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An Approximate Solution for a Second Order Elliptic Inverse Coefficient Problem with Nonlocal Integral

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Abstract
This article studies the nonlocal inverse boundary value problem for a rectangular domain, a second-order, elliptic equation and a two-dimensional equation. The main objective of the article is to find the unidentified coefficient and provide a solution to the problem. The two-dimensional second-order, convection equation is solved directly using the finite difference method (FDM). However, the inverse problem was successfully solved the MATLAB subroutine lsqnonlin from the optimization toolbox after reformulating it as a nonlinear regularized least-square optimization problem with a simple bound on the unknown quantity. Considering that the problem under study is often ill-posed and that even a small error in the input data can have a large impact on the outcome, Tikhonov's regularization technique is used to obtain stable and regularized results.
Keywords: inverse problem, two-dimensional parabolic equation, overdetermination condition, finite difference schemes, Tikhonov technique.

1. Introduction
Partial differential equations play an important role in many areas of everyday life. In the fields of engineering, design, construction and medicine, for example, high-speed computers have greatly influenced the development of numerical methods for solving partial differential equations, advancing and modernizing them compared to analytical methods. Researchers continue to observe steady progress in this field. According to mathematicians, it is the first inverse (ill-posed) problem in mathematics, (Hadamard, 1902). However, this problem has changed. In the twentieth century, practical applications are constantly being researched, which is of great interest for solving elliptic problems. It has been studied by many researchers. In [1], the approximate solution of the inverse elliptic problem with the Dirichlet condition was obtained using the finite difference method. In [2], we have used the Schwarz-alternating method. The numerical solution of the inverse problem for the multidimensional elliptic equation with overdetermination is obtained using a finite difference method in [3], a finite element method in [4], an integral equation method in space [5], the Tikhonov regularization method in [6–24], and the Lavrentiev regularization method [25].
The existence and uniqueness of a solution to this problem was proved in [26], and this paper aims to

solve this type of problem numerically.
This article describes a way to solve this type of inverse coefficient problem numerically that is stable. It uses the finite difference method (FDM), an optimization method, and the Tikhonov regularization. The direct numerical solution of the problem was obtained using FDM. Subsequently, we used Tikhonov's approach to stabilize this problem.
Section 2 presents the mathematical formulation of the inverse problem. In Section 2, we present the mathematical formulation of the inverse problem. Section 3 describes the direct finite difference scheme to obtain the numerical solution of a direct problem, along with numerical test examples. While Section 4 describes the numerical approach to solve an inverse problem using the Tikhonov technique, Section 5 provides a regularized solution. Section 5 presents and discusses the quantitative results. The conclusion of the paper can be found in Section 6.

2. Mathematical formulation

Consider the following inverse problem of retrieving an unknown time-dependent potential coefficient $a(y)$ in the two-dimensional second-order elliptic equation:

$$u_{xx}(x,y) + u_{yy}(x,y) = a(y)u(x,y) + f(x,y), \quad (x,y) \in Q \qquad (1)$$

under the boundary conditions

$$u(x,0) = \varphi(x), \qquad u_y(x,Y) = \psi(x), \qquad x \in [0,1], \qquad (2)$$

$$u_x(0,y) = 0, \qquad y \in [0,Y], \qquad (3)$$

with the nonlocal integral condition

$$\int_0^1 u(x,y)dx = 0, \qquad y \in [0,Y], \qquad (4)$$

and additional measurement condition

$$u(0,y) = H(y), \qquad y \in [0,Y], \qquad (5)$$

where the functions $f(x,y), \varphi(x), \psi(x), \omega(x)$ and $H(y)$ are given and $Q = \{(x,y): 0 < x < 1, 0 < y < Y < \infty\}$. The numerical solution of the inverse problem second-order elliptic equation (1) – (5) is written as $\{a(y), u(x,y)\}$ such that $a(y) \in C[0,Y]$ and $u(x,y) \in C^2(Q)$.
The existence and uniqueness of theorems have been established by Y. Mehraliyev in [26] and stated as follows:

2.1 Existence of the inverse problem classical solution.

Assume the following conditions:
E1) $\varphi(x) \in C^2[0,1], \varphi'''(x) \in L_2(0,1), \quad \varphi'(0) = \varphi'(1) = 0;$
E2) $\psi(x) \in C^1[0,1], \psi''(x) \in L_2(0,1), \ \psi'(0) = \psi'(1) = 0;$
E3) $f(x,y), \ f_x(0,y) \in C(Q), f_{xx}(x,y) \in L_2(Q)$ and
$f_x(0,y) = f_x(1,y) = 0, y \in [0,Y];$
E4) $H(y) \in C^2[0,Y], \ H(y) \neq 0, \ y \in [0,Y];$
Theorem 1. Let the assumptions (E1)–(E4) and the condition

$$(\mathcal{A}_1(Y)+\mathcal{A}_2(Y)\ +\ 2)^2\left(\mathcal{B}_1(Y)+\mathcal{B}_2(Y)\right) < \ 1, \qquad (6)$$

Where

$$\mathcal{A}_1(Y) = \|\varphi(x)\|_{L_2(0,1)} + Y\|\psi(x)\|_{L_2(0,1)} + 2Y\sqrt{Y}\,\|f\,(x,y)\|_{L_2(Q)}$$
$$+2\|\varphi'''(x)\|_{L_2(0,1)} + 2\|\psi''(x)\|_{L_2(0,1)} + 2\sqrt{Y}\|f_{xx}\,(x,y)\|_{L_2(Q)},$$

$$\mathcal{A}_2(Y) = \|[H(\,y)]^{-1}\|_{C[0,Y]}\left\{\frac{1}{\sqrt{6}}\left[\|\varphi'''(x)\|_{L_2(0,1)} + \|\psi''(x)\|_{L_2(0,1)} + \sqrt{Y}\,\|f_{xx}\,(x,y)\|_{L_2(Q)}\right]\right.$$
$$\left. + \|H''(\,y) - f(0,y)\|_{C[0,Y]}\right\},$$

$$\mathcal{B}_1(Y) = 2Y(1+Y),\ \ \mathcal{B}_2(Y) = \ \|[H(y)]^{-1}\|_{C[0,Y]}\frac{1}{\sqrt{6}}Y,$$

Be satisfied. Then problem (1)–(5) has a unique classical solution in the ball $K\ =\ K_R(\|u\|_{E_Y^3} \leq\ R\ = (\mathcal{A}_1(Y)+\mathcal{A}_2(Y)\ +\ 2)$ of the space E_Y^3 (E_Y^3 is Banach space).

3. FDM scheme for direct (forward) problem

In the following section, we will solve the direct problem, i.e., when the unknown term $a(y)$ is presumed to be given. In order to solve this problem, we used FDM to find the numerical solution of the nonlocal problem given by equations (1)-(4). We sub divide the domain Q into $M \times N$ mesh with spatial step size $\Delta x = \frac{1}{M}$, and $\Delta y = \frac{Y}{N}$, where M and N are given positive integers. The grid points are given by

$$x_i = i\Delta x, \qquad i = 0,1,\dots,M,$$
$$y_j = \ j\Delta y, \qquad j = 0,1,\dots,N,$$

we denote the discretized from of of the quantities as follows;

$u(x_i,\ y_j) = u_{i,j},\ \ a(y_j) = a_j, f(x_i,\ y_j) = f_{i,j},\ \ \psi(x_i) = \psi_i$ and $\varphi(x_i) = \varphi_i$

for $i = 0,1,\dots,M,\ \ j = 0,1,\dots,N.$

Then the FDM scheme combined with the trapezoidal rule quadrature for nonlocal integral condition, the discrete expression for equation (1), which can be approximated via central FDM expressions.

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} - a_j u_{i,j} = f_{i,j}, \qquad i = 0,1,\dots,M,\ \ j = 0,1,\dots,N.$$

Simplifying the above equation, we get

$$\frac{1}{(\Delta y)^2}\left(u_{i,j-1} + u_{i,j+1}\right) - 2\left(\frac{1}{(\Delta y)^2} + \frac{1}{(\Delta x)^2} + \frac{a_j}{2}\right)u_{i,j} + \frac{1}{(\Delta x)^2}\left(u_{i+1,j} + u_{i-1,j}\right) = f_{i,j},$$

$$i = 0,1,\dots,M, j = 0,1,\dots,N. \qquad (7)$$

The FDM discretizes equations (2) - (3) as

$$u_{i,0} = \varphi_i, \qquad i = 0,1,\dots,M, \qquad (8)$$

$$\frac{u_{i,N+1} - u_{i,N}}{\Delta y} = \psi_i, \qquad i = 0,1,\dots,M, \qquad (9)$$

$$\frac{-3u_{0,j} + 4u_{1,j} - u_{2,j}}{2\Delta x} = 0, \qquad j = 0,1,\dots,N. \qquad (10)$$

The trapezoidal rule discretizes integral condition (4) as.

$$\frac{1}{2M}\left(u_{0,j} + u_{M,j} + 2\sum_{i=1}^{M-1} u_{i,j}\right) = 0\,, \qquad j = 0,1,\dots,N.$$

Since $u_{0,j} = u_{1,j}$ therefore, the above equation becomes.

$$\frac{10}{3}u_{1,j} + \frac{5}{3}u_{2,j} + 2\sum_{i=3}^{M-1} u_{i,j} + u_{M,j} = 0, \qquad j = 0,1,\dots,N, \tag{11}$$

finally, the discrete additional condition (5) as.

$$u_{0,j} = H_j, \qquad j = 0,1,\dots,N. \tag{12}$$

Equations (7)-(11) can be written in linear system at $j = 0,1,\dots N$.

$$\frac{1}{(\Delta y)^2}u_{i,2} + b_1 u_{i,1} + \frac{1}{(\Delta x)^2}\left(u_{i+1,1} + u_{i-1,1}\right) = f_{i,1} - \frac{1}{(\Delta y)^2}\varphi_i, \quad i = 0,1,\dots,M,$$

$$\frac{1}{(\Delta y)^2}\left(u_{i,3} + u_{i,1}\right) + b_2 u_{i,2} + \frac{1}{(\Delta x)^2}\left(u_{i+1,2} + u_{i-1,2}\right) = f_{i,2}, \qquad i = 0,1,\dots,M,$$

$$\frac{1}{(\Delta y)^2}\left(u_{i,4} + u_{i,2}\right) + b_3 u_{i,3} + \frac{1}{(\Delta x)^2}\left(u_{i+1,3} + u_{i-1,3}\right) = f_{i,3}, \qquad i = 0,1,\dots,M,$$

$$\vdots$$

$$\frac{1}{(\Delta y)^2}\left(u_{i,N} + u_{i,N-2}\right) + b_{N-1} u_{i,N-1} + \frac{1}{(\Delta x)^2}\left(u_{i+1,N-1} + u_{i-1,N-1}\right) = f_{i,N-1}, \qquad i = 0,1,\dots,M,$$

$$\left(b_N + \frac{1}{(\Delta y)^2}\right)u_{i,N} + \frac{1}{(\Delta x)^2}\left(u_{i+1,N} + u_{i-1,N}\right) + \frac{1}{(\Delta y)^2}u_{i,N-1} = f_{i,N} - \frac{\Delta y}{(\Delta y)^2}\psi_i \quad i = 0,1,\dots,M,$$

the above system can be written in matrix form as

$$\mathbf{L}U = P, \tag{13}$$

Where,

$b_j = -2\left(\frac{1}{(\Delta y)^2} + \frac{1}{(\Delta x)^2} + \frac{a_j}{2}\right)$, $U = [U_1, \ U_2, \ \dots, \ U_{N-1}, U_N]^T$,

$U_j = \left[u_{1,j}, \ u_{2,j}, \ \dots, \ u_{M-1,j}, u_{M,j}\right]^T$

$$\boldsymbol{L} = \begin{bmatrix} \Gamma_1 & \Lambda & \Theta & \Theta & & \dots & & & \Theta \\ \Lambda & \Gamma_2 & \Lambda & \Theta & & \dots & & & \Theta \\ & & & & \ddots & & & & \\ \Theta & \Theta & \Theta & \Theta & \dots & \Lambda & \Gamma_{N-1} & \Lambda & 0 \\ \Theta & \Theta & \Theta & \Theta & \dots & 0 & \Lambda & \Gamma_N & \Lambda \end{bmatrix}, \quad P = \begin{bmatrix} F_1 - \Lambda\varphi_i \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N - \Lambda\Delta y\psi_i \end{bmatrix},$$

here, **L** be (NM) ×(NM) square matrices, P is a column with the size (NM)× 1, Γ_j and Λ are the (M × M) matrix, Θ be (M × M) zero matrix and F_j is column with the size (M×1),

$$
\Gamma_j = \begin{bmatrix}
\frac{4}{3(\Delta x)^2}+b_j & \frac{1}{3(\Delta y)^2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\frac{1}{(\Delta x)^2} & b_j & \frac{1}{(\Delta x)^2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & \frac{1}{(\Delta x)^2} & b_j & \frac{1}{(\Delta x)^2} & 0 & 0 & \dots & 0 & 0 & 0 \\
 & & & \ddots & & & & & & \\
0 & 0 & 0 & & \dots & & 0 & \frac{1}{(\Delta x)^2} & b_j & \frac{1}{(\Delta x)^2} \\
\frac{10}{3} & \frac{5}{3} & 2 & & & & \dots & 2 & 2 & 1
\end{bmatrix},
$$

$$
\Lambda = \begin{bmatrix}
\frac{1}{(\Delta y)^2} & 0 & 0 & & \dots & & 0 \\
0 & \frac{1}{(\Delta y)^2} & 0 & & \dots & & 0 \\
 & & \ddots & & & & \\
0 & 0 & 0 & \dots & 0 & \frac{1}{(\Delta y)^2} & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0
\end{bmatrix}, F_j = \begin{bmatrix}
f_{1,j} \\ f_{2,j} \\ \ddots \\ f_{M-1,j} \\ 0
\end{bmatrix},
$$

3.1. Example for direct problem

We first investigate the robustness and efficiency of the proposed FDM method for a direct problem. In cases where the unknown coefficient is given, where the exact solution is

$$
u(x,y) = \frac{(\frac{y}{2}-2)\cos(\pi x)}{10}, \quad (x,y) \in Q
$$

and the input data are as follows:

$$
H(y) = \left(\frac{1}{20}\right)(y-4), \quad \varphi(x) = \frac{-\cos(\pi x)}{5}, \quad \psi(x) = \frac{\cos(\pi x)}{20}, \quad (x,y) \in Q,
$$

$$
a(y) = -15y, \quad f(x,y) = \left(\frac{1}{20}\right)(-\pi^2+15y)(y-4)\cos(\pi x), \quad (x,y) \in Q.
$$

Figure 1 presents the absolute error diagram of interior points when sizes of mesh $N = M = \{20, 40, 80\}$. Mesh independence has been attained, as well as numerical solution convergence toward the exact solution and high agreement. **Figure 2**, as the number of discretization rises, the findings for $H(y)$, becomes more accurate and showing a clear convergence.

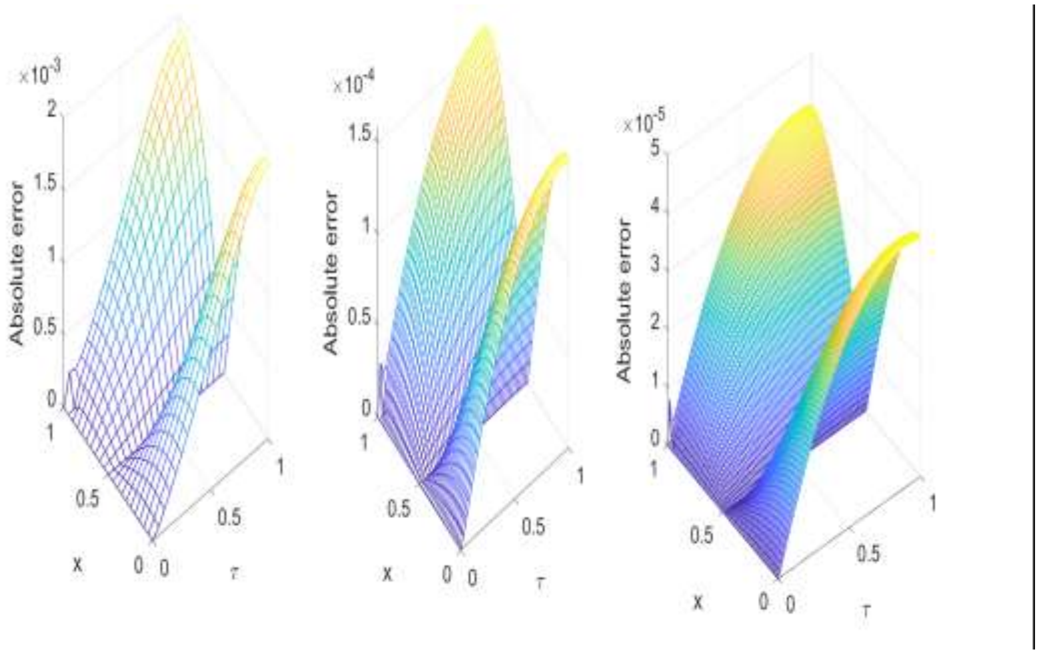


Figure 1. The absolute errors for the direct problem (1)-(4), when sizes of mesh are M = N ∈ {20, 40, 80}.

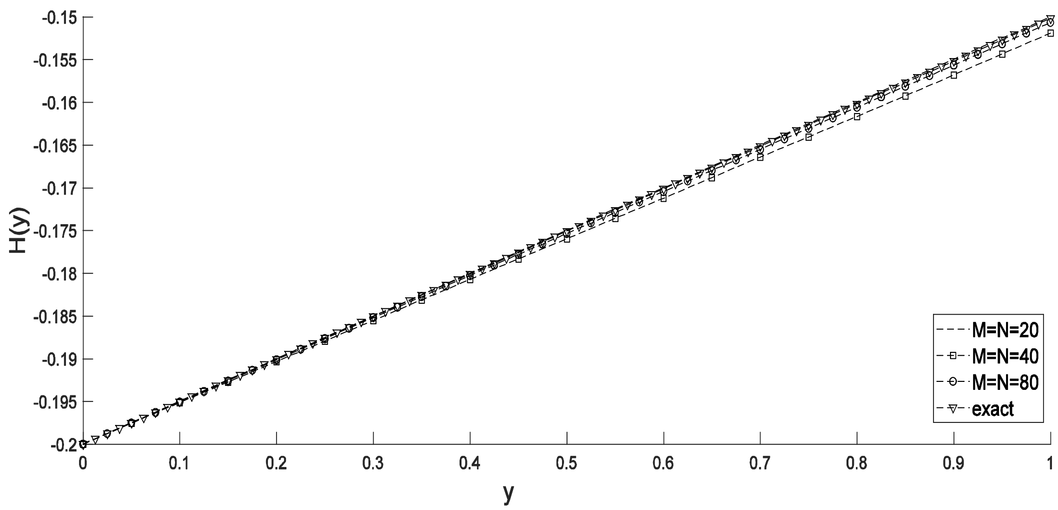


Figure 2. The numerical value and accurate for desired output H(y) with various mesh sizes.

4. Inverse Problem

For the nonlinear inverse problem (1)-(5), we seek a precise and stable identification of $u(x, y)$ and $a(y)$, that mean $a(y)$ be unknown. The one-dimensional second-order elliptic equation together with u (x, y) satisfies the problem given by equations (1)-(5). During the iterative process to solve the inverse problem, we assume that $a(0)$ is a constant starting assumption. Based on the given data, we can calculate this, we can calculate this at $y = 0$. To solve this problem, reformulate it as a nonlinear minimization problem. In other words: We try to find the smallest possible value for the discrepancy between the numerically calculated result and the measured data. Since this is an ill-posed problem, we need to apply Tikhonov regularization for a robust numerical solution in order to obtain stable results. The Tikhonov regularization functional can be derived from condition (5):

$$F(a) = \| u(0, y) - H(y)\|^2{}_{L^2[0,Y]} + \beta\|a(y)\|^2{}_{L^2[0,Y]}, \quad (14)$$

where $\beta > 0$, is a regularization parameter. The discretization of (14) is

$$F(\underline{a}) = \sum_{j=1}^{N}\left[u(0, y_j) - H(y_j)\right]^2 + \beta\sum_{j=1}^{N} a_j^2. \quad (15)$$

The unregularized case, i.e., $\beta = 0$, produces the regular nonlinear least-squares functional, which is inherently unstable when dealing with noisy data. The MATLAB toolbox technique lsqnonlin is used to minimize F under the physical constraint $a > 0$ and does not need the user to provide the gradient of the objective functional (15) [27]. To determine the minimum of a scalar function with several variables, the lsqnonlin subroutine conducts a constrained nonlinear optimization. The subroutine is configured using the following parameters:

• (Maxlter) Maximum number of iterations $= 400$ or $10^2 \times$ (number of variables)

• Solution tolerance (SolTOL) $= 10^{-15}$ and Objective function tolerance (FunTOL) $= 10^{-15}$.

The solution of the inverse problem (1)–(5) is subjected to both accurate and noisy measurement data (5). By including a random error, the noisy data is numerically simulated as follows:

$$H^{\epsilon}(y_j) = H(y_j) + \epsilon_j, \qquad j = 0,1, \dots, N, \quad (16)$$

where ϵ, is random Gaussian normal distribution vectors with mean zero and standard deviations σ, given by

$$\sigma = p \times \max_{y\in[0,Y]} |H(y)|; \quad (17)$$

where p is the percentage of noise. We use the MATLAB bulletin function normrnd to generate the random variables $\underline{\epsilon} = (\epsilon_j)$ and $j = 0,1, \dots, N$.

as follows:

$$\underline{\epsilon} = \text{normrnd}(0, \sigma, N), \quad (18)$$

4.1 Initial guess

As stated previously, to begin the iterative process of solving the inverse problem, an initial estimate is required. The following values for $a(0)$ can be derived from input data.

Consider the nonlinear inverse problem (1)-(5) with unknown coefficient $a(y)$ and from the equation (28) in [26], we have:

$$a(y) = H^{-1}(y)\left\{H''(y) - f(0, y) - \sum_{k=1}^{\infty} \lambda_k^2 u_k(y)\right\}, \qquad y \in [0, Y].$$

From equation (5), since $u_k(y) = u(0, y_k) = H(y_k)$ and $\lambda_k^2 = \pi^2$, therefore the above equation becomes:

$$a(y) = H^{-1}(y)\left\{H''(y) - f(0, y) - \sum_{k=1}^{\infty} \pi^2 H(y_k)\right\}, \qquad y \in [0, Y].$$

the above equations at $y = 0$,we get the first guess:

$$a_0 = H^{-1}(0)\left\{H''(0) - f(0,0) - \sum_{k=1}^{\infty} \pi^2 H(0)\right\}, \quad y \in [0, Y], \quad (19)$$

provided that $H(0)$ did not vanish.

5. Results and discussion

We examine and evaluate the numerically computed results using the FDM in connection with the Tikhonov regularization approach, as described in the previous section. The root mean squares errors (rmse) utilized via the following expression.

$$\text{rmse}(a) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(a_i^{exact} - a_i^{numerical}\right)^2}. \tag{20}$$

Calculated to determine the accuracy of the specified coefficient. For simplicity, we take Y = 1 in all examples.

First, we consider the case where the unknown coefficients are given to test the stability and effectiveness of the proposed FDM scheme for a direct problem, we have a direct problem, and the exact solution is

$$u(x,y) = \frac{-e^{\frac{-7-y}{100}}cos(x\pi)}{9},$$

and the input data are as follows:

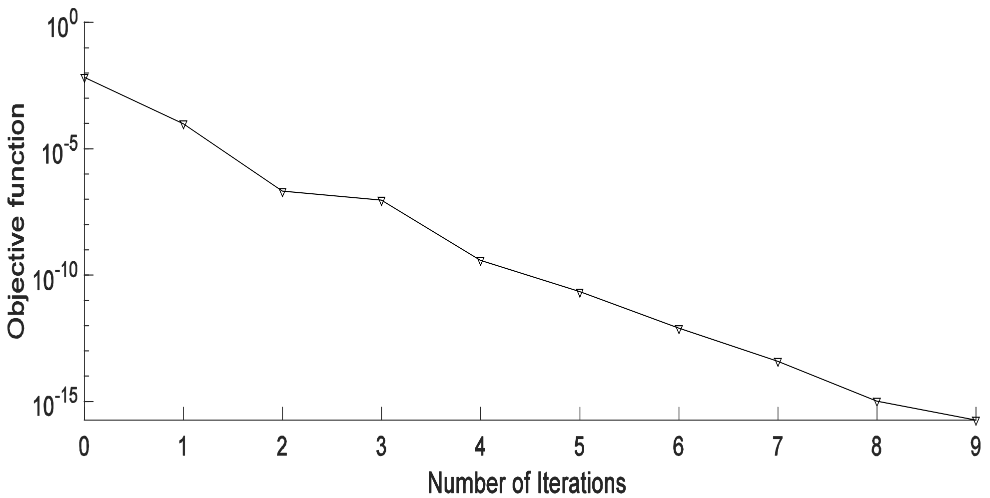
$$a(y) = -10\, e^{-\frac{5y}{100}},$$

$$f(x,\tau) = \frac{e^{\frac{1}{100}(-7-6y)}\left(-100000 + e^{y/20}(-1 + 10000\pi^2)\right)Cos(\pi x)}{90000}$$

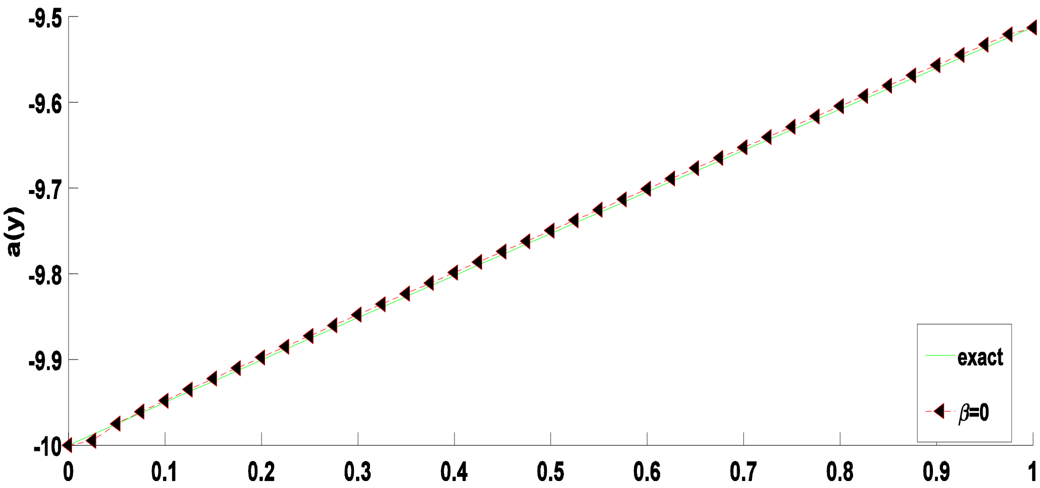
$$\varphi(x) = \frac{-e^{\frac{-7}{100}}cos(x\pi)}{9}, \qquad \psi(x) = \frac{e^{\frac{-7}{100}}cos(x\pi)}{900}, \quad H(y) = \frac{-e^{\frac{-7-y}{100}}}{9}.$$

Next, we consider the inverse elliptic problem (1)-(5) with the coefficient $a(y)$ is unknown. The initial guess was $a_0 = -10$ can be found in the equation (19). It is easy to verify the input data for the conditions of the Theorem 1. Hence, the inverse elliptic problem (1)-(5) with input data above has a unique solution.

We fix M= N = 40 for the numerical investigation started with the situation of no noise included, i.e., $p = 0$ in (17). The objective function (15) represented in **Figure 3(a)**, and a speed declining convergence is seen for achieving a shorter order tolerance $O(10^{-15})$ in just 9 iterations. **Figure 3(b)** shows numerical results for the coefficient $a(y)$ with $\text{rmse}(a) = 0.0034$.



(a)



(b)

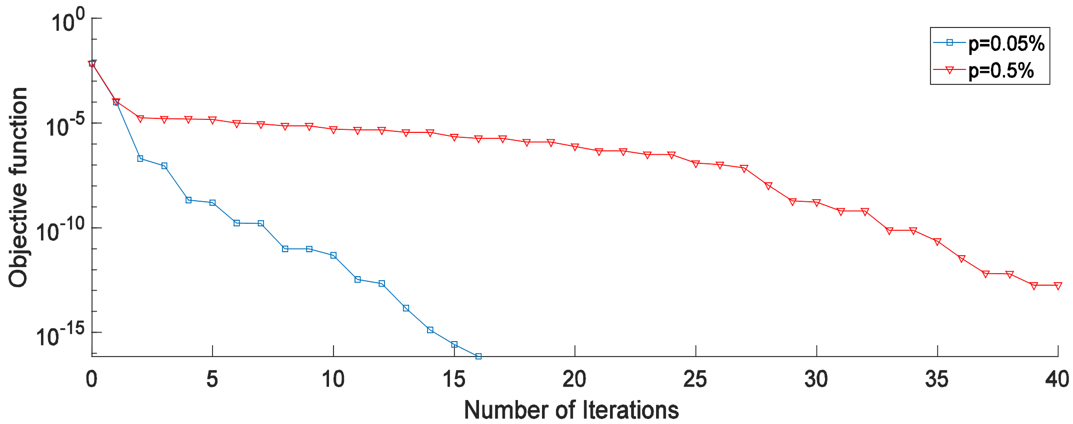
Figure 3. (a) objective function (15) and (b) $\mathrm{a}(\mathrm{y})$ with noise-free and without regularization.

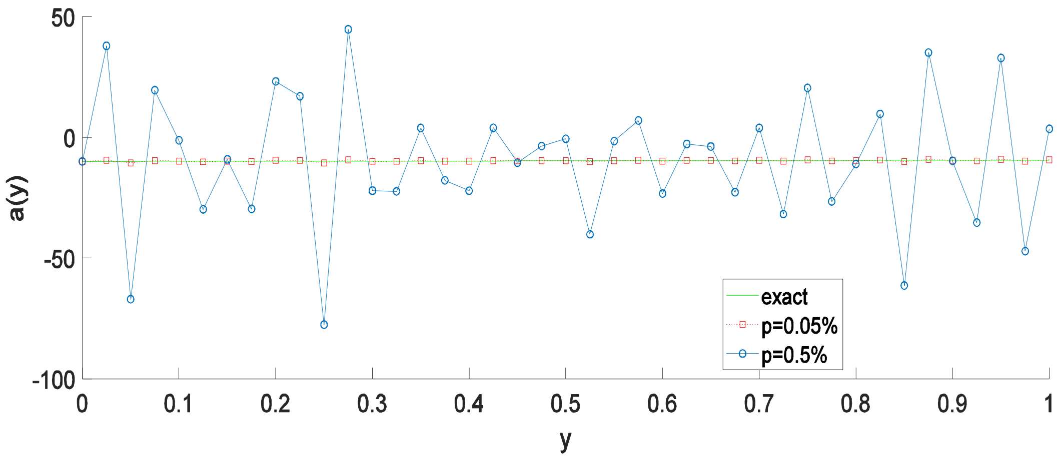
Next, we add $\mathrm{p} \in \{0.05, 0.5\}\%$ noise as in equation (17). The case of noisy data and no regularization is presented in **Figures 4(a)-4(b)**. **Figure 4(a)**, from this figure the monotonic decreasing achieved in about 16 iterations and 40 iterations respectively, and the steady convergence for the rest of iterations to $O(10^{-15})$ in 40 iterations reaching a very low stationary value of order the associated numerical result was found stable and accurate (with $\mathrm{p} = 0.05$, $\mathrm{rmse}(\mathrm{a}) = 0.2750$), but unstable and inaccurate (with $\mathrm{p} = 0.5$, $\mathrm{rmse}(\mathrm{a}) = 27.5397$). This is expected since the problem under investigation is ill-posed problem and small errors (noise) in input data lead to drastic errors in outputs. As seen in **Figure 4(b)**, the numerical solution of the $\mathrm{a}(\mathrm{y})$ stable and accurate at $\mathrm{p} = 0.05$, and with $\mathrm{p} = 0.5$, unstable and diverges from the exact solution but remains on the same path when the value of additive noise increases in equation (17).

In order to restore the deteriorate stability of the coefficient $\mathrm{a}(\mathrm{y})$ a sort of regularization should be applied. The L-curve approach developed by P. Hansen [28], Morozov's discrepancy principle [29], and even just plain old trial and error, as advocated for in [30], are just a few of the available options. By incorporating the penalty terms into equation (15), we employ the Tikhonov regularization technique. We try out different values for the regularization parameter $\beta \in \{10^{-10}, \dots, 10^{-6}\}$, noise of $\mathrm{p} = 0.05\%$ is added to replicate real input data. In **Figures 5(a)**, the monotonic decreasing achieved in about 11 iterations and noise of $\mathrm{p} = 0.05\%$. In **Figures 6(a)**, the monotonic decreasing achieved in about 28 iterations, indicating that the objective function minimization (15) is satisfied**. Figures 5(b)** and **6(b)** depict the unknown potential coefficient $\mathrm{a}(\mathrm{y})$. These figures demonstrate that results are nearly perfectly smooth, particularly in the range [0.3,1], until noise levels increase from 0.05% to 0.5% and instabilities appear. In addition, in **Table 1** $\mathrm{rmse}(\mathrm{a})$ values reveal a reasonable range of values, with the best retrieval occurring at the lowest $\mathrm{rmse}(\mathrm{a})$**.** For additional information, see **Table 1** and **Figure 5-6** for numerical results.

Table 1. The rmse(a) (20) for the noise $p \in \{0.05, 0.5\}$ %, and regularization $\beta \in \{10^{-10}, \dots, 10^{-6}\}$..

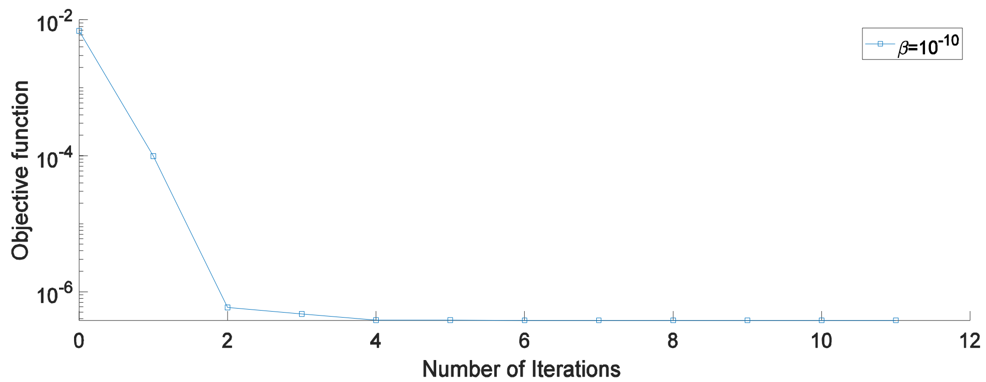
$\mathbf{rmse(a)}$	$\boldsymbol{\beta = 10^{-10}}$	$\boldsymbol{\beta = 10^{-9}}$	$\boldsymbol{\beta = 10^{-8}}$	$\boldsymbol{\beta = 10^{-7}}$	$\boldsymbol{\beta = 10^{-6}}$
$\mathbf{p = 0.05\%}$	**0.2523**	0.3709	0.8006	1.3559	2.0580
$\mathbf{p = 0.5\%}$	21.9455	8.6963	2.1730	**1.3343**	2.0448



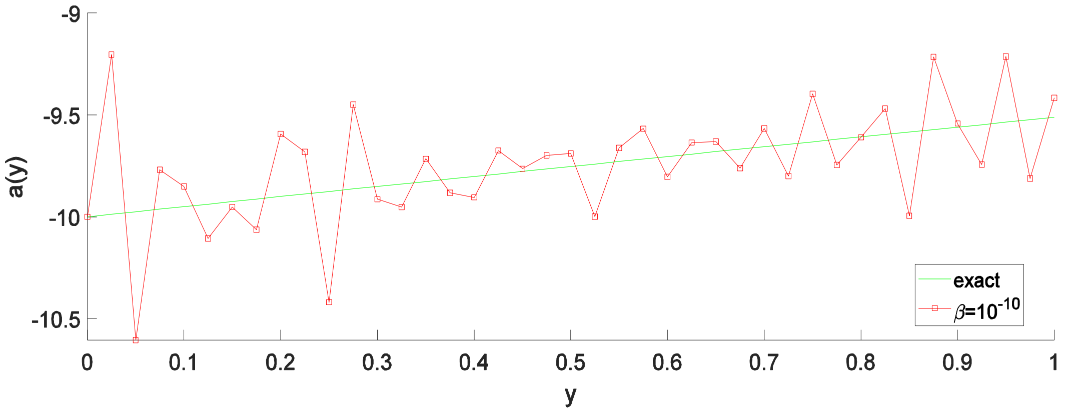


(b)

Figure 4. (a) objective function (15) and (b) a(y), for different noise level $p \in \{0.05, 0.5\}$ % and no regularization.

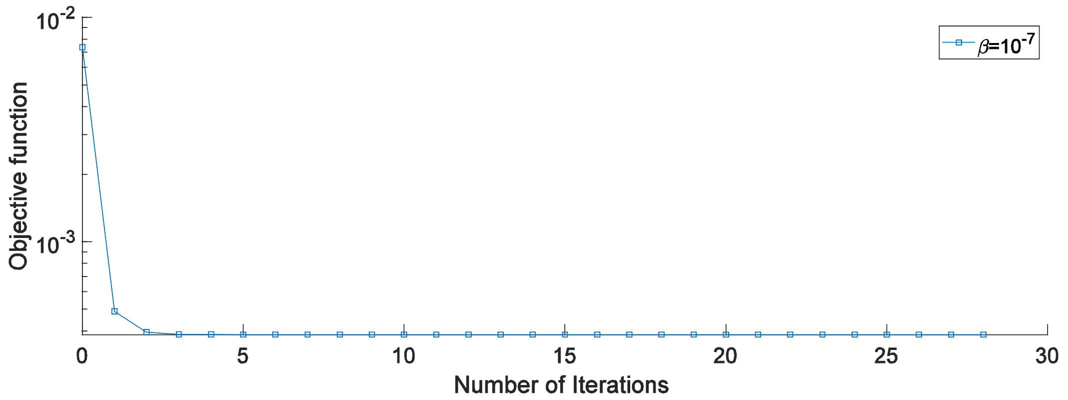


(a)

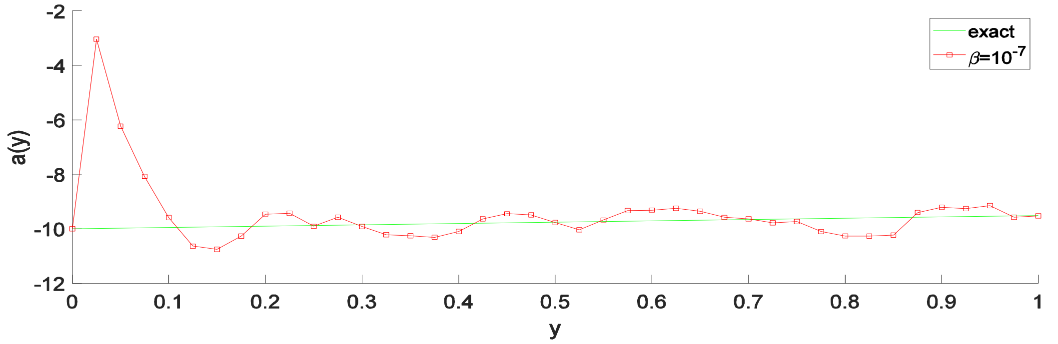


(b)

Figure 5. (a) objective function (15) and (b) $a(y)$, for $p = 0.05\%$ noise and $\beta = 10^{-10}$.



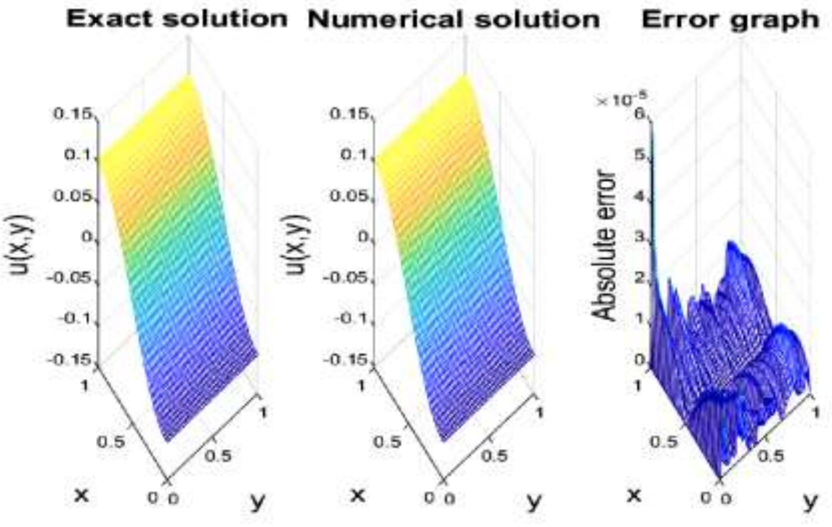
(a)



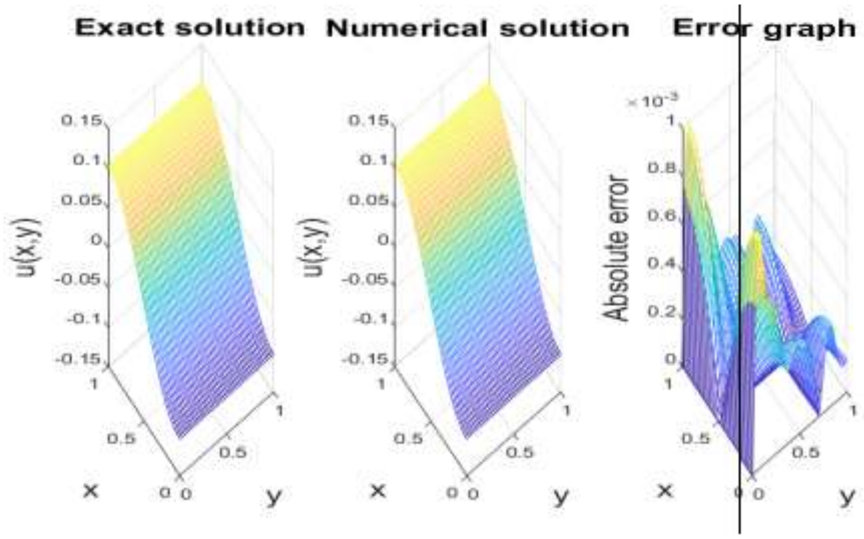
(b)

Figure 6. (a) objective function (15) and (b) $a(y)$, for $p = 0.5\%$ noise and $\beta = 10^{-7}$.

The numerical and exact temperatures $u(x, y)$, with $p = 0.05\%$ noise, $\beta = \{10^{-10}\}$, $p = 0.5\%$ noise, $\beta = \{10^{-10}\}$, as well as the absolute error between them, are illustrated in **Figure 7** and execute arguments obtained.



(a)



(b)

Figure 7. The exact and numerical $u(x, y)$ with (a) $p = 0.05\%$ noise $\beta = 10^{-10}$, (b) $p = 0.5\%$ noise $\beta = 10^{-7}$, as well as the absolute error between them.

6. Conclusions

The finite difference schemes were used for direct two-dimensional second order elliptic inverse problems in conjunction with quadrature by the trapezoidal rule. The instability induced by the ill-posed problem was solved using Tikhonov regularization. The RMS values for noise $p = 0$ and $\beta = 0$ were contrasted for the numerical test problem. It was found that a stable solution with $p = 0.05\%$ noise and the regularization parameter $\beta = 10^{-10}$

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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