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Oscillation Criteria for Solutions of Neutral Differential Equations of the Second-Order Emden-Fowler Type with Forcing Term

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Abstract

In this paper, the oscillation property and the asymptotic behavior of solutions of neutral second-order differential Equations of the Emden-Fowler type were studied under the influence of the coefficients of forces. It has been shown through this research that the coefficients of forces in addition to the Emden-Fowler type have a major role on the oscillation of solutions of neutral Equations. As well as its effect on the convergence and divergence of nonoscillatory solutions. For this purpose, some conditions are obtained to ensure that all solutions of the neutral Equations Emden-Fowler type oscillating or nonoscillating go to ∞, as $t \to \infty$. Some of these conditions are the development of conditions similar to them in some of the well-known results included in the references, for example, condition (8) in this research with condition (4) in (9). The obtained results included some illustrative examples showing that the resulting conditions are easy to apply and guarantee oscillation.

Keywords: Oscillation Criteria, Asymptotic Behavior, Emden-Fowler Type, Neutral Second Order with Forcing Term.

1. Introduction

This paper aims to obtain sufficient conditions to ensure that every solution of the neutral force Equation of the second-order type Emden Fowler oscillates. Consider the Equation**:**

$$\left(\xi(t)\left(\omega'(t)\right)^{\gamma}\right)' + \sum_{i=1}^{n} q_i(t) x^{\gamma}\left(\delta_i(t)\right) \operatorname{sgn}(x) = \sum_{j=1}^{k} r_j(t). \tag{1}$$

$$\omega(t) = x(t) + p(t)x\left(\tau(t)\right). \tag{2}$$

The number γ is a quotient of odd positive integers. $\tau, \delta_i \in \mathrm{C}([t_0, \infty)_{\mathbb{T}}, R), i = 1,2,\dots,n$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \delta_i(t) = \infty$, $\mathrm{sgn}(x) = \pm 1$ if $x \gtrless 0$, $sgn(0) = 0$, $p \in \mathrm{C}([t_0, \infty), R^+)$ and $q_i, r_j \in \mathrm{C}([t_0, \infty), R), i = 1,2,\dots,n, j = 1,2,\dots,k$.
During this research, the following assumptions will be used as needed:

(M_1) $\limsup_{t\to\infty} \int_{t_0}^{t} \left(\frac{1}{\xi(s)}\right)^{\frac{1}{\gamma}} ds = \infty;$

(M_2) $0 < p(t) \le a$, $\xi(t) > 0$

$Q_i(t) = \min_{t \ge t_0}\left\{ q_i(t), \ q_i\big(\tau(t)\big), Q(t) = \min\{Q_i(t), i = 1,2,\dots,n\right\}$, and

$G_i(t) = \max_{t \ge t_0}\left\{ r_i(t), r_i\big(\tau(t)\big), \ i = 1,2,\dots,k\right\}, G(t) = \max\{G_i(t), \ i = 1,2,\dots,k\}.$

(M_3) $\int_T^{\infty} |G(t)| dt < \infty, \ T \ge t_0.$

(M_4) $\int_T^{\infty} |Q(t)| dt = \infty, T \ge t_0$.

The Emden-Fowler Equation has emerged in recent decades as a focus of interest for many researchers, specifically research in oscillation and the asymptotic behavior of the solutions of these Equations, Emden-Fowler Equation has been classified as unconventional Equations and its importance has emerged for its use in many applications, and for this reason many researches have appeared that produce a lot of conditions to ensure that each solution of these Equations oscillates, or that their non-oscillating solutions are convergent. Ahmed *et al.* (1) studied the second-order neutral dynamic linear Equation and established some conditions for the oscillation of every solution of this Equation. Yingzhu *et al.* (2) obtained oscillation conditions of each solution of second-order neutral nonlinear differential Equations. The obtained results in (3) are based on comparison theorems which enable to address the problem of second order Equation oscillation to first order Equation oscillation. Mehta *et al.* (4) investigated the Emden-Fowler Equation of the form $\frac{d}{dt}\left(t^{\alpha} \frac{dw}{dt}\right) = t^{\sigma} w^{\alpha}$, where $w(t) = x(t) \pm q(t)x\big(\tau(t)\big)$, and established some conditions for all solutions to oscillate. Mohamad *et al.* (5, 6) discussed the oscillation property of third order neutral half-linear Equations and established some conditions to insure the oscillation of every solution of these Equations. Moaaz *et al*. (7) and Xu *et al*. (8) obtained oscillation conditions of each solution of second order neutral Emden-Fowler type $(a(t)[(x(t) - p(t)x(\tau(t)))']^{\gamma})' + q(t)x^{\gamma}\big(\sigma(t)\big) = 0$, $\mathrm{t} \ge t_0$. Thandapani *et al.* (9) studied asymptotic properties of the third order quasi-linear neutral functional differential Equation $(a(t)[(x(t) - p(t)x(\tau(t)))'']^{\gamma})' + q(t)x^{\gamma}\big(\sigma(t)\big) = 0$, by using the Riccati transformation, and establishing some conditions which ensure that every solution of that Equation is either oscillatory or converges to zero. Hassan *et al.* (10) he dealt with new standards for the oscillation of half-linear differential Equations developed of the second order, where he concluded that the results obtained work to expand and develop modern standards for the same Equations that have been developed by many authors. Tripathy *et al.* (11) find the necessary and sufficient conditions for volatility one of the impulsive neutral differential system solutions of the second order under certain conditions that ensure the occurrence of oscillation . See Mehta *et al.* (12), and Vidhyaa *et al.* (13) they studied the differential Equations and obtained the oscillation criteria for all the solutions of the neutral differential Equations of the second degree, half-linear $(k(t)((h(t)z'(t))')^{\varepsilon})' + k(t)x^{\varepsilon}(t) = 0, t \geq t_0$, where $z(t) = x(t) + p(t)x(\tau(t))$. Dassios *et al.* (14) he studied the delayed and neutral differential Equations, where he focused on the stability of the important joins because these joins contain delays in each of the state variables and their time derivatives, where the proposed approach consists of model transformation that builds an equivalent set of algebraic differential Equations. Li *et al.* (15) and Marappan *et al.* (16) the oscillatory behavior of solutions of mixed nonlinear neutral differential Equations of the Emden-Fowler type was studied by applying the integral conditions and the integral average method. Baty (17) he studied second-order Lane-Emden-Fowler differential Equations, third-order Emden-Fowler Equations, and fourth-order Lane-Emden-Fowler Equations. He presented numerical methods using neural networks based on physics with the aim of solving higher-order differential Equations. Naeif *et al.* (18) he studied the oscillation and asymptotic behavior of a half-linear three-dimensional neutral system of second order and gave sufficient conditions to ensure oscillation or not. See (19-22) they studied the optimal decomposition method for solving third-order nonlinear Emden-Fowler differential Equations, to avoid the singularity at x=0, by transforming the Emden-Fowler Equation into an integral Volterra Equation.

Our paper was based on article (9), where a more general Equation with a forcing term is used and condition (2) in (9) has been developed into a more general case. A solution $\boldsymbol{x(t)}$ is said to be oscillatory if it has arbitrarily large zeros on $(\boldsymbol{t_0}, \infty)$, otherwise it is said to be nonoscillatory that is either eventually positive or eventually negative (6).

2. Main Results

In this section some results .established for oscillation for every solution of Equation (1). In the beginning, it is shown that every non-oscillatory solution is achieves the following cases**.**

Lemma 2.1: Assume that $q_i(t) \geq 0, \sum_{j=1}^{k} r_j(t) \leq 0$, and (M_1) holds. Let $x(t)$ be a non- oscillatory solution of Equation (1). Then $\omega'(t) > 0$, and either $\lim_{t\to\infty} x(t) = \infty$, or $\lim_{t\to\infty} \xi(t)(\omega'(t))^{\gamma} = 0$.

Proof: Assume that $x(t)$ be eventually positive solution of Equation (1). From (1) it follows that $(\xi(t)(\omega'(t))^{\gamma})' \leq 0$, that is $\xi(t)(\omega'(t))^{\gamma}$ is non-increasing for $t \geq t_0$. we claim that $\xi(t)(\omega'(t))^{\gamma}$ is eventually positive, otherwise if $\xi(t)(\omega'(t))^{\gamma}$ is eventually negative then there is $\mu < 0$ and $t_1 \geq t_0$ such that $\xi(t)(\omega'(t))^{\gamma} \leq \mu < 0, t \geq t_1$, so it follows

$$\omega'(t) \leq \left(\frac{\mu}{\xi(t)}\right)^{\frac{1}{\gamma}}, \quad t \geq t_1. \tag{3}$$

Integrating (3) from t_1 to t we get

$$\omega(t) - \omega(t_1) \leq \mu^{\frac{1}{\gamma}} \int_{t_1}^{t} \left(\frac{1}{\xi(s)}\right)^{\frac{1}{\gamma}} ds. \tag{4}$$

Letting $t \to \infty$, then from inequality (4) yields $\lim_{t\to\infty} \omega(t) = -\infty$, a contradiction. Hence our claim verified and $\xi(t)(\omega'(t))^{\gamma} > 0$, that is $\omega'(t) > 0$, $t \geq t_1 \geq t_0$, then $\lim_{t\to\infty} \xi(t)(\omega'(t))^{\gamma} = l \geq 0$, thus $\xi(t)(\omega'(t))^{\gamma} \geq l$, $t \geq t_1$ or

$$\omega'(t) \geq \left(\frac{l}{\xi(t)}\right)^{\frac{1}{\gamma}}, \quad t \geq t_1. \tag{5}$$

Integrating (5) from t_1 to t, it follows $\omega(t)-\omega(t_1)\geq l^{\frac{1}{\gamma}}\int_{t_1}^{t}\left(\frac{1}{\xi(s)}\right)^{\frac{1}{\gamma}}ds$.

If $l>0$ then $\lim_{t\to\infty}\omega(t)=\infty$, implies that $\lim_{t\to\infty}x(t)=\infty$. If $l=0$ then $\lim_{t\to\infty}\xi(t)(\omega'(t))^{\gamma}=0$.

Lemma 2.2: Assume that $q_i(t)\leq 0,\sum_{j=1}^{k}r_j(t)\geq 0$, and (M_1) holds. Let $x(t)$ be a non-oscillatory solution of Equation (1). Then the following statements hold:

(a) $\omega'(t)>0$, and $\lim_{t\to\infty}x(t)=\infty$.

(b) $\omega'(t)<0$, and $\lim_{t\to\infty}\xi(t)(\omega'(t))^{\gamma}=0$.

Proof: Assume that $x(t)$ be eventually positive solution of Equation (1). From Equation (1) we get $(\xi(t)(\omega'(t))^{\gamma})'\geq 0$ that is $\xi(t)(\omega'(t))^{\gamma}$ is non-decreasing, we have two cases to consider:

1. $\xi(t)(\omega'(t))^{\gamma}>0$;
2. $\xi(t)(\omega'(t))^{\gamma}<0, t\geq t_1\geq t_0$.

Case 1: $\xi(t)(\omega'(t))^{\gamma}>0$ that is $\omega'(t)>0,\ t\geq t_1$ then there exist $\mu>0$ such that $\xi(t)(\omega'(t))^{\gamma}\geq\mu,\ t\geq t_2\geq t_1$

$$\omega'(t)\geq\left(\frac{\mu}{\xi(t)}\right)^{\frac{1}{\gamma}},\ \ t\geq t_2. \tag{6}$$

By integrating (6) from t_2 to t we get $\omega(t)-\omega(t_2)\geq\mu^{\frac{1}{\gamma}}\int_{t_2}^{t}\left(\frac{1}{\xi(s)}\right)^{\frac{1}{\gamma}}ds$.

As $t\to\infty$ it follows $\lim_{t\to\infty}\omega(t)=\infty$, which implies that $\lim_{t\to\infty}x(t)=\infty$.

Case 2: $\xi(t)(\omega'(t))^{\gamma}<0$ that is $\omega'(t)<0, t\geq t_1$ and $\lim_{t\to\infty}\xi(t)(\omega'(t))^{\gamma}=l\leq 0$. we claim that $l=0$ otherwise $l<0$ thus

$$\omega'(t)\leq\left(\frac{l}{\xi(t)}\right)^{\frac{1}{\gamma}},\ \ t\geq t_2\geq t_1\,. \tag{7}$$

By integrating Equation (7) from t_2 to t we get $\omega(t)-\omega(t_2)\leq l^{\frac{1}{\gamma}}\int_{t_2}^{t}\left(\frac{1}{\xi(s)}\right)^{\frac{1}{\gamma}}ds$.

As $t\to\infty$ it follows that $\lim_{t\to\infty}\omega(t)=-\infty$, a contradiction. Then $l=0$.

Theorem 2.1. Assume that $q_i(t)\geq 0,\sum_{j=1}^{k}r_j(t)\leq 0$, $(M_1)-(M_4)$ hold, and for any continuous functions $u(t),v(t),\ uv>0$, there exists $\lambda>0$, such that

$$u^{\gamma}(t)+v^{\gamma}(t)\geq\lambda\big(u(t)+v(t)\big)^{\gamma}. \tag{8}$$

Then every solution of Equation (1) oscillates.

Proof. Assume that Equation (1) has a non-oscillatory solution $x(t)$. For lack of prolongation and repetition, it can be assumed that $x(t)>0,x\big(\tau(t)\big)>0,x\big(\delta_i(t)\big)>0,i=1,2,\dots,n$, for $t\geq t_0$. Let $u(t)=x(t),v(t)=p(t)\,x(\tau(t)),\ \eta(t)=\max\big\{x(t),x\big(\tau(t)\big)\big\}$, then

$(x(t)+p(t)x(\tau(t)))^{\gamma}\leq(x(t)+a\,x(\tau(t)))^{\gamma}\leq(\eta(t)+a\,\eta(t))^{\gamma}=\eta^{\gamma}(t)(1+a)^{\gamma}$.

Since $\eta(t)=\max\big\{x(t),x\big(\tau(t)\big)\big\}$ so there exists $\varepsilon>0$, such that

$x^{\gamma}(t)+p^{\gamma}(t)x^{\gamma}(\tau(t))\geq\varepsilon\eta^{\gamma}(t)=\frac{\varepsilon(1+a)^{\gamma}}{(1+a)^{\gamma}}\eta^{\gamma}(t)=\frac{\varepsilon}{(1+a)^{\gamma}}(\eta(t)+a\,\eta(t))^{\gamma}\geq\frac{\varepsilon}{(1+a)^{\gamma}}(x(t)+p(t)x\big(\tau(t)\big))^{\gamma}$,

Choose $\lambda=\frac{\varepsilon}{(1+a)^{\gamma}}>0$. Hence,

$$x^{\gamma}(t)+p^{\gamma}(t)x^{\gamma}(\tau(t)) \geq \lambda(x(t)+p(t)x(\tau(t)))^{\gamma} = \lambda\omega^{\gamma}(t). \tag{9}$$

From Equation (1) it follows that:

$$(\xi(t)(\omega'(t))^{\gamma})'+\sum_{i=1}^{n} q_i(t)(x(\delta_i(t))^{\gamma}+a^{\gamma}[\xi(\tau(t))(\omega'(\tau(t)))^{\gamma}]' + a^{\gamma}\sum_{i=1}^{n} q_i(\tau(t))(x(\delta_i(\tau(t)))^{\gamma} = \sum_{j=1}^{k}(r_j(t)\ + a^{\gamma} r_j(\tau(t)) \tag{10}$$

$$(\xi(t)(\omega'(t))^{\gamma})'+a^{\gamma}[\xi(\tau(t))(\omega'(\tau(t)))^{\gamma}]'+Q(t)\sum_{i=1}^{n}[x^{\gamma}(\delta_i(t)\ + a^{\gamma}x^{\gamma}(\delta_i(\tau(t))] - \sum_{j=1}^{k}(r_j(t)\ + a^{\gamma} r_j(\tau(t)) \leq 0$$

$$(\xi(t)(\omega'(t))^{\gamma})'+a^{\gamma}[\xi(\tau(t))(\omega'(\tau(t)))^{\gamma}]' + Q(t)\sum_{i=1}^{n}\left[x^{\gamma}(\delta_i(t))+p^{\gamma}(\delta_i(t))x^{\gamma}(\delta_i(\tau(t))\right] - G(t)\sum_{j=1}^{k}(1+\ a^{\gamma}) \leq 0$$

by using(9) the last inequality yields:

$$(\xi(t)(\omega'(t))^{\gamma})'+a^{\gamma}[\xi(\tau(t))(\omega'(\tau(t)))^{\gamma}]'+\lambda Q(t)\sum_{i=1}^{n}\omega^{\gamma}(\delta_i(t)) - kG(t)(1+a^{\gamma}) \leq 0$$

Let $\delta(t) = \min_{t\geq t_1}\{\delta_i(t), i = 1,2 \dots, n\}$, by lemma 2.1, $\omega(t)$ is positive and increasing so there exist a constant $b > 0$, and $t_2 \geq t_1$ such that $\omega(t) \geq b,\ t \geq t_2$. Hence the last inequality leads to:

$$(\xi(t)(\omega'(t))^{\gamma})'+a^{\gamma}[\xi(\tau(t))(\omega'(\tau(t)))^{\gamma}]'+n\lambda Q(t)\omega^{\gamma}(\delta(t)) - kG(t)(1+a^{\gamma}) \leq 0, \tag{11}$$

$$n\lambda Q(t)b^{\gamma} \leq -(\xi(t)(\omega'(t))^{\gamma})' - a^{\gamma}[\xi(\tau(t))(\omega'^{(\tau(t)))^{\gamma}}]' + kG(t)(1+a^{\gamma}), \tag{12}$$

Consequently, by integrating (12) from t_2 to t yields

$$n\lambda b^{\gamma}\int_{t_2}^{t} Q(s)\,ds \leq -\int_{t_2}^{t}(\xi(s)(\omega'(s))^{\gamma})'\,ds - a^{\gamma}\int_{t_2}^{t}[\xi(\tau(s))(\omega'(s))^{\gamma}]'\,ds + k(1+a^{\gamma})\int_{t_2}^{t}G(s)ds$$

$$n\lambda b^{\gamma}\int_{t_2}^{t} Q(s)\,ds \leq \xi(t_2)(\omega'(t_2))^{\gamma} + a^{\gamma}\xi(\tau(t_2))[\omega'^{(\tau(t_2))]^{\gamma}} + k(1+a^{\gamma})\int_{t_2}^{t}G(s)\,ds, \tag{13}$$

Hence by (M_3) it follows from (13) $\int_{t_2}^{\infty} Q(s)\,ds < \infty$, contradicts (M_4).

Theorem 2.2. Assume that $q_i(t) \leq 0, i = 1,2,\dots,n, \sum_{j=1}^{k} r_i(t) \geq 0, \xi'(t) > 0$ on $[t_0,\infty)$. Let $(M_1)-(M_4)$ hold, and for any continuous functions$u(t), v(t),\ uv > 0$, there exists $\lambda > 0$, such that (8) holds, in addition to the condition

$$\limsup_{t\to\infty}\int_{t}^{\alpha(t)}\left[\frac{1}{\xi(s)}\int_{s}^{\alpha(s)}\sum_{i=1}^{n}|q_i(v)|[1-p(\delta_i(v))]^{\gamma}\,dv\right]^{\frac{1}{\gamma}} ds > 1. \tag{14}$$

Then every solution of Equation (1) oscillates.

Proof. Assume that Equation (1) has eventually positive solution $x(\mathrm{t})$, that is $x(t)>0,\ x(\tau(t))>0,\ x(\delta_i(t))>0,\ i=1,2,\dots,n$. From Equation (1) we get $(\xi(t)(\omega'(t))^\gamma)'\geq 0$, based on Lemma 2.2, there are two cases that need to be investigated:

(a) $\omega'(t)>0$, and $\lim_{t\to\infty} x(t)=\infty$.

(b) $\omega'(t)<0$, and $\lim_{t\to\infty} \xi(t)(\omega'(t))^\gamma=0$.

Case (a) proceeding as in the proof of theorem 2.1, we conclude that (11) holds.

Letting

$$z(t)=\xi(t)(\omega'(t))^\gamma. \tag{15}$$

Then $z(t)$ is positive and non-decreasing, hence (11) becomes:

$$z'(t)+a^\gamma\left[z(\tau(t))\right]'+n\lambda Q(t)\omega^\gamma(\delta(t))<kG(t)(1+a^\gamma), \text{for } t\geq t_2 \tag{16}$$

$\omega(t)$ is positive and increasing so there exist a constant $b>0$, and $t_3\geq t_2$ such that $\omega(t)\geq b$, $t\geq t_3$. Therefor (16) reduce to

$$z'(t)+a^\gamma\left[z(\tau(t))\right]'+n\lambda Q(t)b^\gamma<kG(t)(1+a^\gamma), \text{for } t\geq t_3 \tag{17}$$

Integration (17) from t_3 to t, where t is sufficiently large t_3, leads to

$$\int_{t_3}^{t} z'(s)ds+a^\gamma\int_{t_3}^{t}(z(\tau(s)))'ds+n\lambda b^\gamma\int_{t_3}^{t}Q(s)ds<k(1+a^\gamma)\int_{t_3}^{t}G(s)ds$$

Since $z(t)$ is non-decreasing, then the last inequality becomes:

$$z(t)-z(t_3)+a^\gamma z(\tau(t))-a^\gamma z(\tau(t_3))+n\lambda b^\gamma\int_{t_3}^{t}Q(s)\,ds<k\,(1+a^\gamma)\int_{t_3}^{t}G(s)ds,$$

There fore

$$-z(t_3)^\gamma-a^\gamma z(t_3)+n\lambda b^\gamma\int_{t_3}^{t}Q(s)\,ds<k\,(1+a^\gamma)\int_{t_3}^{t}G(s)ds, \tag{18}$$

As $t\to\infty$ a contradiction will be got in (18).

Case (b) in this case

$\omega(t)>0, \omega'(t)<0, \lim_{t\to\infty}\xi(t)(\omega'(t))^\gamma=0, (\xi(t)(\omega'(t))^\gamma)'\geq 0$

Since $\xi'(t)>0$ and $\omega(t)$ is positive decreasing, so it can be conclude that $\omega''(t)\geq 0$, for $t\geq t_2$, $\omega(t)>x(t),\ x(t)=\omega(t)-p(t)x(\tau(t)), x(\delta_i(t))=\omega(\delta_i(t))-p(\delta_i(t))x(\tau(\delta_i(t)))$

So Equation (1) become

$$\left(\xi(t)(\omega'(t))^\gamma\right)'+\sum_{i=1}^{n}q_i(t)[\omega(\delta_i(t))-p(\delta_i(t))x(\tau(\delta_i(t)))]^\gamma=\sum_{j=1}^{k}r_j(t). \tag{19}$$

By integrating (19) from t to $\alpha(t)$, where $\alpha(t)>t$ and $\tau\left(\delta_j\left(\alpha(\alpha(t))\right)\right)<t$,

$\delta_j(t)=\min\{\delta_i(t), i=1,2,\dots,n\}$, We get $-\xi(t)(\omega'(t))^\gamma\geq-\int_t^{\alpha(t)}\sum_{i=1}^{n}q_i(s)[\omega(\delta_i(t))-p(\delta_i(t))\omega(\tau(\delta_i(t)))]^\gamma\,ds$,

$$-\xi(t)\big(\omega'(t)\big)^{\gamma} \geq -\int_{t}^{\alpha(t)} \sum_{i=1}^{n} q_i(s)\omega^{\gamma}(\tau(\delta_i(t)))[1-p(\delta_i(t))]^{\gamma}\, ds,$$

$$\omega'(t) \leq \omega\left(\tau\left(\delta_j(\alpha(t))\right)\right)\left[\frac{1}{\xi(t)}\int_{t}^{\alpha(t)} \sum_{i=1}^{n} q_i(s)[1-p(\delta_i(t))]^{\gamma}\, ds\right]^{\frac{1}{\gamma}}. \tag{20}$$

Where $\tau(\delta_j(t)) = \min\{\tau(\delta_i(t)), i = 1,2,\dots,n]$, integrating (20) from t to $\alpha(t)$ we get

$$\omega(\alpha(t)) - \omega(t) \leq \omega(\tau(\delta_j(\alpha(\alpha(t))))) \int_{t}^{\alpha(t)} \left[\frac{1}{\xi(s)}\int_{s}^{\alpha(s)} \sum_{i=1}^{n} q_i(v)[1-p(\delta_i(v))]^{\gamma}\, dv\right]^{\frac{1}{\gamma}} ds,$$

$$1 \geq \frac{\omega(t)}{\omega\left(\tau\left(\delta_j\left(\alpha(\alpha(t))\right)\right)\right)} \geq -\int_{t}^{\alpha(t)} \left[\frac{1}{\xi(s)}\int_{s}^{\alpha(s)} \sum_{i=1}^{n} q_i(v)[1-p(\delta_i(v))]^{\gamma}\, dv\right]^{\frac{1}{\gamma}} ds.$$

The last inequality contradicts the condition (14), thus case not valid also, hence every solution of Equation (1) oscillates. The proof is complete.

3. Examples

In this section, two examples are given to illustrate the fulfillment of all necessary and sufficient conditions for the results presented in the previous section.

Example 3.1. Consider the following Emden-Fowler Equation:

$$\left[x(t) + \frac{1}{2}x(t-\pi)\right]'' = -\frac{1}{2}x(t-2\pi) - \frac{1}{4}, \quad t \geq 0. \tag{21}$$

Where $\xi(t) = 1$, $p(t) = \frac{1}{2}$, $\tau(t) = t - \pi$, $q_1(t) = Q(t) = \frac{1}{2}$, $r_1(t) = -\frac{1}{2}$, and $\delta(t) = t - 2\pi$, $i = 1,2,\dots,n, \gamma = 1$. In reality $\mathrm{M}_1 - \mathrm{M}_3$ are hold for every $t \geq t_0 = 0$. And

$$\int_{t_0}^{\infty} Q(s)ds = \int_{0}^{\infty} ds = \infty.$$

Then (M_4) is holds for every $t \geq \frac{1}{2}$. Recall that (8) hold for $\lambda = 1$. Hence all the conditions of Theorem 2.1 satisfy that according to Theorem 2.1, each solution of Equation (1) oscillates for example $x(t) = \sin t - \frac{1}{2}$ such as this oscillation solution. See **Figure (1)**

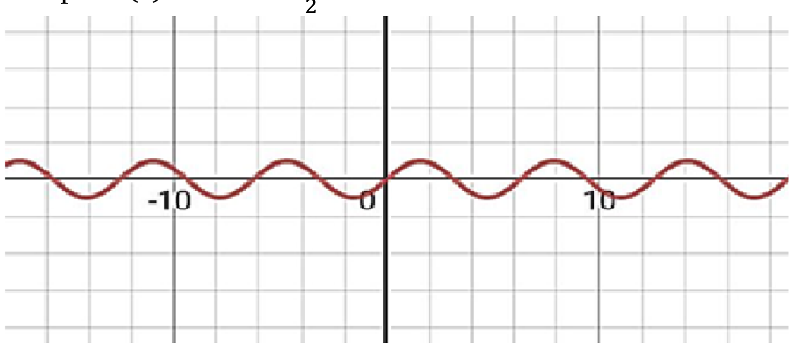

Figure 1. $x(t) = \sin t - \frac{1}{2}$

Example 3. 2. Consider the following Emden-Fowler Equation:

$$[x(t)+2e^{-\pi}x(t-\pi)]''-4e^{-\frac{3\pi}{2}}x\left(t-\frac{3\pi}{2}\right)=e^{-t},\ t\geq 0. \tag{22}$$

Where $\xi(t)=1$, $p(t)=2e^{-\pi}$, $\tau(t)=t-\pi$, $r(t)=e^{-t}$, $q_1(t)=Q(t)=-4e^{-\frac{3\pi}{2}}$, and $\delta(t)=t-\frac{3\pi}{2}$, $\gamma=1$. In reality $\mathrm{M}_1-\mathrm{M}_3$ are hold for every $t\geq t_0=0$. And

$$\int_{t_0}^{\infty}|Q(s)|ds=\int_0^{\infty}4e^{-\frac{3\pi}{2}}ds=\infty.$$

Then (M_4) is holds for every $t\geq 0$. Recall that (8) hold for $\lambda=1$. Hence all the conditions of Theorem 2.2 satisfy that according to Theorem 2.2, each solution of Equation (1) oscillates, for example $x(t)=e^{-t}(\sin t-1)$ is such an oscillatory solution. See **Figure (2)**

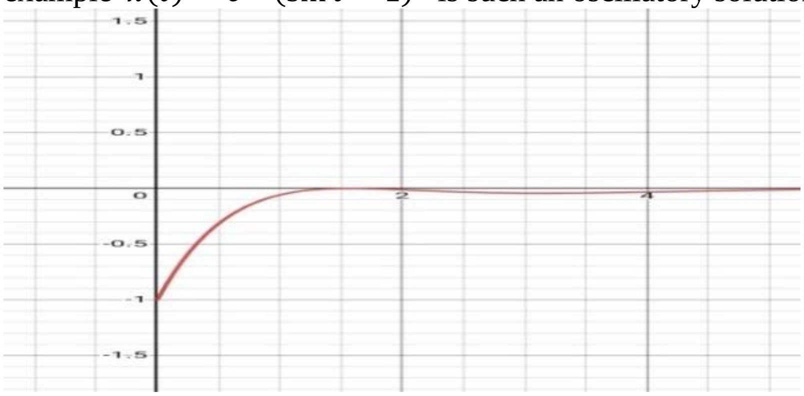

Figure 2. $\varphi(t)=e^{-t}(\sin t-1)$

4. Conclusion

In this paper, we have studied the oscillation property of the solutions of neutral second-order differential Equations of the Emden-Fowler type. Some of the extracted conditions are the development of conditions known in the references, which ensure that either each solution of this Equation oscillates, or each nonoscillatory solution convergence to zero or tends to infinity as $t\to\infty$. Some examples are presented to clarify the results obtained.

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Conflict of Interest

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Ethical Clearance

Ethics of scientific research were carried out in accordance with international conditions.

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