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Perturbation of Weyl's Theorems for Unbounded 2×2 Upper Triangular Operator Matrices

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Abstract

Let $M = [A\ B\ 0\ D\ ]$ be an upper triangular operator matrix which is unbounded and defined on $H \oplus H$, where H is infinite dimensional Hilbert space. This paper is concerned with new spectral properties which defined to other bounded operators. Some sufficient and necessary conditions are given in which these properties are equivalent. We further investigate the relations among Weyl's type theorems and Brodwe's theorems for this type of operator under some conditions. As an application the paper define the plate pending problem equation with henge end, fixed end and free end, after transform it to Hamitonian matrix then calculate the spectrum sets for this matrix which leads to if A has eigenvalues of finite multiplicity, so is M. Inaddition if A, A^* has finite ascent this implies that the Hamiltonian operator M has finite ascent.

Keywords: Browder's Spectrum, Spectral Properties, Upper Triangular Operator Matrices,Weyl's Spectrum, Weyl's Theorems.

1. Introduction

The conception of unbounded operator delivers a non-figurative background for allocating with differential operators, unbounded perceptible in quantum mechanics, and other circumstances. The Weyl's Theorem for bounded hermitian operators was established by Weyl (1). Weyl's Theorem has since been expanded to encompass the class of bounded normal, hyponormal, and Toeplitz operators (2) as well as a number of other non-normal categories of bounded operators. The familiar Weyl's theorem is generalized in such a way. Furthermore, he established this modified version of the traditional Weyl's theorem for limited hyponormal operators in (3). The works in this direction recently been expanded to include the classes of unbounded posinormal operators and unbounded hyponormal operators (4).

For unbounded operators on different spaces such as the space of Banach with non-empty resolvent, the authors introduced the B-Fredholm theory in (5, 15). Weyl's Theorem for the category of paranormal operators on Banach spaces was established by Ramanujan(6), and it was further developed by Aiena and Guillen to include the investigation of Weyl's. Theorem for the perturbation of paranormal operators by algebraic operators and for unbounded compact operators defined on a Banach space are investigated by the authors (7,16,19) , including those by Browder and Weyl. The theory is demonstrated in the final section using examples involving isometrics, analytically Toeplitz operators, semi-shift operators, and weighted right shifts. Both generalized Weyl's theorem and generalized Browder's theorem are susceptible to failure for matrices with two by two operators. In this study, we also investigate the survival of generalized Weyl's, generalized Browder's, generalized a- Weyl's, and generalized a-theorems Browder's for 2D- upper triangular operator matrices on the Banach space (8, 20, 21).

Operator matrices are important to determine the solvability and stability of the underlying systems and are found in various areas of pure and applied mathematics.

If M is a bounded linear operator on a Hilbert space $H = H_1 \oplus H_2$, one always has the following block representation.

$M = (M_{11}\ M_{12}\ M_{21}\ M_{22}\ )$

In addition, if $M_{21} = 0$, then M is an upper triangular operator matrix. There are many publications looking at the spectral properties of upper triangular operator matrices. It's worth mentioning that some authors estimate the set $(\sigma_*(A) \cup \sigma_*(D)) \setminus \sigma_*(M)$ and obtain some sufficient conditions of

$\sigma_*(M) = \sigma_*(A) \cup \sigma_*(D)$

where M the upper triangular operator matrix is acting in a Banach space, and $\sigma_* \in \{\sigma_w, \sigma_e, \sigma_b, \sigma_{ub}, \sigma_{SF_+^-}\}$.

Block operator matrices play a significant role in coupled systems of partial differential equations of mixed order. The study of upper triangular operator matrices and related topics is one of the hottest areas in operator theory. A number of mathematicians have studied upper triangular operator matrices in the past.

2. Preliminaries

In this section, we recall the following concepts, which are used later.

All through this work, H denotes to infinite dimensional complex Hilbert space, $C(H)$ is the set of all closed linear operators defined on H. For an operator $M \in C(H)$, we define $N(M)$ as the kernel of M, while $D(M)$ represents the domain, and $R(M)$ denotes the range of M. The upper semi Fredholm operator is define if $R(M)$ is closed and $n(M) = dim\, dim\, N(M)$ is finite while we say that M is lower semi Fredholm operator if $d(M) = codim\, R(M)$ is finite. A Fredholm operator is upper and lower semi Fredholm operator.

$SF_+(H) = \{M \in C(H): M\ is\ upper\ semi\ Fredholm\}$,

$SF_-(H) = \{M \in C(H): M\ is\ lower\ semi\ Fredholm\}$.

The $index$ of M is defined as $ind\ (M) = n(M) - d(M)$.

An operator $M \in C(H)$ which is Fredholm operator of index 0 is defined as Weyl operator, while $\sigma_w(M) = \{\eta \in C : M - \eta I$ is not weyl$\}$ is used to define the Weyl spectrum of M. In addition we can assign the following notations:

$SF_+^-(H) = \{M \in C(H):\ M \in SF_+(H), ind(M) \leq 0\ \}$

$SF_-^+(H) = \{M \in C(H): M \in SF_-(H), ind\ (M) \geq 0\}$

In (4), Berkani generalized the concept of Fredholm operators to B-Fredholm operators as follows

$\Omega(M) = \{i \in N: \forall j \in N, j \geq i \Rightarrow R(M^i) \cap N(M) \subseteq R(M^j) \cap N(M)\}$

The degree of stable iteration of M is denoted by $dis(M)$ and defined by $dis(M) = inf\ \Omega(M)$ and $dis(M) = \infty$ when $\Omega(M) = \emptyset$.

Furthermore, for $M \in C(H)$ the $B - Fredholm$ operator is upper and lower semi $B - Fredholm$ operator, where M is $upper$ (resp., $lower$) semi $B - Fredholm$ operator if

$\exists\, d \in \Omega(M)$: $dim\{N(M) \cap R(M^d)\}$ is finite and $R(M^d)$ closed and (resp., $codim\ \{R(M) + N(M^n)\}$ is finite), and the index of M is
$ind(M) = dim\{N(M) \cap R(M^d)\} - codim\{R(M) + N(M^d)\}$.
We call $M \in C(H)$ as $B - Weyl$ if it's B- $Fredholm$ operator with $index$ 0 and $\sigma_{BW}(M)$ is used to symbolize the B-$Weyl$ spectrum of A and defined by $\sigma_{BW}(M) = \{\eta \in C : M - \eta\, I$ is not B-weyl $\}$.
Moreover, the ascent $asc(M)$ and descent $dsc(M)$ for $M \in C(H)$ are defined as:
$asc(M) = inf\ \left\{d : N\left(M^d\right) = N\left(M^{d+1}\right)\right\}$ $dsc(M) = inf\ \left\{d : R\left(M^d\right) = R\left(M^{d+1}\right)\right\}$
An operator $M \in C(H)$ is called Browder if it's both $upper$ and $lower$ semi Browder, where $M \in C(H)$ is $upper$ semi- Browder if $asc(M) < \infty$ with M is $upper$ semi- Fredholm and it is $lower$ semi-Browder if $dsc(M) < \infty$ with M is $lower$ semi – Fredholm.
Now, we can define the following spectrum for an operator M as:
$\sigma_{SF_+}(M) = \{\eta \in C : M - \eta I$ not upper semi Fredholm$\}$,
$\sigma_{SF_-}(M) = \{\eta \in C : M - \eta I$ not lower semi Fredholm$\}$,
$\sigma_e(M) = \{\eta \in C : M - \eta I$ not Fredholm$\}$,
$\sigma_{SF_+^-}(M) = \{\eta \in C : M - \eta I \notin\ SF_+^-(H)\}$,
$\sigma_{ub}(M) = \{\eta \in C : M - \eta I$ not upper semi-Browder $\}$,
$\sigma_{Ib}(M) = \{\eta \in C : M - \eta I$ not lower Semi-Browder$\}$ and
$\sigma_b(M) = \{\eta \in C : M - \eta I$ not Browder $\}$, respectively.
Evidently
$\sigma_e(M) \subset \sigma_w(M) \subset \sigma_b(M) = \sigma_e(M) \cup acc\, \sigma(M)$,
where $acc\, \sigma(M)$ denotes the set of accumulation points of the spectrum $\sigma(M)$ of M. approximate point spectrum of M.

Weyl claims that the Wey'l spectrum of a Hermition operator contains exactly all of the points in the spectrum of A with the exception of those points, which are isolate eigenvalues of restricted pluralism, in (9), where he proved the Weyl's theorems for bounded hermition operators. Weyl's theorem has now been extended to other types of bounded operators (10).
Recall that one says that M obeys Weyl's theorem if
$\sigma(M) \setminus \sigma_w(M) = E_0(M)$,
where $E_0(M)$ is the set of isolated points of $\sigma(M)$ which are eigenvalues of finite multiplicity, and that one says that M obeys Browder's theorem if $\sigma_w(M) = \sigma_b(M)$.
We say that M obeys a-Weyl's theorem if
$\sigma_a(M) \setminus \sigma_{SF_+^-}(M) = E_0^a(M)$,
where $E_0^a(M)$is the set of isolated points of $\sigma_a(M)$ which are eigenvalues of finite multiplicity, and that M obeys a-Browder's theorem if $\sigma_{SF_+^-}(M) = \sigma_{ub}(M)$.

Let H be an infinite dimensional Hilbert space. Hamiltonian operator can be defined as densely closed operator matrix
$T = (A\ B\ C\ -A^*) : (D(A) \cap D(C)) \oplus \left(D(B) \cap D(A^*)\right) \to H \oplus H$,
where A is a densely defined closed operator, B and C are self adjoint operators.(see (4)).
For the proof of the main results in the next section, we need the following auxiliary lemmas.

2.1. Lemma

1. M is upper semi- B-Fredholm and $n(M) < \infty$ if and only if M is left Fredholm.
2. M is lower semi- B-Fredholm and $d(M) < \infty$ if and only if M is right Fredholm.

Proof. The proof of this lemma is similar to the proof in bounded case.

2.2. Lemma (see (13)) Let $M = (A\ B\ 0\ D\ ): D(A) \oplus D(D) \subset H \oplus H \longrightarrow H \oplus H$ be a closed operator matrix such that A, D are closed operators with dense domains and B is a closable operator. Then

(1) If A and D are right Fredholm, then M is right Fredholm.

(2) If A and D are left Fredholm, then M is left Fredholm.

(3) If A (resp., D) and M are Fredholm, then D (resp.,A) is Fredholm.

(4) If A (resp., D) and M are Weyl, then D (resp., A) is Weyl.

2.3. Lemma (see (14)) Suppose that either $n(M)$ or $d(M)$ is finite, and that $asc(M)$ is finite then $n(M) \leq d(M)$.

2.4. Lemma (see (14)) Suppose that either $n(M)$ or $d(M)$ is finite, and that $dsc(M) = q$ is finite then $d(M) \leq n(M) + dim\, H/D(M^q)$. In particular, $d(M) \leq n(M)$ if $D(M) = H$.

2.5. Lemma (see (14))

a) If $asc(M)$ and $dsc(M)$ are finite, then $asc(M) \leq dsc(M)$. If also $D(M) = H$, then $asc(M) = dsc(M)$.

b) Suppose that $asc(M)$ is finite and that $n(M) = d(M) < \infty$. Then $dsc(M) = asc(M)$.

c) Suppose that $D(M) = H$, that $dsc(M)$ is finite, and that $n(M) = d(M) < \infty$. Then $asc(M) = dsc(M)$.

2.6. Lemma (see (11))

If M is linear operator on a vector space X then the following hold:

1. If $acs\ (M) < \infty$ then $n(M) \leq d(M)$.
2. If $dsc(M) < \infty$ then $d(M) \leq n(M)$.

3. Results

In this part of paper, we define some spectral properties for unbounded upper triangular operator matrices, these properties are defined for operators in bounded case (see (4), (5), (12) and (13)). Furthermore, we effort some necessary and sufficient conditions to obtain the equivalence among them and among the Weyl type theorems such as Weyl's, a-Weyl's, Browder's and a-Browder's. Before we proceed, we need to define the following spectrums:

$\sigma_{p_\infty}(.) = \{\eta \in \sigma_p(.): n(.-\eta I) = \infty\}$;

$\sigma_{p_\infty}(.^*)^* = \left\{\eta^- \in C: \eta \in \sigma_{p_\infty}(.^*)\right\}$;

$\sigma_{p_+}(.) = \{\eta \in \sigma_p(.): n(.-\eta I) > d(.-\eta I)\}$ for an operator (.) and

$\sigma_R(M) = \{\eta \in C: R(M - \eta I)$ is not closed $\}$.

3.1. Theorem If $\sigma_{P_\infty}(D) \cap \sigma_{P_\infty}(A^*)^* = \sigma_{P_\infty}(A) \cap \sigma_{P_\infty}(D^*)^* = \emptyset$ and $\sigma_{P_+}(A^*)^* \backslash \sigma_{SF_-}(A^*)^* = \sigma_{P_+}(D^*)^* \backslash \sigma_{SF_-}(D^*)^* = \emptyset$ with $\sigma(M) = \sigma_a(M)$ then M obeys Weyl's theorem if and only if M obeys a-Weyl's theorem.

Proof. Since $\sigma(M) = \sigma_a(M)$ this would imply that $E_0(M) = E_0^a(M)$.To prove the equivalence, it sufficient to show that $\sigma_w(M) = \sigma_{SF_-^+}(M)$. Let $\eta \notin \sigma_{SF_-^+}(M)$, to prove $\eta \notin \sigma_w(M)$ i.e., $M - \eta I$ is weyl operator. Since $\sigma_{P_\infty}(D) \cap \sigma_{P_\infty}(A^*)^* = \sigma_{P_\infty}(A) \cap \sigma_{p_\infty}(D^*)^* = \emptyset$ these conditions imply that A and D are Fredholm operators, by Lemma 2.2 (1) and (2), we get M is Fredholm operator. Clearly $ind(A - \eta I) \geqslant 0$, $ind(D - \eta I) \geqslant 0$ Since $\sigma_{p_+}(A^*)^* \backslash \sigma_{SF_-}(A^*)^* = \emptyset$ and $\sigma_{P_+}(D^*)^* \backslash \sigma_{SF_-}(D^*)^* = \emptyset$, thus $ind(M - \eta I) = ind(A - \eta I) + ind(D - \eta I) \geqslant 0$, but $ind(M - \eta I) \leqslant 0$ then $ind(M - \eta I) = 0$ which leads to $\eta \notin \sigma_w(M - \eta I)$. The proof is completed.

3.2. Theorem M obeys a-Weyl's theorem if and only if M obeys a-Browder's theorem, and $E_0^a(M) \cap \sigma_R(M) \ = \emptyset, E_0^a(M) \cap \sigma_{asc}(M) \ = \emptyset$

Proof. The proof is similar to the proof in bounded case (see (16)).

3.3. Theorem M obeys Weyl's theorem if and only if M obeys Browder's theorem, and $E_0(M) \cap \sigma_{p_\infty}(M^*)^* = \emptyset$

Proof. The proof is similar to the proof in bounded case (see (16)).

3.4. Definition (6) For $M \in C(H)$, we say M obeys

1. (w) if $\sigma_a(M) \setminus \sigma_{SF_+^-}(M) = E_o(M)$
2. Property (gw) if $\sigma_a(M) \setminus \sigma_{SBF_+^-}(M) = E(M)$
3. Property (b) if $\sigma_a(M) \setminus \sigma_{SF_+^-}(M) = \sigma(M) \setminus \sigma_b(M)$
4. Property (gb) if $\sigma_a(M) \setminus \sigma_{SBF_+^-}(M) = \pi(M)$

3.5. Definition. For $M \in C(H)$, we say M obeys

1. (am) if $\sigma_a(M) \setminus \sigma_b(M) = E_0^a(M)$
2. Property (sz) if $\sigma(M) \setminus \sigma_{SF_+^-}(M) = E(M)$
3. Property (asz) if $\sigma(M) \setminus \sigma_{SF_+^-}(M) = \pi(M)$

3.6. Theorem

Let M obeys property (gw) with $\sigma(M) = \sigma_a(M)$ and $\sigma_{P_\infty}(A) \cup \sigma_{P_\infty}(D) = \emptyset$ then M obeys (sz) property.

Proof. Let $\eta \in \sigma(M) \setminus \sigma_{SF_+^-}(M)$, then $\eta \in \sigma(M)$ and $ind(M - \eta I) \leqslant 0$, by lemma 2.1.(1), we have $M - \eta I$ is upper Semi B-Fredholm with $n(M - \eta I) < \infty$, since by assumption $\sigma(M) = \sigma_a(M)$, then $\eta \in \sigma_a(M \setminus \sigma_{SBF_+^-}(M)$, but M obeys (gw) property, thus $\eta \in E(M)$.

For the reverse inclusion, let $\eta \in E(M)$, then $\eta \in \sigma(M)$, by assumption: $\sigma_{\rho_\infty}(A) \cup \sigma_{p_\infty}(D) = \emptyset$ and $(D - \eta I)$ are right Fredholm with $ind(A - \eta I)$+ $ind(D - \eta I) \leqslant 0$, then by Lemma $2 \cdot 1(1)$, we have $(M - \eta I)$ is right Fredholm with $ind(M - \eta I) = ind(A - \eta I) + ind(D - \eta I) \leqslant 0$, then $\eta \notin \sigma_{SF_+^-}(M)$ thus, $\eta \in \sigma(M) \setminus \sigma_{SF_+^-}(M)$. Then M obeys Property (sz).

3.7. Theorem

Let M obeys property (gb) with $\sigma(M) = \sigma_a(M)$ and $\sigma_{p_\infty}(A) \cup \sigma_{p_\infty}(B) = \emptyset$, then M obeys property (asz).

Proof. The proof of this theorem is similar to the Proof of theorem 3.6.

3.8. Theorem If $\sigma(M) = \sigma_a(M)$ and $D(M) = H$, then M obeys (w) property with $E_0(M) \cap \sigma_{asc}(M) = \emptyset$ and $E_0(M) \cap E_{dsc}(M) = \emptyset$ if and only if M obeys property (b) with $E_0(M) \cap \sigma_R(M) = \emptyset$

Proof. To prove (M) obeys property (b) with $E_0(M) \cap E_R(M) = \emptyset$ we need to proof $\sigma_a(M) \setminus \sigma_{SF_+^-}(M) = \sigma(M) \setminus \sigma_b(M)$.

Let $\eta \in \sigma_a(M) \setminus \sigma_{SF_+^-}(M)$, then $\eta \in \sigma_a(M) = \sigma(M)$ and $(M - \eta I)$ is upper semi Fredholm with $ind(M - \eta I) \leqslant 0$, to prove $\eta \notin \sigma_b(M)$, i.e., $(M - \eta I)$ is Browder operator. Since by assumption $E_0(M) \cap \sigma_{asc}(M) = \emptyset$ and $E_0(M) \cap \sigma_{asc}(M) = \emptyset$ then $asc(M - \eta I) < \infty$ and $dsc(M - \eta I) < \infty$, by Lemma 2.4. we get $d(M - \eta I) \leqslant n(M - \eta I) < \infty$, thus $\eta \notin \sigma_b(M)$ then $\eta \in \sigma(M) \setminus \sigma_b(M)$, thus $\sigma(M) \setminus \sigma_{SF_+^-}(M) \subseteq \sigma(M) \setminus \sigma_b(M)$, but M obeys property (w) , i.e., $\eta \in E_0(M)$ then $E_0(M) \cap \sigma_R(M) = \emptyset$.

Now, Let $\eta \in \sigma(M) \setminus \sigma_b(M)$, then $\eta \in \sigma(M) = \sigma_a(M)$ and $M - \eta I$ is Browder i.e., $(M - \eta I)$ is Fredholm operator with finite ascent and finite descent, then by definition of Fredholm operator: $M - \eta I$ is upper and lower semi Fredholm operator. Since $asc(M - \eta I)$ and $d(M - \eta I) < \infty$, then by lemma 2.3 and lemma 2.4 we get $n(M - \eta I) = d(M - \eta I)$ thus $M - \eta I$ is belong to $\sigma_{SF_+^-}(M)$, thus $\eta \notin \sigma_{SF_+^-}(M)$, from all of that we get $\eta \in \sigma_a(M) \setminus \sigma_{SF_+^-}(M)$ then (M) obeys property (b).

To proof the reverse direction, i.e., to proof M obeys property (w) we need to show that $\sigma_a(M)\backslash\sigma_{SF_+^-}(M) = E_0(M)$ let $\eta \in \sigma_a(M)\backslash\sigma_{SF_+^-}(M)$, since M obeys (w) property then $\eta \in \sigma_a(M) \setminus \sigma_{SF_+^-}(M) = \pi_0(M) = \{ \eta \in \pi(M): n(M-\eta I) < \infty\}$, then $\eta \in$ iso $\sigma(M)$. Since $M - \eta I$ is upper semi Fredholm operator with $ind(M-\eta I) \leq 0$ and $asc(M-\eta I) < \infty$ and $dsc(M-\eta I) < \infty$ with $D(M) = H$. then $n(M-\eta I) = d(M-\eta I) < \infty$, also $n(M-\eta I) = d(M-\eta I)$ must be larger than zero, if $n(M-\eta I) = d(M-\eta I) = 0$ then $M-\eta I$ is one-one mapping of $D(M)$ onto all of H. The inverse $(M-\eta I)^{-1}$ is then closed and hence bounded, thus $\eta \notin \sigma(M)$, which is Contradiction. Hence $n(M-\eta I) > 0$, then $\eta \in E_0(M)$.

Now, assume $\eta \in E_0(M)$, To prove $\eta \in \sigma_a(M)\backslash\sigma_{SF_+^-}(M)$ Since $\eta \in E_0(M)$ then $\eta \in$ iso$\sigma(M)$ which imply $\eta \in \sigma(M) = \sigma_a(M)$. To prove $M-\eta I$ is belong to $\sigma_{SF_+^-}(M)$. Since M obeys (b) property with $E_0(M) \cap \sigma_R(M) = \emptyset$ and $\eta \in E_0(M)$, then $0 < n(M-\eta I) < \infty$ and $R(M - \eta I)$ is closed with $M-\eta I$ is Browder operator i.e., $\eta \notin \sigma_{SF_+^-}(M)$ and $E(M) \cap \sigma_{asc}(M) = \emptyset$ and $E_0(M) \cap \sigma_{dsc}(M) = \phi$. The proof is completed.

3.9. Theorem If M is upper triangular unbounded operator matrix with $E_0^a(M) \cap \sigma_{asc}(M) = \emptyset$ and $\sigma_{p_\infty}(A) \cup \sigma_{p_\infty}(D) = \emptyset$, $\sigma_{p_+}(A^*)^*\backslash\sigma_{SF_-}(A^*)^* = \sigma_{p_+}(D^*)^*\backslash\sigma_{SF_-}(D^*)^* = \emptyset$, then M obeys property (am).

Proof. Let $\eta \in \sigma_a(M) \setminus \sigma_b(M)$, to prove $\eta \in E_0^a(M)$ i.e., to prove $\eta \in E_0^a(M) = \{\eta \in iso\sigma_a(M): 0 < n(M-\eta I) < \infty\}$.

Since $\sigma_a(M) \setminus \sigma_b(M) = \sigma_a(M) \setminus (\sigma_b(M) \cup acc\,\sigma(M))$
$= \sigma_a(\mu)\backslash\sigma_w(\mu) \subset E_0^a(\mu)$, since $\eta \notin \sigma_b(M)$ then we get $0 < n(M-\eta I) = d(M-\eta I) < \infty$ thus $\eta \in E_0^a(M)$.

Let $\eta \in E_0^a(M)$, to prove $\eta \in \sigma_a(M)\backslash\sigma_b(M)$ Since $\eta \in E_0^a(M)$, then $\eta \in \sigma_a(M)$, it remains to prove $\eta \in \sigma_b(M)$, since $E_0^q(M) \cap \sigma_{asc}(M) = \emptyset$ then $asc(M-\eta I) < \infty$. From lemma 4 (see (9)) and $\sigma_{p_\infty}(A) \cup \sigma_{p_\infty}(D) = \emptyset$ we see that $ind(A-\eta I) + ind(D-\eta I) \leqslant 0$, and since $\sigma_{p_+}(A^*)^*\backslash\sigma_{SF_-}(A^*)^* = \sigma_{p_+}(D^*)^*\backslash\sigma_{SF_-}(D^*)^* = \emptyset$, we have $ind(A-\eta I) \geq 0$ and $ind(D - \eta I) \geq 0$ Hence $ind(M-\eta I) = 0$then by lemma $2 \cdot 5 \cdot (b)$, we get $asc(M) = dsc(M) < \infty$, thus $\eta \notin \sigma_b(M)$. Then M obeys (am) property.

3.10. Theorem If M obeys Weyl's theorem and $D(M) = H$ with $E_0(M) \cap \sigma_{asc}(M) = \emptyset$ and $E_0(M) \cap \sigma_{dsc}(M) = \emptyset$ then M obeys (am) property.

Proof. By using the same steps in theorem 3.9 one can show $\sigma_a(M)\backslash\sigma_b(M) \subset E_0^a$. It remains to proof $E_0^a(M) \subset \sigma_a(M)\backslash\sigma_b(M)$.Let $\eta \in E_0^a(M)$ then $\eta \in \sigma_a(M)$ also $\eta \in E_0(M)$ but M obeys Weyl's theorem then $\sigma(M)\backslash\sigma_w(M) = E_0(M)$ i.e., $M-\eta I$ is Weyl theorem. By assumptions we have $E_0^a(M) \cap \sigma_{asc}(M) = \emptyset$ and $E_0^a(M) \cap \sigma_{dsc}(M) = \emptyset$ these would imply that $asc(M-\eta I)$ and $dsc(M-\eta I)$ are finite then $(M-\eta I)$ is Browder operator. Then the proof is completed.

3.11. Example In this example, we tried to apply the results in (16) and (17) and some results in this paper to the Hamiltonian operator matrix by applying the plate bending problem.

Assume the plate bending problem

$$\Delta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w = 0$$

With x *and* y from 0 to 1.

For the y-direction: at $y = 0$ we have $w = 0$ (Hinge end)

at $y = 0.7$ we have $w = 0$ *and* $\frac{\partial w}{\partial y}$=0 (Fixed end)

at $y = 1$ we have w =given function and $\frac{\partial^2 w}{\partial y^2} = 0$ (Free end)

For the x-direction: $w, \frac{\partial w}{\partial x}$ are given function at $x = 0$ to $x = 1$.
the problem can be described by the following Hamiltonian system (18)
$$\frac{\partial}{\partial x}(w\, \vartheta\, \Phi\, \mu\,) = \left(0\,1\,0\,0\; -\frac{\partial^2}{\partial y^2}\;0\,0\; -\frac{1}{D}\;0\,0\,0\;\frac{\partial^2}{\partial y^2}\;0\,0\; -1\,0\right)(w\, \vartheta\, \Phi\, \mu\,),$$
and the corresponding Hamiltonian operator matrix is given by
$$H = \left(0\,1\,0\,0\; -\frac{d^2}{dy^2}\;0\,0\; -\frac{1}{D}\;0\,0\,0\;\frac{d^2}{dy^2}\;0\,0\; -1\,0\right) =: (A\; B\; 0\; -A^*\,)$$
with domain is $D(A) \oplus D(A^*) \subset L_2(0,1) \oplus L_2(0,1)\,, A = AC[0,1]$, and
$$A = \left(0\,1\; -\frac{d^2}{dy^2}\;0\right), B = \left(0\,0\,0\; -\frac{1}{D}\right)\; D(A) = \{(w\,\vartheta\,) \in H : \omega(0) = 0, \omega' \in A, \omega'' \in H\}$$
With some simple calculation, we have $\sigma_{P_+}(A^*)^* \backslash \sigma_{SF_-}(A^*)^* = \emptyset, \sigma_{p_\infty}(A^*) = \emptyset, -\sigma_{p_+}(A^*) \cap \sigma_{p_+}(A^*)^* = \emptyset$, $-\sigma_{p_+}(A)^* \cap \sigma_{p_+}(A) = \emptyset$ and $\sigma_{asc}\,(A^*) = \emptyset$. Then from Propositions 4.1, 4.2 and 4.3in (5), and from Propositions 10 and 11 in (10), we have
$\sigma_\star(H) = -\sigma_\star(A^*) \cup \sigma_\star(A)$,
where $\sigma_\star \in \{\sigma_e, \sigma_w, \sigma_b, \sigma_{SF_+^-}, \sigma_{lb}\,\}$.
now, by theorem 3.1 we found that $\sigma_w(H) = \sigma_{SF_-^+}(H)$ if $\sigma_a(H) = \sigma(H)$.

4. Conclusion

In this paper, other spectral properties are introduced and studied for the upper triangular operator matrices. Furthermore, Weyl's type theorems and Browder's theorems are also proved under certain conditions. Finally, as an application the paper study the plate bending problem and calculate the spectrum sets denoted by $\sigma_\star$ where $\sigma_\star \in \{\sigma_e, \sigma_w, \sigma_b, \sigma_{SF_+^-}, \sigma_{lb}\,\}$.

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Conflict of Interest

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