç^{[2]}$, we obtain $Ç^{[2]} = \left(€_3 Ç_2^{[2]}, Ç_2^{[2]}, €_4 Ç_2^{[2]}\right)^T$, where $Ç_2^{[2]}$ any real numbers that are not zero, with $€_3, €_4$ *w*hich are mentioned in the state of the theorem .

Now, consider:

$$\frac{\partial f}{\partial r_8}(\acute{Y}, r_8) = f_{r_8}(\acute{Y}, r_8) = \left(\frac{\partial f_1}{\partial r_8}, \frac{\partial f_2}{\partial r_8}, \frac{\partial f_3}{\partial r_8}\right)^{\mathrm{T}} = (0\,,0\,,j)^T$$

So, $$\frac{\partial f}{\partial r_8}(\mathrm{A}_2\,, r_8) = (0\,,0,0\,)^T.$$

That's why $\left(Ç^{[2]}\right)^T f_{r_8}(\mathrm{A}_2, \ddot{r}_8) = 0.$

Therefore , according to Sotomayor 's theorem there can be no saddle-node bifurcation. Although the first need for transcritical bifurcation has been satisfied. Now, since

$$Df_{r_8}(A_2, \ddot{r}_8)\dot{\rho}^{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} €_1\dot{\rho}_2^{[2]} \\ \dot{\rho}_2^{[2]} \\ €_2\dot{\rho}_2^{[2]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -€_2\dot{\rho}_2^{[2]} \end{pmatrix}$$

$$\left(Ç^{[2]}\right)^T\left[Df_{r_8}(A_2, \ddot{r}_8)\dot{\rho}^{[2]}\right] = \left(€_3Ç_2^{[2]}, Ç_2^{[2]}, €_4Ç_2^{[2]}\right)\begin{pmatrix} 0 \\ 0 \\ -€_2\dot{\rho}_2^{[2]} \end{pmatrix} = -€_2€_4Ç_2^{[2]}\dot{\rho}_2^{[2]} \neq 0.$$

Moreover, by substituting $\dot{\rho}^{[2]}$ in (23), we get:

$$D^2f(A_2, \ddot{r}_8)\left(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}\right) = \begin{bmatrix} -2€_1(\dot{\rho}_2^{[2]})^2\,(€_1 + r_1 + r_2€_2) \\ 2(\dot{\rho}_2^{[2]})^2(r_3€_1 - r_4€_2) \\ 2€_2(\dot{\rho}_2^{[2]})^2(r_6€_1 + r_7) \end{bmatrix}.$$

Hence, it is obtained that: $\left(€_3Ç_2^{[2]}, Ç_2^{[2]}, €_4Ç_2^{[2]}\right)$

$$\left(Ç^{[2]}\right)^T D^2f(A_2, \ddot{r}_8)\left(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}\right) = 2\vartheta(\dot{\rho}_2^{[2]})^2Ç_2^{[2]}$$

Where, $€_1$, $€_3$ and $€_4$ are mentioned in the theorem's state.

As a result of condition (26), we get that:

$$\left(Ç^{[2]}\right)^T D^2f(A_2, \ddot{r}_8)\left(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}\right) \neq 0\,.$$

Thus, according to Sotomayor's theorem system (2) has a transcritical bifurcation at the equilibrium point $A_2 = \left(\bar{\mathrm{h}}, \bar{i}, 0\right)$ with the parameter $\ddot{r}_8 = r_6\bar{h} + r_7\bar{i}$.

4.3 Local bifurcation analysis near $A_3 = (\bar{\bar{h}}, 0, \bar{\bar{j}})$:

Theorem (4): Assume that the following criteria are fulfilled:

$$r_4 > \frac{r_3\,\bar{\bar{h}} - r_5}{\bar{\bar{j}}}, \tag{27}$$

$$\grave{\varepsilon}_2 \neq 0, \tag{28}$$

$$\mathbb{R} \neq 0 \tag{29}$$

where:

$$\grave{\varepsilon}_1 = \frac{r_1r_6\bar{\bar{h}} + r_7\bar{\bar{j}}\left(1 - 2\bar{\bar{h}} - r_2\right)}{\bar{\bar{h}}\left[r_2 + r_1\left(r_6\bar{\bar{h}} - r_8\right)\right]}, \qquad \grave{\varepsilon}_2 = \frac{1 - r_2\,\bar{\bar{j}} - (2 - r_2\grave{\varepsilon}_1)\bar{\bar{h}}}{r_1\,\bar{\bar{h}}}, \qquad \grave{\varepsilon}_3 = \frac{r_7\bar{\bar{j}}}{r_1\bar{\bar{h}}},$$

$$\mathbb{R} = Ç_3^{[3]}[€_2(r_6€_1 + r_7) - €_1€_3(€_1 + r_1 + r_2€_2)] + Ç_2^{[3]}[r_3€_1 - r_4€_2].$$

Then system (2) at the equilibrium point $A_3 = (\bar{\bar{h}}, 0, \bar{\bar{j}})$ with the parameter value: $\ddot{r}_4 = r_3\bar{\bar{h}} - r_4\bar{\bar{j}}$ has a transcritical bifurcation, but a saddle $-$ node cannot occur at A_3.

Proof: The characteristic equation represented by eq. $(3.a)$ of $s$ystem (2) at the equilibrium point A_3 has zeroeigenvalue (say $\lambda_{3i} = 0$) at $r_4 = \ddot{r}_4$ and the Jacobian matrix J_3 with 4parameter $r_4 = \ddot{r}_4$ becomes:

$\ddot{J}_3 = J_3(r_4 = \ddot{r}_4) = [\tilde{\eta}_{ij}]_{3\times 3}$,

Where: $\tilde{\eta}_{ij} = \eta_{ij}$ for all i , j $= 1,2,3$ except $\tilde{\eta}_{ij} = r_3\bar{\bar{h}} - \ddot{r}_4\bar{\bar{J}} - r_4 = 0$.

Let $\dot{\rho}^{[3]} = \left((\dot{\rho}_2^{[3]}, \dot{\rho}_2^{[3]}, \dot{\rho}_3^{[3]}\right)^T$ be the eigenvector which follows the eigenvalue be the eigenvector which follows the eigenvalue. $\lambda_{3s} = 0$. Thus $\left(\ddot{J}_3 - \lambda_{3s}I\right)\dot{\rho}^{[3]} = 0$, which gives:

$\dot{\rho}^{[3]} = \left(\dot{\rho}_1^{[3]}, \grave{\varepsilon}_2\, \dot{\rho}_1^{[3]}, \grave{\varepsilon}_1\, \dot{\rho}_1^{[3]}\right)^T$, where $\dot{\rho}_1^{[3]}$ any nonzero number with, $\grave{\varepsilon}_1$ which are mentioned in the state of the theorem.

Let $Ç^{[3]} = \left(Ç_1^{[3]}, Ç_2^{[3]}, Ç_3^{[3]}\right)^T$ become the eigenvector linked to the eigenvalue $\lambda_{3s} = 0$ of the matrix $\ddot{J}_3$. Then we have $\left(\ddot{J}_3^{\,T} - \lambda_{3i}I\right)Ç^{[3]} = 0$.

By solving this equation for $Ç^{[3]}$, we obtain:

$Ç^{[3]} = \left(\grave{\varepsilon}_3\, Ç_3^{[3]}, Ç_2^{[3]}, Ç_3^{[3]}\right)^T$ where $Ç_3^{[3]}$ any nonzero number with $\grave{\varepsilon}_3$ those are referred to in the theorem's state. Now, consider:

$$\frac{\partial f}{\partial r_4}(\acute{Y}, r_4) = f_{r_4}(\acute{Y}, r_4) = \left(\frac{\partial f_1}{\partial r_4}, \frac{\partial f_2}{\partial r_4}, \frac{\partial f_3}{\partial r_4}\right)^{\mathrm{T}} = (0, -ij, 0)^T$$

So, $$\frac{\partial f}{\partial r_4}(A_3, r_4) = (0, 0, 0)^T.$$

And hence $\left(Ç^{[3]}\right)^T f_{r_4}(A_3, \ddot{r}_4) = 0$.

Therefore, the saddle-node bifurcation is ruled out by Sotomayor's theorem**.** While the first transcritical bifurcation condition has been satisfied. Now, since$\dot{\rho}^{[3]} = [\dot{\rho}_1^{[3]}, \grave{\varepsilon}_2\, \dot{\rho}_1^{[3]}, \grave{\varepsilon}_1\, \dot{\rho}_1^{[3]}]$

$$Df_{r_4}(A_3, \ddot{r}_4)\dot{\rho}^{[3]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -j & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{\rho}_1^{[3]} \\ \grave{\varepsilon}_2\, \dot{\rho}_1^{[3]} \\ \grave{\varepsilon}_1\, \dot{\rho}_1^{[3]} \end{pmatrix} = \begin{pmatrix} 0 \\ -j\, \grave{\varepsilon}_2\, \dot{\rho}_1^{[3]} \\ 0 \end{pmatrix}$$

$$\left(Ç^{[2]}\right)^T\left[Df_{r_4}(A_2, \ddot{r}_4)\dot{\rho}^{[2]}\right] = \left(€_3 Ç_3^{[3]}, Ç_2^{[3]}, Ç_3^{[3]}\right)\begin{pmatrix} 0 \\ -j\, \grave{\varepsilon}_2\, \dot{\rho}_1^{[3]} \\ 0 \end{pmatrix} = -\grave{\varepsilon}_2\, Ç_2^{[3]}\bar{\bar{J}}\, \dot{\rho}_1^{[3]}.$$

So, by condition (28), we obtain that:

$$\left(Ç^{[2]}\right)^T\left[Df_{r_4}(A_2, \ddot{r}_4)\dot{\rho}^{[2]}\right] \neq 0$$

Moreover, by substituting $\dot{\rho}^{[3]}$ in (23)**,** we get**:**

$$D^2 f(A_3, \ddot{r}_4)\left(\dot{\rho}^{[3]}, \dot{\rho}^{[3]}\right) = \begin{bmatrix} -2€_1(\dot{\rho}_2^{[2]})^2 (€_1 + r_1 + r_2€_2) \\ 2(\dot{\rho}_2^{[2]})^2(r_3€_1 - r_4€_2) \\ 2€_2(\dot{\rho}_2^{[2]})^2(r_6€_1 + r_7) \end{bmatrix}.$$

Hence, it is obtained that: $\left(€_3 Ç_3^{[3]}, Ç_2^{[3]}, Ç_3^{[3]}\right)$

$\left(Ç^{[2]}\right)^T D^2 f(A_3, \ddot{r}_4)\left(\dot{\rho}^{[3]}, \dot{\rho}^{[3]}\right)=2(\dot{\rho}_2^{[2]})^2\ \mathbb{R}$

Where, $€_1$, $€_2$, $€_3$ and $\mathbb{R}$ are mentioned in the state of the theorem. As a result of condition (29), we get that: $\left(Ç^{[2]}\right)^T D^2 f(A_2, \ddot{r}_8)\left(\dot{\rho}^{[2]}, \dot{\rho}^{[2]}\right) \neq 0$.

Thus, according to Sotomayor's theorem system (2) has a transcritical bifurcation at the equilibrium point $A_3 = (\bar{\bar{h}}, 0, \bar{\bar{j}})$ with the parameter $\ddot{r}_4 = r_3\bar{\bar{h}} - r_4\bar{\bar{j}}$.

4.4 Local bifurcation analysis near $A_4 = \left(\tilde{h}, \tilde{\imath}, \tilde{j}\right)$:

Theorem (5): Assume that the following criteria are fulfilled:

$€_1 = \frac{-r_3}{r_6}$, $€_2 = \frac{r_8 - r_6\tilde{h} - (r_7\tilde{\imath} + \tilde{j}€_1)}{r_6\ \tilde{j}}$, $€_3 = \frac{-r_3\ \tilde{\imath}}{r_6\ \tilde{j}}$,

$€_4 = \frac{(r_6\tilde{h} + r_7\tilde{\imath} - r_8)\ €_3 - r_4\ \tilde{j}}{r_2\ \tilde{h}} \neq 0,$ (30)

$1 = 2\ \tilde{h} + r_1\tilde{\imath} + r_2\ \tilde{j},$ (31)

$€_2€_4\ [€_2 + r_1€_1 + r_2] \neq €_1[r_3€_2 + r_4] + €_4[r_6€_2 + r_7€_1].$ (32)

Then system (2) at the equilibrium point $A_4 = \left(\tilde{h}, \tilde{\imath}, \tilde{j}\right)$ with the parameter value:

$$\ddot{r}_1 = \frac{1 - 2\ \tilde{h} - r_2\ \tilde{j}}{\tilde{\imath}} ,$$

has a saddle $-$ node bifurcation, but neither a transcritical nor a pitchfork bifurcation at A_4.

Proof: The characteristic equation given by eq. (4.a) if $H_4 = 0$ and A_4 becomes a non-hyperbolic equilibrium point, of system (2) having zero eigenvalue (say $\lambda_{4h} = 0$).
The Jacobian matrix for system (2) at equilibrium point E_4 with parameter$(r_1 = \ddot{r}_1)$

clearlybecomes: $\ddot{J}_4 = J_4(r_1 = \ddot{r}_1) = \left[\bar{\bar{n}}_{ij}\right]_{3\times3}$,
for all i, j =1, 2, 3 except $\tilde{n}_{11}$ which is given by:where, $\bar{\bar{n}}_{ij} = \tilde{n}_{11}$

$\tilde{n}_{11} = 1 - 2\tilde{h} - \ddot{r}_1\ \tilde{\imath} - r_2\ \tilde{j}$

Let $\dot{\rho}^{[4]} = \left(\dot{\rho}_1^{[4]}, \dot{\rho}_2^{[4]}, \dot{\rho}_3^{[4]}\right)^T$ be the eigenvector that follows the eigenvalue $\lambda_{4h} = 0$.
Thus $\left(\ddot{J}_4 - \lambda_{4h}I\right)\dot{\rho}^{[4]} = 0$, which gives:

$\dot{\rho}^{[4]} = \left(€_2\dot{\rho}_3^{[4]}, €_1\dot{\rho}_3^{[4]}, \dot{\rho}_3^{[4]}\right)^T$, where $\dot{\rho}_3^{[4]}$ any non-zero value that has the $€_1$ and $€_2$ conditions given in the theorem.

Let $Ç^{[4]} = \left(Ç^{[4]}, Ç^{[4]}, Ç^{[4]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{4i} = 0$ of the matrix $\ddot{J}_4$. Next, we have $\left(\ddot{J}_4^{\,T} - \lambda_{4i}I\right)Ç^{[4]} = 0$. By solving this equation for $\emptyset^{[4]}$, we obtain:

any nonzero number with $€_3$ and $€_4$ which are $Ç^{[4]} = \left(€_4Ç_2^{[4]}, Ç_2^{[4]}, €_3Ç_2^{[4]}\right)^T$ where $Ç_2^{[4]}$ mentioned in the state of the theorem.
Now,

$$\frac{\partial f}{\partial r_1} = f_{r_1}(\acute{Y}, r_1) = \left(\frac{\partial f_1}{\partial r_1}, \frac{\partial f_2}{\partial r_1}, \frac{\partial f_3}{\partial r_1},\right)^T = (-hi, 0, 0)^T,$$

So, $f_{a_1}(A_4, \ddot{r}_1) = (-\tilde{h}\,\tilde{\imath}, 0, 0)^T$, and hence, it is obtained that:

$$\left(\c{C}^{[4]}\right)^T f_{r_1}(A_4, \ddot{r}_1) = -\tilde{h}\,\tilde{\imath}\,€_4 \c{C}_2^{[4]}.$$

In the context of condition (30), we thus have that: $\left(\c{C}^{[4]}\right)^T f_{r_1}(A_4, \ddot{r}_1) \neq 0.$

$$D^2 f(A_4, \ddot{r}_1)\left(\dot{\rho}^{[4]}, \dot{\rho}^{[4]}\right) = \begin{bmatrix} -2\,€_2\,(\dot{\rho}_3^{[4]})^2[€_2 + r_1€_1 + r_2] \\ 2\,€_1\,(\dot{\rho}_3^{[4]})^2[r_3€_2 + r_4] \\ 2\,(\dot{\rho}_3^{[4]})^2[r_6€_2 + r_7€_1] \end{bmatrix}.$$

Hence, it is obtained that:

$$\left(\c{C}^{[4]}\right)^T D^2 f(A_4, \ddot{r}_1)\left(\dot{\rho}^{[4]}, \dot{\rho}^{[4]}\right)$$

$$= -2\,(\dot{\rho}_3^{[4]})^2 \c{C}_2^{[4]}\,(€_2€_4\,[€_2 + r_1€_1 + r_2] - €_1[r_3€_2 + r_4] - €_4[r_6€_2 + r_7€_1]).$$

Therefore, in accordance with condition (32) we get that:

$$\left(\c{C}^{[4]}\right)^T D^2 f(A_4, \ddot{r}_1)\left(\dot{\rho}^{[4]}, \dot{\rho}^{[4]}\right) \neq 0$$

Sotomayor's theorem is used to show that system (2) has a saddle-node bifurcation $A_4 = (\tilde{h}, \tilde{\imath}, \tilde{\jmath})$ at $\ddot{r}_1$.

5. Numerical Simulation

This section numerically analyzes System (2)'s dynamical behavior for a given set of parameters and various initial point sets. These are the study's objectives, including:

1. Analyze the impact of changing the value of each parameter on the system's dynamic behavior (2).
2. Verify the analytical results that were found.

The following hypothetical collection of parameters is found to satisfy the stability requirements of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Figure $\mathbf{5.1}\ (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$,

$$\boldsymbol{r_1 = 0.05},\quad \boldsymbol{r_2 = 0.1},\quad \boldsymbol{r_3 = 0.03},\quad \boldsymbol{r_4 = 0.1},$$
$$\boldsymbol{r_6 = 0.06},\quad \boldsymbol{r_7 = 0.07},\quad \boldsymbol{r_8 = 0.1}\quad \boldsymbol{r_5 = 0.01}, \tag{5.1}$$

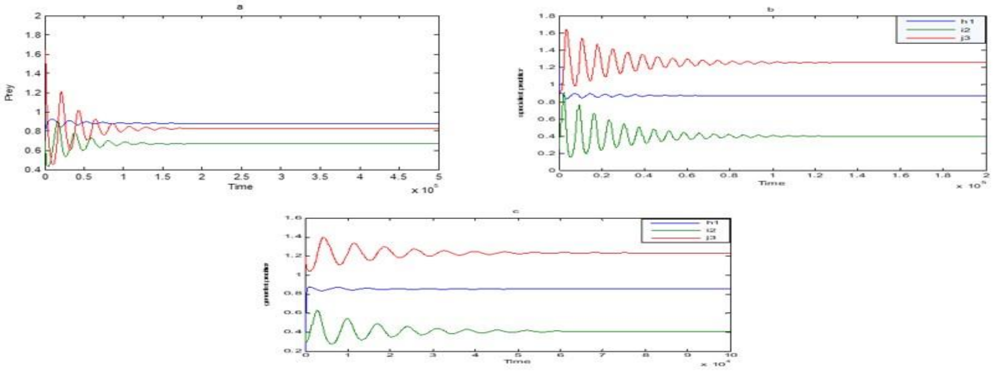
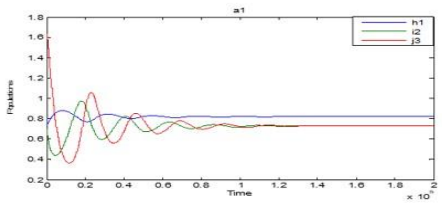


Figure1: The time series of the solution of system (2) started from the three different initial point (0.8, 0.7, 1.64), (1.2, 0.7, 1.64) and (0.2, 0.4, 0.64) for the data given by (5.1), **(a)** the trajectories of **h** as a function of time, **(b)** the trajectories of **i** as a function of time, **(c)** the trajectories of **j** as a function of time.

As the solution of system (2) approaches asymptotically to the positive equilibrium point A_4= (0.4544, 0.6567, 2. 1722) beginning with three distinct starting points, **Figure1.** clearly demonstrates that system (2) has a globally asymptotically stable, and this is supporting our obtained analytical results. We will now talk about how system's (2) parameter settings affect the system's dynamical behavior. The system is numerically solved for the data in (5.1) by changing one parameter at a time, sometimes even two, and the results are shown below.

The system will get closer to the point of positive equilibrium A_4 in the interior of the positive quadrant of the hij-plane as shown in **Figure2. a1** when the predation rate on a prey is varied from the specialist predator in the range $0.0001 < r_1 \leq 0.595$ while maintaining other parameters as data given in (5.1), r_1 has a usual value of 0.15. As shown in **Figure2.a2** for average value r_1=0.9, it is seen that the solution of system (2) approaches asymptotically to the equilibrium point A_2 in the range $0.595 < r_1 < 2$



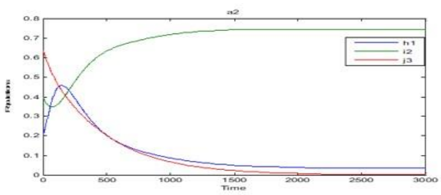


Figure2.a1: The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point A_4= (0.8182, 0.7275, 0.7271) in the interior of R^3_+. For the data in (5.1) with $r_1 = 0.15$. **Figure2.a2:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point A_2= (0.3333, 0.7407, 0) in the interior of R^3_+. For the data in (5.1) with $r_1 = 0.9$.

Additionally, changing the specialist predator's mortality death rate between 0.0001 and 0.03 while maintaining the other parameters according to the data in (5.1) results in the specialized predator going extinct, and **Figure3.e1** illustrates how system (2) approaches asymptotically to the positive equilibrium pointA_4 with a typical value of r_5=0.01, however, for a typical value of r_5=0.1, as shown in **Figure3.e2**, at 0.03< r_5 <0.15 approaches asymptotically to the axial equilibrium point A_1, additionally, for a typical value ofr_5=0.5, **Figure3.e3** shows that for 0.15≤ r_5 <0.6 approaches asymptotically to the equilibrium point A_3.

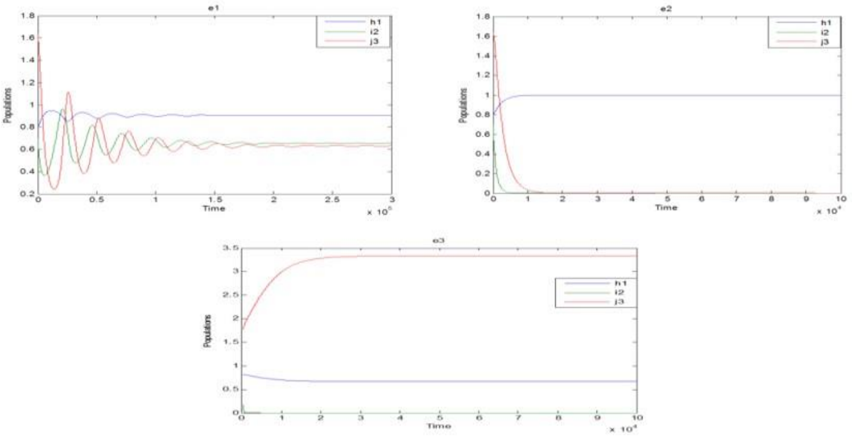


Figure3.e1: The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point A_4= (0.99043, 0.6535, 0.6304) in the interior of R^3_+. For the data in (5.1) with $r_5 = 0.01$. **Figure3.e2:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point A_1= (1, 0, 0) in the interior of R^3_+. For the data in (5.1) with $r_5 = 0.1$. **Figure3.e3** time series of the solution of system (2) approaches asymptotically to point A_3= (0.6667, 0, 3.3333).

The system (2) still approaches asymptotically to the equilibrium point A_2 despite changing the parameter r_6 which represents the conversion rate from the prey to the generalist predator in the range $0.0001 \leq r_6 < 0.015$ this causes extinction in the prey, however in additional for 0.015≤ r_6<0.1 approaches asymptotically to the positive equilibrium point A_4, as shown in **Figure4**. $\boldsymbol{g2}$, for typical value r_6=0.08.

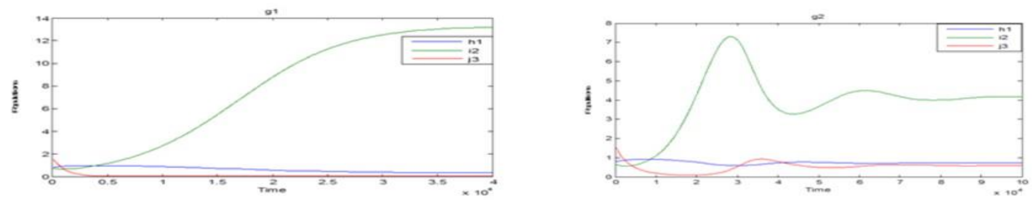


Figure4.g1: The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point A_2= (0.3392, 13.2187, 0) in the interior of R^3_+. For the data in (5.1) with $r_6 = 0.013$. **Figure4.g2:** The time series of the solution of system (2) approaches asymptotically to the positive equilibrium point A_4= (0.7311, 4.1682, 0.6044) in the interior of R^3_+. For the data in (5.1) with $r_6 = 0.08$

6. Conclusion:

This study used an ecological mathematical model that includes a predator-prey model and a food web, as well as a population of prey and a population of specialized predators as refuges. Additionally, this model includes linear types of functional reactions for the predation of creatures that were not protected. Starting with the hypothetical set of data provided by eq. (5.1), system (2) has been numerically solved for several sets of initial points and various sets of parameters, and the following observations are obtained:

1. When approaching globally stable locations via Int. R_+^3techniques system (2) only has two types of attractors. The system (2) approaches asymptotically to the globally stable positive point$A_4 = (\ 0.4544, 0.6567\ ,\ \ 2.1722\ )$for the set parameter value specified in (5.1).
2. The positive equilibrium point A_4 being approached by the solution of system (2) as the assault rate on a victim from the specialized predator r_1 increases to 0.15 while maintaining the other parameters as in eq. (5.1), but if r_1= 0.59 it can be seen that the solution to system (2) approaches the equilibrium point A_2 asymptotically; r_1= 0.595 is a bifurcation point.
3. The trajectory changed from the axial point $\mathrm{A}_1$to the equilibrium point A_2 in the range $0.0001 \leq \mathrm{r}_3 < 0.006$, and from the equilibrium point A_2 to the equilibrium point A_4 in the range $0.006 \leq \mathrm{r}_3 < 0.0125$ consequently the bifurcation points for the parameter r_3 are at r_3= 0.006 and r_3 = 0.01 respectively.
4. As the attack rate of generalist predator on specialist predator r_4 keeping the rest of parameters as in eq.(5.1), the solution of system (2) approaches to the positive point A_4, if r_4 =1.5, it is shown that the solution of system (2) approaches asymptotically to the equilibrium point A_3, indicating that r_4=1.5 is a bifurcation point.
5. The natural death rate of specialist predator r_5 the solution of system (2) advances from the positive equilibrium point A_4 to the axial equilibrium point $\mathrm{A}_1$with $0.03 < \mathrm{r}_5 < 0.15$ keeping the other parameters as in eq.(5.1), and from the axial equilibrim point A_1 to the equilibrium point A_3 in the ring 0.15≤ r_5 <0.6, the parameter r_5 = 0.045 is bifurcation point.
6. As the conversion rate of prey to the generalist predator r_6 decreasing, and keeping the rest parameters values as in eq. (5.1) the solution of system (2) approaches the equilibrium point A_4, while for $0.0001 \leq \mathrm{r}_6 < 0.015$, the generalist predator population revives and then the trajectory changed from the point A_2 to the positive equilibrium point A_4, while for $0.015 \leq \mathrm{r}_6 < 0.1$ thus, the parameter $r_7 =\ 0.015$ is a bifurcation point.
7. In light of the conversion rate of specialist predator to the generalist predator r_7 decreasing to 0.0002 and keeping the rest parameters values as in eq. (5.1) the solution of system (2) approaches the equilibrium point A_2, while for $0.00001 \leq r_7 < 0.005$ the generalist predator population revives and when 0.005≤ r_7 <0.2, the trajectory changed from the point A_2 to the positive equilibrium point A_4, the parameter r_7 =0.005 is a bifurcation point.

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Conflict of Interest

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