δ-Hollow Modules

Inaam M. A. Hadi
Sameeah Hasoon Aidi*
Department of Mathematics/ College of Education for Pure Science, (Ibn Al-Haitham)/ University of Baghdad

Received in : 8 May 2014 , Accepted in: 22 June 2014

Abstract

Let R be a commutative ring with unity and M be a non zero unitary left R-module. M is called a hollow module if every proper submodule N of M is small ($N \ll M$), i.e. $N + W \neq M$ for every proper submodule W in M. A δ-hollow module is a generalization of hollow module, where an R-module M is called δ-hollow module if every proper submodule N of M is δ-small ($N \ll_{\delta} M$), i.e. $N + W \neq M$ for every proper submodule W in M with $\frac{M}{W}$ is singular. In this work we study this class of modules and give several fundamental properties related with this concept.

Key Words: Small submodule, δ-small submodule, hollow module, δ-hollow module, singular module, nonsingular module.

* This paper is a part of the thesis submitted by the second author.

Introduction

Throughout this article all rings are commutative rings with identity, and all modules are unitary left R-module. A proper submodule L of a module M is called small (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. A module M is called hollow if every proper submodule of M is called small, [1]. As a generalization of the concept small submodule, Zhou in [2] introduce the concept δ-small submodule, where a submodule N of an R-module M is called δ-small (denoted by $N \ll_{\delta} M$) if whenever $N + K = M$ and M/K is singular module, then $K = M$. In fact an R-module M is called singular (non singular) if $Z(M) = \{m \in M: \operatorname{ann}_R(m)$ is an essential ideal of $R\} = M$ ((0)), [3], and a submodule N of an R-module M is called essential in M (denoted by $N \leq_e M$ or $N \xrightarrow{*} M$) if $N \cap W \neq (0)$ for any non zero $W \leq M$, [4]. The concept of δ-hollow module appeared in [5], where an R-module M is called δ-hollow, if every proper submodule of M is a δ-small in M. Hence hollow module is δ-hollow, but the converse is not true.

The aim of this work is to give a comprehensive study of the class of δ-hollow modules. It is of interest to know how far the old theories of hollow module extend to the new situation.

1- Preliminary

In this section, we give some definitions and propositions which are useful in our work.

Definition 1.1:

A non zero module M is called a hollow module if every proper submodule N of M is a small submodule of M ($N \ll M$) that is $N + W \neq M$ for every $W < M$, [1].

Definition 1.2:

Let M be an R-module. A submodule A of a module M is called a δ-small submodule of M (denoted by $A \ll_{\delta} M$) if $M \neq A + B$ for any proper c-singular B of M, (where B is c-singular if $\frac{M}{B}$ is singular module), see [2]..

An R-module M is called δ-hollow if every proper submodule is δ-small in M, [5].

An R-module M is called semisimple if every submodule of M is a direct summand of M [3], [4].

Proposition 1.3: [2]

Let M be an R-module and A be a submodule of M. Then the following are equivalent

(1) $A \ll_{\delta} M$.

(2) If $M = A + B$, then $M = Y \oplus B$, for projective semisimple submodule Y of A.

(3) If $M = A + B$ with $\frac{M}{B}$ Goldie torsion, then $M = B$, where R-module M is called Goldie torsion if $Z_2[M] = M$, and $Z_2(M)$ is defined by $\frac{Z_2(M)}{Z(M)} = \frac{M}{Z(M)}$, (see [3]).

Proposition 1.4: [2]

(1) Let A and B be submodules of an R-module M such that $A \leq B$. If $A \ll_{\delta} B$ then $A \ll_{\delta} M$.

(2) Let A and B be submodules of an R-module M such that $A \leq B$. If $B \ll_{\delta} M$ then $A \ll_{\delta} M$.

(3) Let A and B be submodules of an R-module M such that $A \leq B$, then $B \ll_{\delta} M$ if and only if $A \ll_{\delta} M$ and $\frac{B}{A} \ll_{\delta} \frac{M}{A}$.

(4) Let M and N be an R-modules and let $f : M \longrightarrow N$ be a homeomorphism. If A is a submodule of M such that $A \ll_{\delta} M$, then $f(A) \ll_{\delta} N$.

(5) Let A and B be submodules of an R-module M. Then $A + B \ll_{\delta} M$ if and only if $A \ll_{\delta} M$ and $B \ll_{\delta} M$.

(6) Let $M = M_1 \oplus M_2$ be an R-module, let $A_1 \leq M_1$ and $A_2 \leq M_2$. Then $A_1 \oplus A_2 \ll_{\delta} M_1 \oplus M_2$ if and only if $A_1 \ll_{\delta} M_1$ and $A_2 \ll_{\delta} M_2$.

Proposition 1.5: [6]

Let A and B be submodules of an R-module M such that $A \leq B$. If B is a direct summand of M and $A \ll_{\delta} M$ then $A \ll_{\delta} B$.

Recall that an R-module M is called indecomposable if the only direct summand of M are (0), M, [4].

An R-module M is called a prime module if $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$, for each non zero submodule of M, [7].

A non zero R-module is called uniform module if every non zero submodule of M is essential in M ($N \leq_e M$), [4].

Proposition 1.6:

Let M be an R-module, then:

(1) Let A be a proper submodule of an indecomposable R-module M. Then $A \ll_{\delta} M$ if and only if $A \ll M$, [6, proposition 1.2.13].

(2) Let A be a submodule of singular R-module M. Then $A \ll_{\delta} M$ if and only if $A \ll M$, [6, proposition 1.2.14].

(3) Let M be torsion module over an integral domain R and A be a submodule of M. Then $A \ll_{\delta} M$ if and only if $A \ll M$, [6, corollary 1.2.16], where an R-module over integral domain R is called torsion if
$T(M) = \{m \in M : \exists\, r \in R/\{0\}, rm = 0\} = M$, [4].

(4) Let M be a prime R-module with $Z(M) \neq 0$ and A be a proper submodule of M. Then $A \ll_{\delta} M$ if and only if $A \ll M$, [6, proposition 1.2.17].

(5) Let M be a uniform R-module and A be a submodule of M. Then $A \ll_{\delta} M$ if and only if $A \ll M$, [6, proposition 1.2.18].

(6) Let M be an R-module. Then every non singular semisimple submodule A of M is δ-small in M, [6, proposition 1.2.3].

2- Basic Properties of δ-Hollow Modules

In this section, we give the basic properties about δ-hollow modules. We see that under certain conditions hollow modules and δ-hollow modules are equivalent. Also we noticed that some properties of hollow modules can be generalized to δ-hollow modules.

Remarks and Examples 2.1:

(1) It is clear that Z_6 is non singular semisimple, so every submodule of Z_6 is non singular semisimple, hence every submodule is small by proposition (1.6 (6)). Thus Z_6 is a δ-hollow module. But Z_6 is not hollow. Also notice that Z_6 is decomposable.

(2) It is clear that every hollow module is a δ-hollow module.
Hence each of the Z-module Z_4, Z_8 and Z_{p^∞} are δ-hollow.

(3) Z_{12} as Z-module is not a δ-hollow module since $<\bar{3}> \oplus <\bar{4}> = Z_{12}$ and $\frac{Z_{12}}{<\bar{4}>} \cong Z_4$ and Z_4 is a singular Z-module. However $<\bar{4}> \neq Z_{12}$.

By using proposition (1.6) hollow modules and δ-hollow modules are coincident under certain class of modules.

Theorem 2.2:

Let M be an R-module. Then:

(1) If M is an indecomposable module, then M is a hollow module if and only if M is a δ-hollow module.

(2) If M is a singular module, then M is a hollow module if and only if M is a δ-hollow module.

(3) If M is a prime module with $Z(M) \neq 0$, then M is a hollow module if and only if M is a δ-hollow module.

(4) If M is a uniform R-module then M is a hollow module if and only if M is a δ-hollow module.

(5) If M is a torsion module over a commutative integral domain R then M is a hollow module if and only if M is a δ-hollow module.

Proposition 2.3:

Epimorphic image of δ-hollow module is δ-hollow.

Proof:

Let M be a δ-hollow module, let M' be a module and let $f : M \longrightarrow M'$ be an epimorphism. Suppose N' is a proper submodule of M' with N' + K' = M' and $\frac{M'}{K'}$ is singular. This implies $f^{-1}(N) \lneqq M$ because if $f^{-1}(N) = M$ then $ff^{-1}(N') = f(M) = M'$ and N' = M' which is a contradiction. Thus $f^{-1}(N') \ll_{\delta} M$. Also N'+K'= M' implies that $f^{-1}(N') + f^{-1}(K') = M$.

To check $\frac{M}{f^{-1}(K)}$ is a singular R-module. We show that $Z(\frac{M}{f^{-1}(K)}) = \frac{M}{f^{-1}(K)}$.

Let $m+f^{-1}(K')\in\dfrac{M}{f^{-1}(K)}$. We must prove $\underset{R}{\operatorname{ann}}(m+f^{-1}(K'))\underset{e}{\leq} R$. Since $f(m)+K'\in\dfrac{M'}{K'}$ (which is singular). Thus $f(m)+K'\in Z(\dfrac{M'}{K'})$.

So $\underset{R}{\operatorname{ann}}(f(m)+K')\underset{e}{\leq} R$. Let J be any ideal of R, $J\neq 0$ so $\underset{R}{\operatorname{ann}}(f(m)+K')\cap J\neq 0$. Thus there exists $j\in J$, $j\neq 0$ and $j(f(m)+K')=0_{\frac{M}{K'}}$, then $jf(m)+K'=K'$. Thus $jf(m)\in K'$, which implies $f(jm)\in K'$, hence $jm\in f^{-1}(K')$. Thus $j(m+f^{-1}(K'))=f^{-1}(K')=0_{\frac{M}{f^{-1}(K')}}$, that is $j\in\underset{R}{\operatorname{ann}}(m+f^{-1}(K'))\cap J$, and hence $\underset{R}{\operatorname{ann}}(m+f^{-1}(K'))\underset{e}{\leq} R$.

Thus $m+f^{-1}(K')\in Z(\dfrac{M}{f^{-1}(K')})$, that is $\dfrac{M}{f^{-1}(K')}$ is singular. Since $f^{-1}(N')\ll_{\delta} M$ and $\dfrac{M}{f^{-1}(K')}$ is singular, we get $f^{-1}(K')=M$ (since M is δ-hollow). It follows that $f(f^{-1}(K'))=f(M)=M'$, hence $K'=M'$. Thus M' is a δ-hollow module.

Corollary 2.4:

Let M be an R-module. If M is a δ-hollow module then $\dfrac{M}{N}$ is a δ-hollow module for every proper submodule N of M.

Proof:

Let N be a proper submodule of a δ-hollow M. Let $\pi: M\longrightarrow\dfrac{M}{N}$ be the natural epimorphism, then $\dfrac{M}{N}$ is a δ-hollow module by proposition (2.3).

Corollary 2.5:

A direct summand of a δ-hollow module is a δ-hollow module.

Proof:

Let M be a δ-hollow R-module and N be a direct summand of M. Hence $M=N\oplus K$ for some submodule K of M. By second isomorphism theorem $\dfrac{M}{K}\simeq N$. But $\dfrac{M}{K}$ is δ-hollow by corollary (2.4). Thus N is δ-hollow.

Proposition 2.6:

Let M be an R-module and $K\ll_{\delta} M$. If $\dfrac{M}{K}$ is a δ-hollow module then M is a δ-hollow module.

Proof:

Let $N<M$ with $M=N+L$, where L is a submodule of M and $\dfrac{M}{L}$ is singular R-module then $\dfrac{M}{K}=\dfrac{N+L}{K}=\dfrac{N+K}{K}+\dfrac{L+K}{K}$. But $\dfrac{M}{K}\Big/\dfrac{(L+K)}{K}\cong\dfrac{M}{L+K}$ by third fundamental theorem.

We shall prove $\dfrac{M}{L+K}$ is singular.

Let $\bar{m} \in \frac{M}{L+K}$, so $\bar{m} = m + (L + K)$. But $\underset{R}{ann}(m + L) \leq \underset{R}{ann}(m + (L + K))$, since if $r \in \underset{R}{ann}(m + L)$, then $rm + L = 0_{\frac{M}{L}} = L$ implies that $rm \in L \leq L + K$. Therefore $rm + (L + K) = L + K = 0_{\frac{M}{L+K}}$, which implies that $r \in \underset{R}{ann}(m + (L + K))$. But $\frac{M}{L}$ singular implies $\underset{R}{ann}(m + L) \underset{e}{\leq} R$, hence $\underset{R}{ann}(m + (L + K)) \underset{e}{\leq} R$. Thus $m + (L + K) \in Z(\frac{M}{L+K})$, so $\frac{M}{L+K}$ is singular. Hence $\frac{L+K}{K} = \frac{M}{K}$ since $\frac{M}{K}$ is δ-hollow. It follows that L + K = M. But $K \ll_{\delta} M$, implies L = M. Therefore M is δ-hollow.

3- δ-Hollow Modules and Other Related Modules

In this section, we give some relationships between δ-hollow modules and other related modules.

Let M be a module, then:

M is called amply supplemented module if for any two submodules U and V of M with U + V = M, V contains a supplement of U in M, where a submodule A of M is called a supplement of B ($B \leq M$) if M = A + B and $A \cap B \ll A$. Equivalently A is a supplement of B if A + B = M and B is a minimal element in the set of submodules $L \leq M$ with B + L = M, [8].

Recall that every hollow module is amply supplemented, see [11,proposition (1.3.5)].

We shall give analogus statement for δ-hollow, but first recall that an R-module is called δ-amply supplemented if for any two submodules U and V of M with U + V = M, V contains a δ-supplemented of U in M, where a submodule N of M is called δ-supplement of a submodule W of M if N + W = M, $N \cap W \ll_{\delta} N$, [9], [10].

Proposition 3.1:

Every δ-hollow module is a δ-amply supplemented.

Proof:

Let U proper submodule of M and U + M = M. Since U + M = M and $\frac{M}{M} = (0)$ singular and $U \cap M = U$. But $U \ll_{\delta} M$, since M is δ-hollow.

Recall that a submodule N of a module M is called δ-coclosed in M (briefly $N \underset{scc}{\leq} M$) if $\frac{N}{K}$ is singular and $\frac{N}{K} \ll_{\delta} \frac{M}{K}$ implies N = K for any submodule K of M contained in N, [12].

Proposition 3.2:

Let M be a module and L be a non zero submodule of M which is δ-hollow, then either L is δ-small submodule of M or a δ-coclosed submodule of M, but not both.

Proof:

Suppose L is not δ-coclosed submodule of M, so there exists K < L such that $\frac{L}{K} \ll_{\delta} \frac{M}{K}$ and $\frac{L}{K}$ is singular . But L is δ-hollow and K < L, hence $K \ll_{\delta} L$ and $\frac{L}{K} \ll_{\delta} \frac{M}{K}$. Hence

$L \ll_\delta M$. If L δ-coclosed submodule of M, and suppose that $L \ll_\delta M$ then $\frac{L}{(0)} \ll_\delta \frac{M}{(0)} = M$, and hence $L = 0$, which is a contradiction.

Proposition 3.3:

Every non zero δ-coclosed submodule of a δ-hollow module is δ-hollow.

Proof:

Let M be a δ-hollow module and let N be a non zero submodule of M such that $N \underset{\delta cc}{\leq} M$. To show that N is δ-hollow.

Let $L < N \leq M$ then $L < M$ and so $L \ll_\delta M$. But $L < N$ and N is δ-coclosed implies that $L \ll_\delta N$ by [12, corollary (2.6)]. Thus N is a δ-hollow module.

Proposition 3.4:

Let M be a δ-hollow module and let N be a direct summand of M. Then N is δ-hollow.

Proof:

Let A be a proper submodule of N. Since M is δ-hollow, $A \ll_\delta M$ and by proposition (1.5) $A \ll_\delta N$. Therefore N is δ-hollow.

Proposition 3.5:

Let M be a singular R-module, let $N \ll_\delta M$. If $\frac{M}{N}$ is a finitely generated R-module, then M is finitely generated.

Proof:

As $\frac{M}{N}$ is finitely generated, $\frac{M}{N} = R(x_1 + N) + \ldots + R(x_n + N)$ for some $x_1, \ldots, x_n \in M$. We claim that $M = Rx_1 + \ldots + Rx_n$. Let $m \in M$ then $m + N = r_1(x_1 + N) + \ldots + r_n(x_n + N)$, so that $m - r_1x_1 - \ldots - r_nx_n \in N$. This implies $m = r_1x_1 + \ldots + r_nx_n + n$ for some $n \in N$. Thus $M = \langle x_1, \ldots, x_n \rangle + N$. But $M / \langle x_1, \ldots, x_n \rangle$ is singular (since M is singular) and $N \ll_\delta M$ by hypothesis $M = \langle x_1, \ldots, x_n \rangle$.

Corollary 3.6:

Let M be a singular R-module and N be a proper submodule of module M. If M is a δ-hollow module and $\frac{M}{N}$ is finitely generated then M is finitely generated.

Proof:

It is clear by proposition (3.5).

Corollary 3.7:

Let M be an R-module with every factor of M is singular and let $N < M$. If M is a δ-hollow and $\frac{M}{N}$ finitely generated, then M is finitely generated.

Proof:

It is clear by proposition (3.5).

Note:

Let M be an R-module. If every non zero factor of M is indecomposable, then by [13,41.4(1)] M is hollow module, which implies that M is δ-hollow. But the converse is not true, for example Z_6 as Z_6-module is δ-hollow but does not imply that every non zero factor of Z_6 is indecomposable, since $\frac{Z_6}{(0)} \cong Z_6$ is not indecomposable.

Recall that, an R-module. M is called δ-lifting, if for every submodule N of M, there exist submodules K, K' ≤ M such that M = K ⊕ K' with K ≤ N and $N \cap K' \ll_{\delta} M$, [8].

It is clear that every lifting is δ-lifting.

Proposition 3.8:

Every indecomposable and δ-lifting module is δ-hollow.

Proof:

Let M be indecomposable and δ-lifting module and N be a proper submodule of M. Since M is δ-lifting, then M= K ⊕ K' where K ≤ N and $N \cap K' \ll_{\delta} K'$. But M is indecomposable, so K' = 0 and M = K. Then M ≤ N < M which is a contradiction. Hence K' = M and so $N \cap K' = N \cap M = N$. Thus $N \ll_{\delta} M$. It follows that M is δ-hollow.

Proposition 3.9:

Every δ-hollow module is δ-lifting.

Proof:

Let N be a proper submodule of δ-hollow module M, then M = (0) ⊕ M and {0} ≤ N where $N \cap M = N \ll_{\delta} M$. Thus M is δ-lifting.

Proposition 3.10:

Let M be an R-module. Then the following statements are equivalent:

(1) M is indecomposable and δ-lifting.
(2) M is δ-hollow and indecomposable.
(3) M is hollow.

Proof:

(1) ⇒ (2) Let N < M. Since M is δ-lifting then M = K ⊕ K' with K ≤ N and $N \cap K' \ll_{\delta} M$. As M is indecomposable, then K' = 0 or K = 0.
If K' = 0, then K = M, which implies that M ≤ N. That is a contradiction. So K = 0, hence K' = M and $N \cap K' = N \cap M = N \ll_{\delta} M$. Thus M is δ-hollow.

(2) ⇒ (3) It is clear by proposition (2.2(1)).

(3) ⇒ (1) If M is hollow, then M is indecomposable by [11,proposition 1.3.9]. But M is hollow, hence M is lifting by [11, proposition 1.3.16], which implies that M is δ-lifting.

The following is needed for the next result.

Definition 3.11: [2]

A pair (P, f) is a δ-projective cover of an R-module M, if P is a projective module and f:P⟶M is an epimorphism and $\ker f \ll_{\delta} P$.

Proposition 3.12:

Let (P, f) be δ-projective cover of M. Then M is δ-hollow if and only if P is δ-hollow.

Proof:

(⇒) Since M is δ-hollow module and since $f:P \longrightarrow M$ is an epimorphism, then $\frac{P}{\ker f} \cong M$ by the first fundamental theorem and hence $\frac{P}{\ker f}$ is δ-hollow. But $\frac{P}{\ker f}$ is δ-hollow and $\ker f \ll_\delta P$. So by proposition (2.6), P is δ-hollow.

(⇐) Let N be a proper submodule of M, then $f^{-1}(N)$ is a proper submodule of P. Since P is δ-hollow, then $f^{-1}(N) \ll_\delta P$, and hence $f f^{-1}(N) \ll_\delta M$ by proposition (1.3(4)). But $f f^{-1}(N) = N$, so $N \ll_\delta M$. Thus M is δ-hollow.

References

1. Fleury, P. (1974), Hollow Modules and Local Endomorphism Rings, Pac.J.Math., 53, 379-385.
2. Zhou, Y.Q. (2000)Generalizations of Perfect Semiperfect and Semiregular Rings, Algebra Collog, 7, 305-318.
3. Goodearl ,K.R. (1976), Ring Theory, Nonsingular Rings and Modules, Marcel Dekkel.
4. Kasch, F. (1982) Modules and Rings, Academic Press, Inc-London.
5. Nematollah, M.J. (2009) On δ-Supplemented Submodules, Tarbiat Modlen Univ., 20th Seminar on Algebra 2-3 ordibeheshst, 1388 (Apr 22-23), 155-158.
6. Hassan, S.S. (2011) Some Generalizations of δ-lifting Modules, M.Sc. Thesis, University of Baghdad.
7. Desale,G.and Nicholoson,W.K.(1981)Endoprimitive Ring, J.of Algebra, 70: 548-560.
8. Mohamed,S.H. and Muller,B.J. (1990)Continuous and Discrete Modules, Cambridge Univ. Press, Cambridge.
9. Wang, Y. (2007)δ-Small Submodules and δ-Supplemented Modules, International J. Math. And Mathematical Sciences, 1-8.
10. Kosan ,M.T. (2007) δ-lifting and δ-supplemented Modules, Algebra Colloguim, H (1), 53-60.
11. Ali, P.M.H. (2005) Hollow Modules and Semihollow Modules, M.Sc. Thesis, Univ. of Baghdad.
12. Lomp, C. and Büyükasik, E. (2009) when δ-Semiperfect Rings are Semiperfect, Turk.J.Math., 33, pp.1-8.
13. Wisbauer, R. (1991) Foundations of Modules and Rings Theory, Gordon and Breach Science Publisher Reading.

المقاسات المجوفه من النمط δ

إنعام محمد علي هادي
سميعه حسون عيدي
قسم الرياضيات / كلية التربية للعلوم الصرفة(ابن الهيثم) / جامعة بغداد

استلم البحث8 ايار 2014،قبل البحث في 22حزيران 2014

الخلاصة

لتكن R حلقة إبدالية ذات محايد وليكن M مقاساً غير صفري أيسر أحادي على R. يُدعى M مقاساً مجوفاً اذا كان كل مقاس جزئي فعلي N في M مقاساً جزئياً صغيراً $(N \ll M)$ ، هذا يعني $M \neq N + W$ لكل W مقاس جزئي فعلي في M. ندرس المقاس المجوف من النمط – δ،تعميماً للمقاس المجوف، إذ يُدعى M مقاساً مجوفاً من النمط – S اذا كان كل مقاس جزئي فعلي N في M مقاس جزئي صغير من النمط - δ $(N \ll_\delta M)$هذا يعني ان $M \neq N + W$ لكل مقاس جزئي فعلي W من M بحيث $\frac{M}{W}$ مقاس منفرد. ندرس في هذا العمل الصنف من المقاسات ونعطي العديد من الخواص الاساسية المتعلقة بهذا المفهوم.

الكلمات المفتاحية : مقاس جزئي صغير ، مقاس جزئي صغيرة من النمط δ، مقاس مجوف ، مقاس مجوف من النمط δ،
مقاس منفرد ، مقاس غير منفرد.