Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2025, 38(2)

The Local Bifurcation of Food Chain Prey-Predator Model with Crowley-Martin-Type of Functional Response

Asmaa Abdulhussein Aziz[1*] and Azhar Abbas Majeed [2]
[1,2]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
*Corresponding Author.

Received: 4 June 2023 Accepted: 26 February 2024 Published: 20 April 2025
doi.org/10.30526/38.2.3553

Abstract

The conditions under which the local bifurcation including (saddle-node, transcritical, and pitchfork) of all stable points in a model of a food chain occur are examined in this paper. The Growly-Martin model and the emotion of fear have both been important factors in the development of a functional reaction in a food chain model, It has been shown that a transcritical bifurcation and a pitchfork bifurcation can be discovered near to each of the sites $A1$, A2, $A3 and\ A4$, and that a saddle-node bifurcation can be found close to the point where positive equilibrium is located. In conclusion, a numerical simulation was run in order to illustrate how the proposed model may display bifurcation in its behavior. This was done in order to show how the impact of parameters on the dynamics of the proposed model.

Keywords: Transcritical pitchfork**,** Sotomayor's theorem, Local bifurcation, Global bifurcation.

1. Introduction

Bifurcation theory is the scientific study of how a moving system's structure changes over time. Changes that can be qualitative or visual in integral family curves, vector fields, differential equation solutions. Bifurcation happens when a small, smooth change in the values of a system's parameters (bifurcation parameters) causes a sudden shift in its "specific" or "topological" behavior. This term is used a lot when mathematicians study systems that change over time (1-5). There are two major categories of bifurcations: local and global. Local bifurcations like the saddle node, transcritical pitchfork, period-doubling flip, Hopf, and Neimark secondary Hopf bifurcation, which can all be studied purely by analyzing how their local stability properties change, happen as parameters cross critical thresholds. Global bifurcation happens when stability and bigger invariant sets, like periodic orbits, collide. Contrary to local bifurcations, this alters the structure of the pathways in phase space in a non-local manner. Topological shifts are widespread because they can have an impact on areas over incredibly large distances; for instance, the homoclinic (6-8).Pamuk and Cay (9) looked at the steady state with respect to Hopf bifurcation of a feedback diffusion method, which is used to control how vascular cells and inhibitors talk to each other. Mukherjee and Maji (10)

established the criteria for local bifurcations for all equilibrium points and Hopf for positive equilibrium points in a prey-predator model with prey haven. While Majeed and Ali (11) presented a model of bifurcation with refuge that included a predator-structured stage food chain. In recent years, many researchers have studied the importance of bifurcation. Majeed and Alabacy (12) found the conditions for local bifurcation at all equilibrium points a prey-predator-refuge-harvesting model. Since then, lots of researchers have considered the fear effect of predators in the dynamical study on prey-predator models (13-14). Many other academics have studied the local bifurcation in recent years, including (15-19).
Finally, in this work, the occurrence of the local bifurcation (LB) of the proposed system have been discussed.

2. Materials and Methods

2.1. Model formulation

Consider the following system that given in (20).

$$\begin{aligned}
\frac{dR_1}{dt} &= \frac{r_1R_1}{1+K_1R_2} - m_1R_1^2 - \frac{\beta_1R_1R_2}{(1+a_1R_1)(1+a_2R_2)} = f_1(R_1\,.R_2\,.R_3).\\
\frac{dR_2}{dt} &= \frac{r_2R_2}{1+K_2R_3} - m_2R_2^2 + \frac{l_1R_1R_2}{(1+a_1R_1)(1+a_2R_2)} - \frac{\beta_2R_2R_3}{(1+a_3R_2)(1+a_4R_3)} - d_1R_2 = f_2\,(R_1\,.R_2\,.R_3) \qquad (1)\\
\frac{dR_3}{dt} &= \frac{l_2R_2R_3}{(1+a_3R_2)(1+a_4R_3)} - d_2R_3 = f_3\,(R_1\,.R_2\,.R_3).
\end{aligned}$$

Following is a table describing the positive parameters of system (1).

Table 1. The system's parameters (1):

Parameters	Parameters Description
$R_I I. = 1.2.3$	THE DENSITY OF PREY . INTERMEDIATE PREDATOR AND TOP – PREDATOR AT TIME T. RESPECTIVELY
$R_I I. = 1.2$	THE INTRINSIC GROWTH RATE OFTHE PREY AND THE INTERMEDIATE PREDATOR RESPECTIVELY
$M_{I.} I = 1.2$	THE RATE OF INTERNAL COMPETITION FOR THE PREY AND THE INTERMEDIATE PREDATOR RESPECTIVELY
$K_I. I = 1.2$	THE RATE OF FEAR OF PREY AND INTERMEDIATE PREDATOR RESPECTIVELY
$B_I. I = 1.2$	ATTACK RATE FOR PREY AND INTERMEDIATE PREDATOR RESPECTIVELY
$L_I. I = 1.2$	FOOD TRANSFER RATE FOR INTERMEDIATE PREDATOR AND TOP PREDATOR RESPECTIVELY
$A_I. I = 1.3$	HANDLING TIME OF PREY AND INTERMEDIATE PREDATOR RESPECTIVELY
$A_I. I = 2.4$	MAGNITUDE OF DISTRUBANCE AMONG INTERMEDIATE PREDATOR AND TOP PREDATOR RESPECTIVELY
$D_I. I = 1.2$	THE NATURAL DEATH RATE OF INTERMEDIATE PREDATOR AND TOP PREDATOR RESPECTIVELY

2.2. Local bifurcation analysis

In this subsection, the local bifurcation of the model (1) has been examined, with a particular emphasis on the changes that occur around each equilibrium point as a result of changes in the parameter values governing the dynamic behavior. With the assistance of Sotomayor's theorem, our mission is to develop higher order conditions with the intention of ensuring the appearance of the most typical of the area's local bifurcations.
Now, according to Jacobean matrix $J(R_1.R_2.R_3)$ of the system (1) which is given in (16) as follows:

$$J = [u_{ij}]_{3\times3} \qquad (2)$$

where:

$$u_{11} = \frac{r_1}{1+K_1R_2} - 2m_1R_1 - \frac{\beta_1R_2}{(1+a_1R_1)^2(1+a_2R_2)}\text{‘}\ u_{12} - \left(\frac{r_1R_1K_1}{(1+K_1R_2)^2} + \frac{\beta_1R_1}{(1+a_1R_1)(1+a_2R_2)^2}\right)\text{‘}\ u_{13} = 0\text{‘}$$

$$u_{21} = \frac{l_1R_2}{(1+a_1R_1)^2(1+a_2R_2)}\text{‘}\ u_{22} = \frac{r_2}{1+K_2R_3} - 2m_2R_2 + \frac{l_1R_1}{(1+a_1R_1)(1+a_2R_2)^2} - \frac{\beta_2R_3}{(1+a_3R_2)^2(1+a_4R_3)}d_1\text{‘}$$

$$u_{23} = -\frac{r_2R_2K_2}{(1+K_2R_3)^2} - \frac{\beta_2R_2}{(1+a_3R_2)(1+a_4R_3)^2}\text{‘}\ u_{31} = 0\text{‘}u_{32} = \frac{l_2R_3}{(1+a_3R_2)^2(1+a_4R_3)}\text{‘}$$

$u_{33} = \frac{l_2 R_2}{(1+a_3R_2)(1+a_4R_3)^2} - d_2.$

For any non- zero vector T $=(t_1.t_2.t_3)^T$:

$D^2F_\mu(X.\mu)(T.T) = [Y_{i1}]_{3\times1}.$ (3)

$Y_{11} = 2(\left(\frac{\beta_1R_2a_1}{(1+a_1R_1)^3(1+a_2R_2)} - m_1\right){t_1}^2 - \left(\frac{r_1K_1}{(1+K_1R_2)^2} + \frac{\beta_1}{(1+a_1R_1)^2(1+a_2R_2)^2}\right)t_1t_2 + \left(\frac{r_1{K_1}^2}{(1+K_1R_2)^3} + \frac{\beta_1a_2}{(1+a_1R_1)(1+a_2R_2)^3}\right){t_2}^2R_1$,

$Y_{21} = -2\left(\frac{l_1R_2a_1}{(1+a_1R_1)^3(1+a_2R_2)}{t_1}^2 - \frac{l_1}{(1+a_1R_1)^2(1+a_2R_2)^2}t_1t_2 - (\frac{r_2{K_2}^2R_2}{(1+K_2R_3)^3} + \frac{\beta_2\,a_4R_2}{(1+a_3R_2)(1+a_4R_3)^3}){t_3}^2 + \left(m_2 + \frac{l_1R_1a_2}{(1+a_1R_1)(1+a_2R_2)^3} - \frac{\beta_2R_3a_3}{(1+a_3R_2)^3(1+a_4R_3)}\right){t_2}^2 + \left(\frac{r_2K_2}{(1+K_2R_3)^2} + \frac{\beta_2}{(1+a_3R_2)^2(1+a_4R_3)^2}\right)t_2t_3\right)$,

$Y_{31} = -2(\frac{l_2R_3a_3}{(1+a_3R_2)^3(1+a_4R_3)}{t_2}^2 - \frac{l_2}{(1+a_3R_2)^2(1+a_4R_3)^2}t_2t_3 + \frac{l_2R_2a_4}{(1+a_3R_2)(1+a_4R_3)^3}{t_3}^2)$,

and $D^3F_\mu(X.\mu)(T.T.T) = [W_{i1}]_{3\times1}$, (4)

$W_{11} = 6\left[\left(\frac{r_1{K_1}^2}{(1+K_1R_2)^3} + \frac{\beta_1a_2}{(1+a_1R_1)^2(1+a_2R_2)^3}\right)t_1{t_2}^2 + \left(\frac{t_2}{1+a_2R_2} - \frac{R_2a_1t_1}{1+a_1R_1}\right)\frac{\beta_1a_1{t_1}^2}{(1+a_1R_1)^3(1+a_2R_2)} - \left(\frac{r_1{K_1}^3R_1}{(1+K_1R_2)^4} + \frac{\beta_1{a_2}^2R_1}{(1+a_1R_1)(1+a_2R_2)^4}\right){t_2}^3\right]$.

$W_{21} = 6[\left(\frac{r_2{K_2}^2}{(1+K_2R_3)^3} + \frac{\beta_2\,a_4}{(1+a_3R_2)^2(1+a_4R_3)^3}\right)t_2{t_3}^2 + \left(\frac{R_2a_1t_1}{1+a_1R_1} - \frac{t_2}{1+a_2R_2}\right)\frac{l_1a_1{t_1}^2}{(1+a_1R_1)^3(1+a_2R_2)} + \left(\frac{l_1R_1{a_2}^2}{(1+a_1R_1)(1+a_2R_2)^4} - \frac{\beta_2R_3{a_3}^2}{(1+a_3R_2)^4(1+a_4R_3)}\right){t_2}^3 + \left(\frac{\beta_2\,a_4t_3}{(1+a_3R_2)^3(1+a_4R_3)^2} - \frac{l_1a_2t_1}{(1+a_1R_1)^2(1+a_2R_2)^3}\right){t_1}^2 - \left(\frac{r_2{K_2}^3R_2}{(1+K_2R_3)^4} + \frac{\beta_1{a_2}^2R_1}{(1+a_1R_1)(1+a_2R_2)^4}\right){t_3}^3\right]$,

$W_{31} = 6\left[\left(\frac{a_3R_3t_2}{(1+a_3R_2)} - \frac{t_3}{(1+a_4R_3)}\right)\frac{l_2a_3{t_2}^2}{(1+a_3R_2)^3(1+a_4R_3)} + \left(\frac{a_4R_2t_3}{(1+a_4R_3)} - \frac{t_2}{(1+a_3R_2)}\right)\frac{l_2a_4{t_3}^2}{(1+a_3R_2)(1+a_4R_3)^3}\right]$,

where $X = (R_1.R_2.R_3)$ and (μ) any parameter.

Theorem (1): System (1) with the parameter value $d`_1 = d_1 = r_2 + \frac{l_1r_1}{m_1+a_1r_1}$, has a transcritical and pitchfork bifurcation at $A_1 = (\frac{r_1}{m_1}.0.0)$, if the next conditions hold:

$Z`_1 \neq Z`_2,$ (5)

$Z`_3 \neq Z`_4.$ (6)

Where:

$Z`_1 = \left(\frac{l_1N}{(1+a_1R_1)^2(1+a_2R_2)^2} + \frac{\beta_2R_3a_3}{(1+a_3R_2)^3(1+a_4R_3)}\right),$

$Z`_2 = \left(\frac{l_1R_2a_1N^2}{(1+a_1R_1)^3(1+a_2R_2)} + m_2 + \frac{l_1R_1a_2}{(1+a_1R_1)(1+a_2R_2)^3}\right),$

$Z`_3 = \left(\frac{l_1R_2{a_1}^2N^3}{(1+a_1R_1)^4(1+a_2R_2)} + \frac{l_1R_1{a_2}^2}{(1+a_1R_1)(1+a_2R_2)^4}\right),$

$Z`_4 = \left(\frac{l_1a_1N^2}{(1+a_1R_1)^3(1+a_2R_2)^2} + \frac{l_1a_2N}{(1+a_1R_1)^2(1+a_2R_2)^3} + \frac{\beta_2R_3{a_3}^2}{(1+a_3R_2)^4(1+a_4R_3)}\right).$

Proof: By using the Jacobian matrix in equation (1.7) in (20) $J`_1 = J_1(A_1.d`_1) = \left[q`_{ij}\right]_{3\times3}$, where $q`_{ij} = q_{ij}$, except $q`_{22} = 0$ .Then, the characterizing equation for$J`_1$ has an eigenvalue of zero, which is λ_{1R_2} at $d`_1 = d_1$,

Now, let $T^{[1]} = (t_1^{[1]}.t_2^{[1]}.t_3^{[1]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{1R_2} = 0$. Thus,$(J`_1 - \lambda_{1R_2}I)T^{[1]} = 0.$ this gives:

$t_1^{[1]} = Nt_2^{[1]}.t_3^{[1]} = 0. where\ N = -\left(\frac{r_1K_1}{m_1} + \frac{\beta_1}{m_1+a_1r_1}\right)$and $t_2^{[1]}$any real number that is not zero.

Let $M^{[1]} = (m_1^{[1]}.m_2^{[1]}.m_3^{[1]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{1R_2} = 0$ of the matrix $J`_1^T$.Then. $\left(J`_1^T - \lambda_{1R_2} I\right) M^{[1]} = 0$.

By solving this equation for. $M^{[1]} = \left(0.m_2^{[1]}.0\right)^T$.where $m_2^{[1]}$ any real number that is not zero.

Now consider this: $\frac{\partial f}{\partial d_1} = f_{d_1}(X.d_1) = \left(\frac{\partial f_1}{\partial d_1}.\frac{\partial f_2}{\partial d_1}.\frac{\partial f_3}{\partial d_1}\right)^T = (0.-R_2.0)^T$.

So, $f_{d_1}(A_1.d`_1) = (0.0.0)^T$ and hence $\left(M^{[1]}\right)^T f_{d_1}(A_1.d`_1) = 0$

Using Sotomayor's theorem, it is impossible to satisfy the saddle-node bifurcation condition. The first condition for transcritical bifurcation is therefore satisfied. Now

$$Df_{d_1}(X.d_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where, $Df_{d_1}(X.d_1)$ represents the derivative of $f_{d_1}(X.d_1)$ with respect to $X = (R_1.R_2.R_3)^T$. Furthermore, it is observed that:

$$Df_{d_1}(A_1.d_1)T^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Nt_2^{[1]} \\ t_2^{[1]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -t_2^{[1]} \\ 0 \end{bmatrix}$$

$$\left(M^{[1]}\right)^T \left[Df_{d_1}(A_1.d`_1)T^{[1]}\right] = \left(0.m_2^{[1]}.0\right)^T \left(0.-t_2^{[1]}.0\right) = -t_2^{[1]} m_2^{[1]} \neq 0.$$

By substituting $T^{[1]}$ in equation (3) we get:

$$D^2F_\mu(A_1.d`_1)\left(T^{[1]}.T^{[1]}\right) = \left(Y`_{ij}\right). \tag{7}$$

$Y`_{11} = 2{t_2}^2[\left(\frac{R_2 a_1 N^2}{(1+a_1R_1)^2} - \frac{N}{(1+a_1R_1)(1+a_2R_2)} + \frac{R_1 a_2}{(1+a_2R_2)^2}\right)\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)} + \left(\frac{K_1R_1}{(1+K_1R_2)} - N\right)\frac{r_1K_1}{(1+K_1R_2)^2} - m_1N^2],$

$Y`_{21} = 2{t_2}^2[\left(\frac{N}{(1+a_1R_1)(1+a_2R_2)} - \frac{R_1a_2}{(1+a_2R_2)^2} - \frac{l_1R_2a_1N^2}{(1+a_1R_1)^2}\right)\frac{l_1}{(1+a_1R_1)(1+a_2R_2)} - m_2 + \frac{\beta_2R_3a_3}{(1+a_3R_2)^3(1+a_4R_3)}],$

$Y`_{31} = \frac{-2{t_2}^2 l_2R_3a_3}{(1+a_3R_2)^3(1+a_4R_3)}$,

Hence, it was obtained

$$\left(M^{[1]}\right)^T\left[D^2F_\mu(A_1.d`_1)\left(T^{[1]}.T^{[1]}\right)\right] = 2\left(t_2^{[1]}\right)^2 m_2^{[1]}(Z`_1 - Z`_2) \neq 0.$$

This indicates that the system (1) exhibits a transcritical bifurcation at A_1 with a parameter $d`_1 = d_1$, If condition (5) not satisfied then.

By substituting $T^{[1]}$ in equation (4) we get:

$$D^3F_\mu(A_2.d`_1)\left(T^{[1]}.T^{[1]}.T^{[1]}\right) = \left(W`_{ij}\right), \tag{8}$$

$W`_{11} = 6{t_2}^3[(N - \frac{K_1R_1}{(1+K_1R_2)})\frac{r_1{K_1}^2}{(1+K_1R_2)^3} + (\frac{a_1N^2}{(1+a_1R_1)^2(1+a_1R_1)} + \frac{a_2N}{(1+a_1R_1)(1+a_2R_2)^2} - \frac{R_2{a_1}^2N^3}{(1+a_1R_1)^3} - \frac{{a_2}^2R_1}{(1+a_2R_2)^2})\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)}],$

$W`_{21} = 6{t_2}^3[(\frac{R_2{a_1}^2N^3}{(1+a_1R_1)^3} - \frac{a_1N^2}{(1+a_1R_1)^2(1+a_2R_2)} - \frac{a_2N}{(1+a_1R_1)(1+a_2R_2)^2} + \frac{R_1{a_2}^2}{(1+a_2R_2)^3})\frac{l_1}{(1+a_1R_1)(1+a_2R_2)} - \frac{\beta_2R_3{a_3}^2}{(1+a_3R_2)^4(1+a_4R_3)}],$

$W`_{31} = \frac{6{t_2}^3 l_2R_3{a_3}^2}{(1+a_3R_2)^4(1+a_4R_3)},$

$$\left(M^{[1]}\right)^T\left[D^3F_\mu(A_2.d`_1)\left(T^{[1]}.T^{[1]}.T^{[1]}\right)\right] = 6\left(t_2^{[1]}\right)^3 m_2^{[1]}(Z`_3 - Z`_4) \neq 0$$

Which grantee that there is pitch fork bifurcation at A_1 where $d`_1 = d_1$.

Theorem (2): System (1) with the parameter value $\bar{d}_2 = d_2 = \frac{l_2\bar{R}_2}{1+a_3\bar{R}_2}$,has a transcritical and pitchfork bifurcation at $A_2 = (0.\bar{R}_2.0)$. if the next conditions hold:

$\bar{Z}_1 \neq \bar{Z}_2$. (9)

$\bar{Z}_3 \neq \bar{Z}_4$. (10)

Where:

$\bar{Z}_1 = \frac{l_2\Delta}{(1+a_3R_2)^2(1+a_4R_3)^2}$, $\bar{Z}_2 = (\frac{l_2R_3a_3\Delta^2}{(1+a_3R_2)^3(1+a_4R_3)} + \frac{l_2R_2a_4}{(1+a_3R_2)(1+a_4R_3)^3})$,

$\bar{Z}_3 = \frac{l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{R_3{a_3}^2\Delta^2}{(1+a_3R_2)^3} + \frac{R_2{a_4}^2}{(1+a_4R_3)^3}\right)$, $\bar{Z}_4 = \frac{l_2\Delta}{(1+a_3R_2)^2(1+a_4R_3)^2}(\frac{a_4}{(1+a_4R_3)} + \frac{a_3\Delta}{(1+a_3R_2)})$.

Proof: By using the Jacobian matrix in equation (1.8) in (20) $\bar{J}_2 = J_2(A_2.\bar{d}_2) = [\bar{a}_{ij}]_{3\times 3}$, where $\bar{a}_{ij} = a_{ij}$, except $\bar{a}_{33} = 0$.Then, the characterizing equation for$\bar{J}_2$ has an eigenvalue of zero, which is λ_{2R_3} at $\bar{d}_2 = d_2$.

Now, let $T^{[2]} = (t_1^{[2]}.t_2^{[2]}.t_3^{[2]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{2R_2} = 0$. Thus,$(\bar{J}_2 - \lambda_{2R_2}I)T^{[2]} = 0$. this gives:$t_1^{[2]} = 0.\ t_2^{[2]} = \Delta t_3^{[2]}$.

where :

$\Delta = -\left(\frac{r_2R_2K_2}{r_2-d_1} + \frac{\beta_2R_2}{(1+a_3R_2)(r_2-d_1)}\right)$, *and* $t_3^{[2]}$ any real number that is not zero.

Let $M^{[2]} = (m_1^{[2]}.m_2^{[2]}.m_3^{[2]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{2R_2} = 0$ of the matrix $\bar{J}_2^T$.Then. $(\bar{J}_2^T - \lambda_{2R_2}I)\,M^{[2]} = 0$.

By solving this equation for . $M^{[2]} = \left(0.0.m_3^{[2]}\right)^T$. and $m_3^{[2]}$ any real number that is not zero.

Now consider this: $\frac{\partial f}{\partial d_2} = f_{d_2}(X.d_2) = \left(\frac{\partial f_1}{\partial d_2}.\frac{\partial f_2}{\partial d_2}.\frac{\partial f_3}{\partial d_2}\right)^T = (0.0.-R_3)^T$.

So, $f_{d_2}(A_2.\bar{d}_2) = (0.0.0)^T$ and hence $(M^{[2]})^T f_{d_2}(A_2.\bar{d}_2) = 0$.

Using Sotomayor's theorem, it is impossible to satisfy the saddle-node bifurcation condition. The first condition for transcritical bifurcation is therefore satisfied. Now

$$Df_d(X.d_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

where, $Df_{d_2}(X.d_2)$ represents the derivative of $f_{d_2}(X.d_2)$ with respect to $X = (R_1.R_2.R_3)^T$. Furthermore, it is observed that:

$$Df_{d_2}(A_2.d_2)T^{[2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ \Delta t_3^{[2]} \\ t_3^{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -t_3^{[2]} \end{bmatrix}$$

$$(M^{[2]})^T[Df_{d_2}(A_2.\bar{d}_2)T^{[2]}] = \left(0.0.m_3^{[2]}\right)\left(0.0.-t_3^{[2]}\right)^T = -m_3^{[2]}t_3^{[2]} \neq 0$$

By substituting $T^{[2]}$*in* equation (3) we get:

$D^2F_\mu(A_2.\bar{d}_2)(T^{[2]}.T^{[2]}) = (\bar{Y}_{ij})$. (11)

$\bar{Y}_{11} = 2\Delta^2R_1t_3^{[2]}\left(\frac{r_1{K_1}^2}{(1+K_1R_2)^3} + \frac{\beta_1a_2}{(1+a_1R_1)(1+a_2R_2)^3}\right)$,

$\bar{Y}_{21} = 2\,t_3^{[2]}[(\frac{R_3a_3\Delta^2}{(1+a_3R_2)^2} + \frac{a_4R_2}{(1+a_4R_3)^2} - \frac{\Delta}{(1+a_1R_1)(1+a_2R_2)})\frac{\beta_2}{(1+a_1R_1)(1+a_2R_2)} - \frac{l_1R_1a_2\Delta^2}{(1+a_1R_1)(1+a_2R_2)^3} +$
$(\Delta - \frac{K_2R_2}{(1+K_2R_3)})\frac{r_2K_2}{(1+K_2R_3)^2} - m_2\Delta^2]$,

$\bar{Y}_{31} = 2t_3^{[2]}\frac{l_2}{(1+a_1R_1)(1+a_2R_2)}(\frac{1}{(1+a_1R_1)(1+a_2R_2)} - \frac{R_3a_3\Delta^2}{(1+a_3R_2)^2} - \frac{R_2a_4}{(1+a_4R_3)^2})$,

Hence, it was obtained.

$\left(M^{[2]}\right)^T\left[D^2F_\mu\left(A_2.\bar{d}_2\right)\left(T^{[2]}.T^{[2]}\right)\right] = 2\left(t_3^{[2]}\right)^2 m_3^{[2]}(\bar{Z}_1 - \bar{Z}_2) \neq 0.$

This indicates that the system (1) exhibits a transcritical bifurcation at A_2 with a parameter $\bar{d}_2 = d_2$. If condition (9) not satisfied, then.

By substituting $T^{[2]}$ in equation (4) we obtain:

$D^3F_\mu\left(A_2.\bar{d}_2\right)\left(T^{[2]}.T^{[2]}.T^{[2]}\right) = \left(\bar{W}_{ij}\right).$ (12)

$\bar{W}_{11} = -6\Delta^2\left(t_3^{[2]}\right)^3\left(\frac{\beta_1 a_2{}^2 R_1}{(1+a_1R_1)(1+a_2R_2)^4} + \frac{r_1K_1{}^3R_1}{(1+K_1R_2)^4}\right),$

$\bar{W}_{21} = 6\left(t_3^{[2]}\right)^3\left[\left(\Delta - \frac{K_2R_2}{(1+K_2R_3)}\right)\frac{r_2K_2{}^2}{(1+K_2R_3)^3} + \left(\frac{l_1\Delta^3}{(1+a_2R_2)^3} - \frac{\beta_1}{(1+a_2R_2)^3}\right)\frac{a_2{}^2R_1}{(1+a_1R_1)(1+a_2R_2)} + \left(\frac{a_4\Delta}{(1+a_3R_2)(1+a_4R_3)} + \frac{a_4}{(1+a_4R_3)^2} - \frac{R_3a_3{}^2\Delta^2}{(1+a_3R_2)^2}\right)\frac{\beta_2\Delta}{(1+a_3R_2)^2(1+a_4R_3)}\right],$

$\bar{W}_{31} = \frac{6\left(t_3^{[2]}\right)^3 l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{R_3a_3{}^2\Delta^2}{(1+a_3R_2)^3} + \frac{R_2a_4{}^2}{(1+a_4R_3)^3} - \frac{a_4\Delta}{(1+a_3R_2)(1+a_4R_3)^2} - \frac{a_3\Delta^2}{(1+a_3R_2)^2(1+a_4R_3)}\right).$

$\left(M^{[2]}\right)^T\left[D^3F_\mu\left(A_2.\bar{d}_2\right)\left(T^{[2]}.T^{[2]}.T^{[2]}\right)\right] = 6\left(t_3^{[2]}\right)^3 m_3^{[2]}(\bar{Z}_3 - \bar{Z}_4) \neq 0$

Which grantee that there is pitch fork bifurcation at A_2 where $\bar{d}_2 = d_2$.

Theorem (3): System (1) with the parameter value $\hat{d}_2 = d_2 = \frac{l_2\hat{R}_2}{1+a_3\hat{R}_2}$, has a transcritical and pitchfork bifurcation at $A_3 = (\hat{R}_1.\hat{R}_2.0)$, if the next conditions hold:

$\hat{Z}_1 \neq \hat{Z}_2.$ (13)

$\hat{Z}_3 \neq \hat{Z}_4.$ (14)

Where:

$\hat{Z}_1 = \frac{l_2\emptyset_2}{(1+a_3R_2)^2(1+a_4R_3)^2}$ ، $\hat{Z}_2 = \frac{l_2R_3a_3\emptyset_2{}^2}{(1+a_3R_2)^3(1+a_4R_3)} + \frac{l_2R_2a_4}{(1+a_3R_2)(1+a_4R_3)^3})$ ،

$\hat{Z}_3 = \frac{l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{R_3a_3{}^2\emptyset_2{}^3}{(1+a_3R_2)^3} + \frac{R_2a_4{}^2}{(1+a_4R_3)^3}\right)$ ، $\hat{Z}_4 = \frac{l_2\emptyset_2}{(1+a_3R_2)^2(1+a_4R_3)^2}\left(\frac{a_4}{(1+a_4R_3)} + \frac{a_3\emptyset_2}{(1+a_3R_2)}\right).$

Proof: By using the Jacobian matrix inequation (1.9) in (20) $\hat{J}_3 = J_3\left(A_3.\hat{d}_2\right) = \left[\hat{d}_{ij}\right]_{3\times3}$, where $\hat{d}_{ij} = d_{ij}$, except $\hat{d}_{33} = 0$ .Then, the characterizing equation for $\hat{J}_3$ has an eigenvalue of zero, which is λ_{3R_3} at $\hat{d}_2 = d_2$.

Now, let $T^{[3]} = (t_1^{[3]}.t_2^{[3]}.t_3^{[3]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{3R_3} = 0$. Thus,$\left(\hat{J}_3 - \lambda_{3R_3}I\right)T^{[3]} = 0.$ this gives: $t_1^{[3]} = \emptyset_1 t_3^{[3]}.t_2^{[3]} = \emptyset_2 t_3^{[3]}$.

where: $\emptyset_1 = \frac{d_{23}d_{12}}{d_{11}d_{22}-d_{12}d_{21}}$ ، $\emptyset_2 = \frac{d_{23}d_{11}}{d_{12}d_{21}-d_{11}d_{22}}$.

and $t_3^{[3]}$ any real number that is not zero.

Let $M^{[3]} = (m_1^{[3]}.m_2^{[3]}.m_3^{[3]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{3R_3} = 0$ of the matrix $\hat{J}_3^T$.Then. $\left(\hat{J}_3^T - \lambda_{3R_3}I\right)M^{[3]} = 0$.

By solving this equation for . $M^{[3]} = \left(0.0.m_3^{[3]}\right)^T$,and $m_3^{[3]}$ any real number that is not zero.

Now consider this:$\frac{\partial f}{\partial d_2} = f_{d_2}(X.d_2) = \left(\frac{\partial f_1}{\partial d_2}.\frac{\partial f_2}{\partial d_2}.\frac{\partial f_3}{\partial d_2}\right)^T = (0.0.-R_3)^T$.

So, $f_{d_2}\left(A_3.\hat{d}_2\right) = (0.0.0)^T$and hence $\left(M^{[3]}\right)^T f_{d_2}\left(A_3.\hat{d}_2\right) = 0.$

Using Sotomayor's theorem, it is impossible to satisfy the saddle-node bifurcation condition. The first condition for transcritical bifurcation is therefore satisfied. Now

$$Df_{d_2}(X.d_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

where, $Df_{d_2}(X.d_2)$ represents the derivative of $f_{d_2}(X.d_2)$ with respect to $X=(R_1.R_2.R_3)^T$. Furthermore, it is observed that:

$$Df_{d_2}(A_3.d_2)T^{[3]}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}\emptyset_1 t_3^{[3]}\\ \emptyset_2 t_3^{[3]}\\ t_3^{[3]}\end{bmatrix}=\begin{bmatrix}0\\0\\-t_3^{[3]}\end{bmatrix}$$

$$\left(M^{[3]}\right)^T\left[Df_{d_2}\left(A_3.\hat{d}_2\right)T^{[3]}\right]=\left(0.0.m_3^{[3]}\right)\left(0.0.-t_3^{[3]}\right)^T=-m_3^{[3]}t_3^{[3]}\neq 0$$

By substituting $T^{[3]}$ in equation (3) we get:

$$D^2F_\mu\left(A_3.\hat{d}_2\right)\left(T^{[3]}.T^{[3]}\right)=\left(\hat{Y}_{ij}\right). \tag{15}$$

$$\hat{Y}_{11}=2\left(t_3^{[3]}\right)^2[\left(\frac{R_2a_1{\emptyset_1}^2}{(1+a_2R_2)^2}-\frac{\emptyset_1\emptyset_2}{(1+a_1R_1)(1+a_2R_2)}+\frac{a_2R_2{\emptyset_2}^2}{(1+a_2R_2)^2}\right)\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)}+\left(\frac{K_1R_1{\emptyset_1}^2}{(1+K_1R_2)}-\emptyset_1\emptyset_2\right)\frac{r_1K_1}{(1+K_1R_2)^2}-m_1{\emptyset_1}^2],$$

$$\hat{Y}_{21}=2\left(t_3^{[3]}\right)^2[\left(\frac{\emptyset_1\emptyset_2}{(1+a_1R_1)(1+a_2R_2)}-\frac{R_2a_1{\emptyset_1}^2}{(1+a_1R_1)^2}-\frac{R_1a_2{\emptyset_2}^2}{(1+a_2R_2)^2}\right)\frac{l_1}{(1+a_1R_1)(1+a_2R_2)}+\left(\frac{a_3R_3{\emptyset_2}^2}{(1+a_3R_2)^2}+\frac{\emptyset_2}{(1+a_3R_2)(1+a_4R_3)}+\frac{a_4R_2}{(1+a_4R_3)^2}\right)\frac{\beta_2}{(1+a_3R_2)(1+a_4R_3)}-m_2{\emptyset_2}^2+\left(\frac{K_2R_2}{(1+K_2R_3)}-\emptyset_2\right)\frac{r_2K_2}{(1+K_2R_3)^2}],$$

$$\hat{Y}_{31}=\frac{2\left(t_3^{[3]}\right)^2l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{\emptyset_2}{(1+a_3R_2)(1+a_4R_3)}-\frac{R_3a_3{\emptyset_2}^2}{(1+a_3R_2)^2}-\frac{R_2a_4}{(1+a_4R_3)^2}\right).$$

Hence, it was obtained

$$\left(M^{[3]}\right)^T\left[D^2F_\mu\left(A_3.\hat{d}_2\right)\left(T^{[3]}.T^{[3]}\right)\right]=2\left(t_3^{[3]}\right)^2m_3^{[3]}\left(\hat{Z}_1-\hat{Z}_2\right)\neq 0.$$

This indicates that the system (1) exhibits a transcritical bifurcation at A_2 with a parameter $\hat{d}_2=d_2$. If condition (13) not satisfied then.

By substituting $T^{[3]}$ in equation (4) we get:

$$D^3F_\mu\left(A_3.\hat{d}_2\right)\left(T^{[3]}.T^{[3]}.T^{[3]}\right)=\left(\widehat{W}_{ij}\right), \tag{16}$$

$$\widehat{W}_{11}=6\left(t_3^{[3]}\right)^3\left[\left(\frac{a_2\emptyset_1{\emptyset_2}^2}{(1+a_1R_1)(1+a_2R_2)^2}-\frac{{\emptyset_2}^3{a_2}^2R_1}{(1+a_2R_2)^3}-\frac{R_2{a_1}^2{\emptyset_1}^3}{(1+a_1R_1)^3}\frac{a_1{\emptyset_1}^2\emptyset_2}{(1+a_1R_1)^2(1+a_2R_2)}\right)\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)}+\left(\emptyset_1-\frac{K_1R_1\emptyset_1}{(1+K_1R_2)}\right)\frac{r_1{K_1}^2{\emptyset_2}^2}{(1+K_1R_2)^3}\right],$$

$$\widehat{W}_{21}=6\left(t_3^{[3]}\right)^3[\left(\emptyset_2-\frac{K_2R_2}{(1+K_2R_3)}\right)\frac{r_2{K_2}^2}{(1+K_2R_3)^3}-\frac{\beta_1{a_2}^2R_1}{(1+a_1R_1)(1+a_2R_2)^4}+\left(\frac{R_2{a_1}^2{\emptyset_1}^3}{(1+a_1R_1)^3}+\frac{a_1{\emptyset_1}^2\emptyset_2}{(1+a_1R_1)^2(1+a_2R_2)}-\frac{a_2\emptyset_1{\emptyset_2}^2}{(1+a_1R_1)(1+a_2R_2)^2}+\frac{R_1{a_2}^2{\emptyset_2}^3}{(1+a_2R_2)^3}\right)\frac{l_1}{(1+a_1R_1)(1+a_2R_2)}+\left(\frac{a_4\emptyset_2}{(1+a_3R_2)(1+a_4R_3)^2}+\frac{a_4{\emptyset_2}^2}{(1+a_3R_2)(1+a_4R_3)^2}-\frac{R_3{a_3}^2{\emptyset_2}^3}{(1+a_3R_2)^3}\right)\frac{\beta_2 I}{(1+a_3R_2)^2(1+a_4R_3)}],$$

$$\widehat{W}_{31}=\frac{6\left(t_3^{[3]}\right)^3l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{R_3{a_3}^2{\emptyset_2}^3}{(1+a_3R_2)^3}+\frac{R_2{a_4}^2}{(1+a_4R_3)^3}-\frac{a_4\emptyset_2}{(1+a_3R_2)(1+a_4R_3)^2}-\frac{a_3{\emptyset_2}^2}{(1+a_3R_2)^2(1+a_4R_3)}\right).$$

Hence,

$$\left(M^{[3]}\right)^T\left[D^3F_\mu\left(A_3.\hat{d}_2\right)\left(T^{[3]}.T^{[3]}.T^{[3]}\right)\right]=6\left(t_3^{[3]}\right)^3m_3^{[3]}\left(\hat{Z}_3-\hat{Z}_4\right)\neq 0.$$

By condition (14). Which grantee that there is pitch fork bifurcation at A_3 where $\bar{d}_2=d_2$.

Theorem (4): System (1) with the parameter value $\bar{\bar{r}}_1=r_1=\frac{\beta_1\bar{\bar{R}}_2(1+K_1\bar{\bar{R}}_2)}{(1+a_2\bar{\bar{R}}_2)}$, has a transcritical and pitchfork bifurcation at $A_4=(0.\bar{\bar{R}}_2.\bar{\bar{R}}_3)$, if the next conditions hold:

$$\bar{\bar{Z}}_1\neq\bar{\bar{Z}}_2. \tag{17}$$

$$\bar{\bar{Z}}_3\neq\bar{\bar{Z}}_4. \tag{18}$$

Where:

$$\bar{\bar{Z}}_1=\frac{r_1{K_1}^2{I_1}^2}{(1+K_1R_2)^3}+\frac{\beta_1a_2{I_1}^2}{(1+a_1R_1)(1+a_2R_2)^3}+\frac{\beta_1R_2a_1}{(1+a_1R_1)(1+a_2R_2)^3},$$

$\bar{\bar{Z}}_2 = \left(m_1 + \frac{r_1K_1I_1}{(1+K_1R_2)^2} + \frac{\beta_1I_1}{(1+a_1R_1)^2(1+a_2R_2)^2}\right),$

$\bar{\bar{Z}}_3 = (\frac{r_1K_1{}^2}{(1+K_1R_2)^3}I_1{}^2 + \frac{\beta_1a_1}{(1+a_1R_1)^3(1+a_2R_2)^2}I_1 + \frac{\beta_1a_2}{(1+a_1R_1)^2(1+a_2R_2)^3}I_1{}^2 - \frac{\beta_1R_2a_1{}^2}{(1+a_1R_1)^4(1+a_2R_2)}),$

$\bar{\bar{Z}}_4 = (\frac{\beta_1R_2a_1{}^2}{(1+a_1R_1)^4(1+a_2R_2)} + \frac{r_1K_1{}^3R_1}{(1+K_1R_2)^4}I_1{}^3 + \frac{\beta_1a_2{}^2R_1}{(1+a_1R_1)(1+a_2R_2)^4}I_1{}^3).$

Proof: By using the Jacobian matrix from equation (1.10) in (20) $\bar{\bar{J}}_4 = J_4(A_4.\bar{\bar{r}}_1) = [\bar{\bar{e}}_{ij}]_{3\times3}$, where $\bar{\bar{e}}_{ij} = e_{ij}$. except $\bar{\bar{e}}_{11} = 0$ .Then, the characterising equation for $\bar{\bar{J}}_4$ has an eigenvalue of zero, which is $\lambda_{4R_1} = 0$ at $\bar{\bar{r}}_1 = r_1$.

Now, let $T^{[4]} = (t_1^{[4]}.t_2^{[4]}.t_3^{[4]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{4R_1} = 0$. Thus,$(\bar{\bar{J}}_4 - \lambda_{4R_1}I)T^{[4]} = 0.$ this gives:

$t_2^{[4]} = I_1t_1^{[4]}.t_3^{[4]} = I_2t_1^{[4]}.$

where:

$I_1 = \frac{e_{21}e_{33}}{e_{23}e_{32}-e_{22}e_{33}}, \ I_2 = \frac{e_{32}e_{21}}{e_{33}e_{22}-e_{23}e_{32}}$.

and $t_1^{[4]}$ any real number that is not zero.

Let $M^{[4]} = (m_1^{[4]}.m_2^{[4]}.m_3^{[4]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{4R_1} = 0$ of the matrix $\bar{\bar{J}}_4^T$.Then. $(\bar{\bar{J}}_4^T - \lambda_{4R_1}I)\,M^{[4]} = 0$.

By solving this equation for . $M^{[4]} = \left(m_1^{[4]}.0.0\right)^T$ and $m_1^{[4]}$ any real number that is not zero.

Now consider this: $\frac{\partial f}{\partial r_1} = f_{r_1}(X.r_1) = \left(\frac{\partial f_1}{\partial r_1}.\frac{\partial f_2}{\partial r_1}.\frac{\partial f_3}{\partial r_1}\right)^T = \left(\frac{R_1}{1+K_1R_2}.0.0\right)^T$.

So. $f_{r_1}(A_4.\bar{\bar{r}}_1) = (0.0.0)^T$ and hence $(M^{[4]})^Tf_{r_1}(A_4.\bar{\bar{r}}_1) = 0.$

Using Sotomayor's theorem, it is impossible to satisfy the saddle-node bifurcation condition. The first condition for transcritical bifurcation is therefore satisfied. Now

$$Df_{r_1}(X.r_1) = \begin{bmatrix} \frac{1}{1+K_1\bar{\bar{R}}_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

where, $Df_{r_1}(X.r_1)$ represents the derivative of $f_{r_1}(X.r_1)$ with respect to $X = (R_1.R_2.R_3)^T$. Furthermore, it is observed that:

$$Df_{r_1}(A_4.r_1)T^{[4]} = \begin{bmatrix} \frac{1}{1+K_1R_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} t_1^{[4]} \\ I_1t_1^{[4]} \\ I_2t_1^{[4]} \end{bmatrix} = \begin{bmatrix} \frac{t_1^{[4]}}{1+K_1\bar{\bar{R}}_2} \\ 0 \\ 0 \end{bmatrix}$$

$\left(M^{[4]}\right)^T[Df_{r_1}(A_4.\bar{\bar{r}}_1)T^{[4]}] = \left(m_1^{[4]}.0.0\right)\left(\frac{t_1^{[4]}}{1+K_1\bar{\bar{R}}_2}.0.0\right)^T = \frac{m_1^{[4]}t_1^{[4]}}{1+K_1\bar{\bar{R}}_2} \neq 0,$

By substituting $T^{[4]}$ in equation (3) we get:

$$D^2F_\mu(A_4.\bar{\bar{r}}_1)(T^{[4]}.T^{[4]}) = (\bar{\bar{Y}}_{ij}), \quad (19)$$

$\bar{\bar{Y}}_{11} = 2\left(t_1^{[4]}\right)^2[\left(\frac{R_2a_1I_1{}^2}{(1+a_2R_2)^2} - \frac{I_1I_2}{(1+a_1R_1)(1+a_2R_2)} + \frac{a_2R_2I_2{}^2}{(1+a_2R_2)^2}\right)\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)} - m_1I_1{}^2 + \left(\frac{K_1R_1I_1{}^2}{(1+K_1R_2)} - I_1I_2\right)\frac{r_1K_1}{(1+K_1R_2)^2}],$

$\bar{\bar{Y}}_{21} = 2\left(t_1^{[4]}\right)^2[\left(\frac{I_1I_2}{(1+a_1R_1)(1+a_2R_2)} - \frac{R_2a_1I_1{}^2}{(1+a_1R_1)^2} - \frac{R_1a_2I_2{}^2}{(1+a_2R_2)^2}\right)\frac{l_1}{(1+a_1R_1)(1+a_2R_2)} + \left(\frac{K_2R_2}{(1+K_2R_3)} - I_2\right)\frac{r_2K_2}{(1+K_2R_3)^2} + \left(\frac{a_3R_3I_2{}^2}{(1+a_3R_2)^2} + \frac{I_2}{(1+a_3R_2)(1+a_4R_3)} + \frac{a_4R_2}{(1+a_4R_3)^2}\right)\frac{\beta_2}{(1+a_3R_2)(1+a_4R_3)} - m_2I_2{}^2],$

$\bar{\bar{Y}}_{31} = \frac{2(t_1^{[4]})^2\, l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{l_2}{(1+a_3R_2)(1+a_4R_3)} - \frac{R_3a_3I_2{}^2}{(1+a_3R_2)^2} - \frac{R_2a_4}{(1+a_4R_3)^2}\right).$

Hence, it was obtained

$\left(M^{[4]}\right)^T\left[D^2F_\mu(A_4.\bar{\bar{r}}_1)\left(T^{[4]}.T^{[4]}\right)\right] = 2\left(t_1^{[4]}\right)^2 m_1^{[4]}(Z_1 - Z_2) \neq 0$.

This indicates that the system (1) exhibits a transcritical bifurcation at A_4 with a parameter $\bar{\bar{r}}_1 = r_1$, and no pitch fork bifurcation at A_4 where $\bar{\bar{r}}_1 = r_1$.

By substituting $T^{[4]}$ in equation (4) we get:

$D^3F_\mu(A_4.\bar{\bar{r}}_1)\left(T^{[4]}.T^{[4]}.T^{[4]}\right) = \left(\bar{\bar{W}}_{ij}\right),$ (20)

$\bar{\bar{W}}_{11} = 6\left(t_1^{[4]}\right)^3\Big[\Big(\frac{a_2I_1I_2{}^2}{(1+a_1R_1)(1+a_2R_2)^2} - \frac{I_2{}^3a_2{}^2R_1}{(1+a_2R_2)^3} - \frac{R_2a_1{}^2I_1{}^3}{(1+a_1R_1)^3} + \frac{a_1I_1{}^2I_2}{(1+a_1R_1)^2(1+a_2R_2)}\Big)\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)} + \left(I_1 - \frac{K_1R_1I_1}{(1+K_1R_2)}\right)\frac{r_1K_1{}^2I_2{}^2}{(1+K_1R_2)^3}\Big],$

$\bar{\bar{W}}_{21} = 6\left(t_1^{[4]}\right)^3\Big[\left(I_2 - \frac{K_2R_2}{(1+K_2R_3)}\right)\frac{r_2K_2{}^2}{(1+K_2R_3)^3} - \frac{\beta_1a_2{}^2R_1}{(1+a_1R_1)(1+a_2R_2)^4} + \Big(\frac{R_2a_1{}^2I_1{}^3}{(1+a_1R_1)^3} + \frac{a_1I_1{}^2I_2}{(1+a_1R_1)^2(1+a_2R_2)} - \frac{a_2I_1I_2{}^2}{(1+a_1R_1)(1+a_2R_2)^2} + \frac{R_1a_2{}^2I_2{}^3}{(1+a_2R_2)^3}\Big)\frac{l_1}{(1+a_1R_1)(1+a_2R_2)} + \Big(\frac{a_4I_2}{(1+a_3R_2)(1+a_4R_3)^2} + \frac{a_4I_2{}^2}{(1+a_3R_2)(1+a_4R_3)^2} - \frac{R_3a_3{}^2I_2{}^3}{(1+a_3R_2)^3}\Big)\frac{\beta_2I}{(1+a_3R_2)^2(1+a_4R_3)}\Big],$

$\bar{\bar{W}}_{31} = 6\left(t_1^{[4]}\right)^3\frac{l_2}{(1+a_3R_2)(1+a_4R_3)}\Big(\frac{R_3a_3{}^2I_2{}^3}{(1+a_3R_2)^3} + \frac{R_2a_4{}^2}{(1+a_4R_3)^3} - \frac{a_4I_2}{(1+a_3R_2)(1+a_4R_3)^2} - \frac{a_3I_2{}^2}{(1+a_3R_2)^2(1+a_4R_3)}\Big).$

Hence,

$\left(M^{[4]}\right)^T\left[D^3F_\mu(A_4.\bar{\bar{r}}_1)\left(T^{[4]}.T^{[4]}.T^{[4]}\right)\right] = 6\left(t_1^{[4]}\right)^3 m_1^{[4]}(\bar{\bar{Z}}_3 - \bar{\bar{Z}}_4) \neq 0$

If condition (17). Which grantee that there is pitch fork bifurcation at A_4 where $\bar{\bar{r}}_1 = r_1$.

Theorem (5): Suppose that conditions (1.11b), (1.11e) and, reverse condition (1.11c) in (17) with the following conditions are satisfied:

$r_{11}r_{22} < r_{12}r_{21}.$ (21)

$\tilde{Z}_1 \neq \tilde{Z}_2.$ (22)

Where,

$\tilde{Z}_1 = \frac{l_2\gamma_2}{(1+a_3R_2)^2(1+a_4R_3)^2}$ ‘$\tilde{Z}_2 = \left(\frac{l_2R_3a_3\gamma_2{}^2}{(1+a_3R_2)^3(1+a_4R_3)} + \frac{l_2R_2a_4}{(1+a_3R_2)(1+a_4R_3)^3}\right)$‘ $\gamma_2 = \frac{-\tilde{r}_{33}}{\tilde{r}_{32}} < 0.$

Then the system near $A_5 = \left(\tilde{R}_1.\tilde{R}_2.\tilde{R}_3\right)$. has a saddle–node bifurcation at

$\tilde{d}_2 = d_2 = \frac{\tilde{r}_{11}\tilde{r}_{32}\tilde{r}_{23}}{\tilde{r}_{11}\tilde{r}_{22} - \tilde{r}_{12}\tilde{r}_{21}}$.

Proof: By using the Jacobian matrix in equation (1.11) in (20) $\tilde{J}_5 = J_5\left(A_5.\tilde{d}_2\right) = \left[\tilde{r}_{ij}\right]_{3\times3}$, where $\tilde{r}_{ij} = r_{ij}$. except $\tilde{r}_{33} = \frac{l_2\tilde{R}_2}{(1+a_3\tilde{R}_2)(1+a_4\tilde{R}_3)^2} - \tilde{d}_2$ .Then, the characterising equation for $\tilde{J}_5$ has an eigenvalue of zero, which is (say $\lambda_{5R_3} = 0$) if and if $\rho_3 = 0$.

Now, let $T^{[5]} = (t_1^{[5]}.t_2^{[5]}.t_3^{[5]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{5R_3} = 0$. Thus,$\left(\tilde{J}_5 - \lambda_{5R_3}I\right)T^{[5]} = 0$. this gives: $t_1^{[5]} = \gamma_1 t_3^{[5]}$. $t_2^{[5]} = \gamma_2 t_3^{[5]}$. and $t_3^{[5]}$ any real number that is not zero. $And\ \gamma_1 = \frac{\tilde{r}_{12}\tilde{r}_{33}}{\tilde{r}_{11}\tilde{r}_{32}}$.

Let $M^{[5]} = (m_1^{[5]}.m_2^{[5]}.m_3^{[5]})^T$ be the eigenvector associated with an eigenvalue $\lambda_{5R_3} = 0$ of the matrix $\tilde{J}_5^T$.Then . $\left(\tilde{J}_5^T - \lambda_{5R_3}I\right)M^{[5]} = 0$.

By solving this equation for . $M^{[5]} = \left(\gamma_3m_3^{[5]}.\gamma_4m_3^{[5]}.m_3^{[5]}\right)^T$. where $m_3^{[5]}$ any real number that is not zero, and $\gamma_3 = \frac{\tilde{r}_{21}\tilde{r}_{33}}{\tilde{r}_{11}\tilde{r}_{23}} > 0.$ $\gamma_4 = \frac{-\tilde{r}_{33}}{\tilde{r}_{23}} < 0.$

Now consider this: $\frac{\partial f}{\partial d_2} = f_{d_2}(X.d_2) = \left(\frac{\partial f_1}{\partial d_2}.\frac{\partial f_2}{\partial d_2}.\frac{\partial f_3}{\partial d_2}\right)^T = (0.0.-R_3)^T$.

So. $f_{d_2}(A_5.\tilde{d}_2) = (0.0.-\tilde{R}_3)^T$ and hence $(M^{[5]})^T f_{d_2}(A_5.\tilde{d}_2) = -\tilde{R}_3 m_3^{[5]} \neq 0$.

By substituting $T^{[5]}$ in equation (3) we get:

$$D^2F_\mu(A_5.\tilde{d}_2)(T^{[5]}.T^{[5]}) = (\tilde{Y}_{ij}). \tag{23}$$

$$\tilde{Y}_{11} = 2\left(t_3^{[5]}\right)^2 \left[\left(\frac{K_1R_1\gamma_1{}^2}{(1+K_1R_2)} - \gamma_1\gamma_2\right)\frac{r_1K_1}{(1+K_1R_2)^2} - m_1\gamma_1{}^2 + \left(\frac{R_2a_1\gamma_1{}^2}{(1+a_2R_2)^2} - \frac{\gamma_1\gamma_2}{(1+a_1R_1)(1+a_2R_2)} + \frac{a_2R_2\gamma_2{}^2}{(1+a_2R_2)^2}\right)\frac{\beta_1}{(1+a_1R_1)(1+a_2R_2)}\right],$$

$$\tilde{Y}_{21} = 2\left(t_3^{[5]}\right)^2 \left[\left(\frac{\gamma_1\gamma_2}{(1+a_1R_1)(1+a_2R_2)} - \frac{R_2a_1\gamma_1{}^2}{(1+a_1R_1)^2} - \frac{R_1a_2\gamma_2{}^2}{(1+a_2R_2)^2}\right)\frac{l_1}{(1+a_1R_1)(1+a_2R_2)} + \left(\frac{K_2R_2}{(1+K_2R_3)} - \gamma_2\right)\frac{r_2K_2}{(1+K_2R_3)^2} + \left(\frac{a_3R_3\gamma_2{}^2}{(1+a_3R_2)^2} + \frac{\gamma_2}{(1+a_3R_2)(1+a_4R_3)} + \frac{a_4R_2}{(1+a_4R_3)^2}\right)\frac{\beta_2}{(1+a_3R_2)(1+a_4R_3)} - m_2\gamma_2{}^2\right],$$

$$\tilde{Y}_{31} = \frac{2\left(t_3^{[5]}\right)^2 l_2}{(1+a_3R_2)(1+a_4R_3)}\left(\frac{\gamma_2}{(1+a_3R_2)(1+a_4R_3)} - \frac{R_3a_3\gamma_2{}^2}{(1+a_3R_2)^2} - \frac{R_2a_4}{(1+a_4R_3)^2}\right).$$

Hence, it was obtained.

$$(M^{[5]})^T\left[D^2F_\mu(A_5.\tilde{d}_2)(T^{[5]}.T^{[5]}.T^{[5]})\right] = 2\left(t_3^{[5]}\right)^2 m_3^{[5]}(\tilde{Z}_1 - \tilde{Z}_2) \neq 0.$$

So, by condition (21) system (1) exhibits a saddle-node bifurcation at A_5 for any value of the parameter $\tilde{d}_2 = d_2$ but no pitch fork bifurcation at A_5 where $\tilde{d}_2 = d_2$.

3. Numerical simulation

In this section, the dynamical behavior of the system (1) was investigated. Calculations may be performed for a one set of parameters with a different starting point to examine the analytical results and understand how the parameters impact the dynamic model. **Figure 1. (a-d)**, shows that system (1) has positive solution.

$$r_1 = 0.5.m_1 = 0.5.K_1 = 0.003.\beta_1 = 0.3.l_1 = 0.3.a_1 = 0.1.a_2 = 0.02.d_1 = 0.1.r_2 = 0.5.$$
$$m_2 = 0.4.K_2 = 0.07.\beta_2 = 0.7.l_2 = 0.7.a_3 = 0.01.a_4 = 0.02.d_2 = 0.4. \tag{24}$$

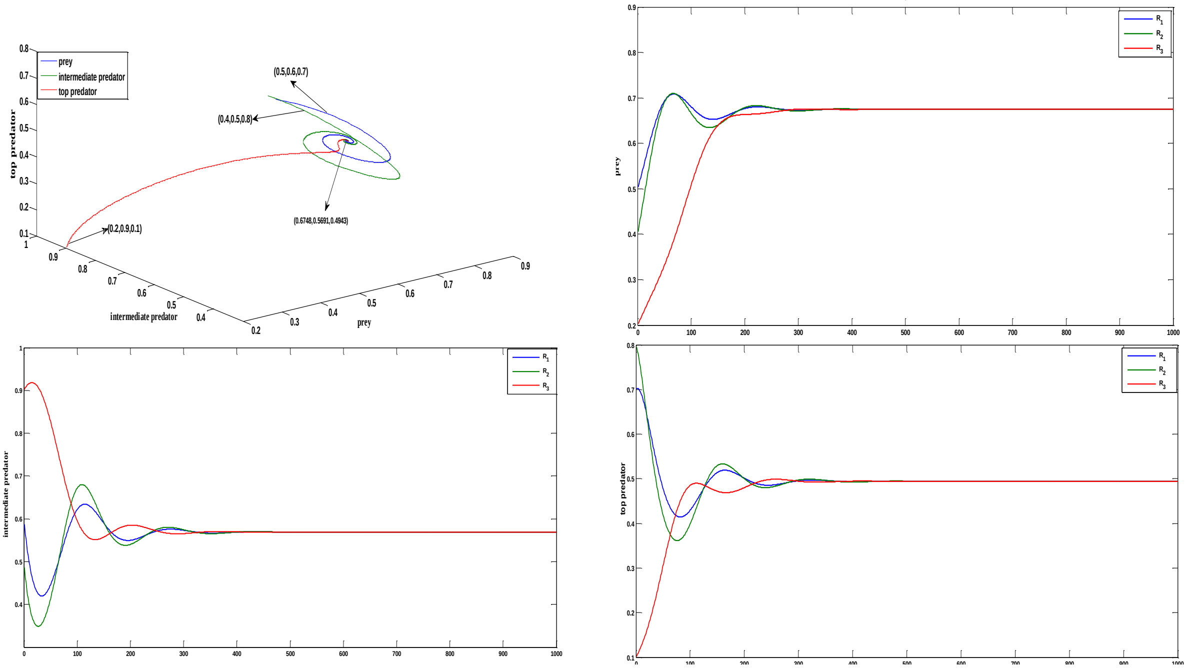

Figure 1. ($\mathbf{a-d}$). Time series of system solution (1) begin with different starting points (0.5.0.6.0.7). (0.4.0.5.0.8). and (0.2.0.9.0.1). (**b**) Path of R_1 depending on time, (**c**) Path of R_2 depending on time,(**d**) Path of R_3 depending on time.

Now, to study the effect of parameters on the dynamic behavior of the system (1), the system (1) was numerically solved to the data in (24) by changing a single parameter each time the results are given in **Table (2).**

The result of changing the value of the parameter r_1 in the range $0.15 \leq r_1 < 3.7$ solution approaches A_4, as described in **Figure 2 ,(a)**for typical value $r_1 = 2$. in the range $0.1 \leq r_1 < 0.15$ the solution approaches to A_5, as described in Figure 2,(b) for typical value $r_1 = 0.14$.

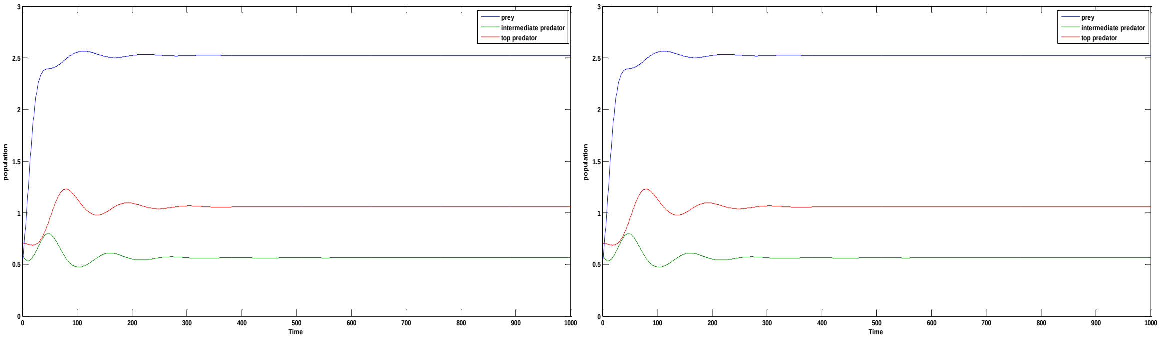

Figure 2. **(a)**.(T.S.) of system solution (1) for the given values in the equation (24) with $r_1 = 2$,which approaches to $A_4 = (0.0.5716.0.2356)$. **(b).** (T.S.) of system solution (1) for the given values in the equation (24) with $r_1 = 0.14$, which approaches to $\mathrm{A}_5 = (2.25226.0.5627.1.0574)$.

The impact of varying the parameter d_1 in the range $0.001 \leq d_1 < 0.475$ the solution approaches to A_5, as shown in **Figure 3, (a**) for typical value $d_1 = 0.2$. in the range $0.475 \leq d_1 < 0.799$ solution approaches A_3, as described in figure 3, (b) for typical value $d_1 = 0.4799$. in the range $0.799 \leq d_1 < 1$ solution approaches A_1, as shown in **Figure 3, (c)** for typical value $d_1 = 0.18$.

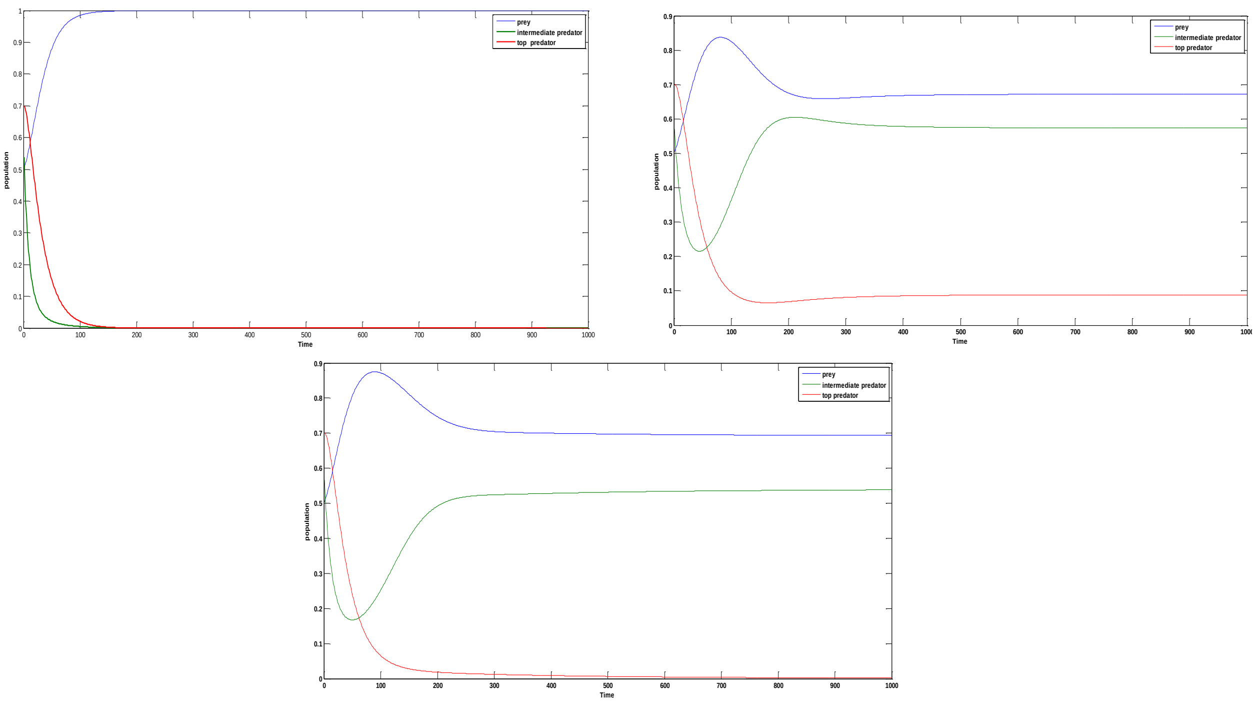

Figure 3. (a),(T.S.) of system solution (1) for the given values in the equation (24) with $d_1 = 0.2$,which approaches to $\mathrm{A}_5 = (0.6981.0.53.0.19)$. (b) (T.S.) of system solution (1) for the given values in the equation (24) with $d_1 = 0.4799$,which approaches to $\mathrm{A}_3 = (0.6932.0.5383.0)$. (c) (T.S.) of system solution (1) for the given values in the equation (24) with $d_1 = 0.81$,which approaches to $\mathrm{A}_1 = (0.999.0.0)$.

For the parameter d_2 in the range $0.1 \leq d_2 < 0.869$ solution approaches A_5, as shown in **Figure 4,(a**) for typical value $d_2 = 0.5$,the range growing further $0.869 \leq d_2 < 1$ solution approaches A_3 as described in **Figure 4,(b)** for typical value $d_2 = 0.921$, in the range

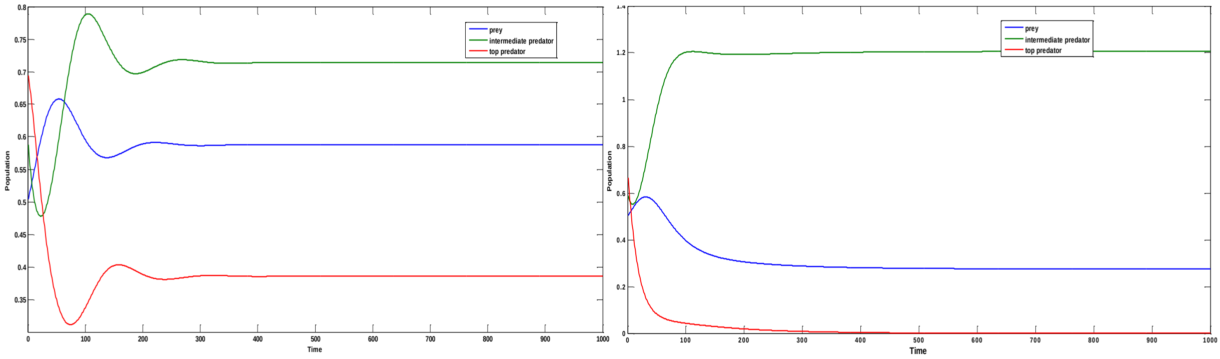

Figure 4. (a) (T.S.) of system solution (1) for the given values in the equation (24) with $d_2 = 0.5$,which approaches to $A_5 = (0.5875.0.7139.0.3857)$. (b) (T.S.) of system solution (1) for the given values in the equation (24) with $d_2 = 0.921$,which approaches to $A_3 = (0.2753.1.2058.0)$.

Table 2. The numerical result.

Range of parameter	*Stable*	*The bifurcation point*
$0.1 \leq r_1 < 0.15$	A_4	$r_1 = 0.15$
$0.15 \leq r_1 < 3.7$	A_5	
$0.0001 \leq r_2 < 0.1$	A_2	$r_2 = 0.1$
$0.1 \leq r_2 < 0.5$	A_5	
$0.003 \leq K_1 \leq 0.999$	A_5	$K_1 = 0.999$
$0.999 \leq K_1 < 1$	A_4	
$0.00007 \leq K_2 < 0.07$	A_5	
$0.0009 \leq m_1 < 0.001$	A_1	$m_1 = 0.001$
$0.001 \leq m_1 < 0.5$	A_5	
$0.4 \leq m_2 < 1.2$	A_5	
$0.3 \leq \beta_1 < 0.99$	A_5	$\beta_1 = 0.99$
$0.99 \leq \beta_1 < 2$	A_1	
$0.7 \leq \beta_2 < 2.5$	A_5	
$0.0001 \leq l_1 < 0.3$	A_5	
$0.1 \leq l_2 < 0.7$	A_5	
$0.0001 \leq a_1 < 1.6$	A_5	
$0.008 \leq a_2 < 2$	A_5	
$0.0006 \leq a_3 < 1.2$	A_5	
$0.0002 \leq a_4 < 3$	A_5	
$0.001 \leq d_1 < 0.475$	A_5	$d_1 = 0.475$
$0.475 \leq d_1 < 0.799$	A_3	$d_1 = 0.799$
$0.799 \leq d_1 < 1$	A_1	
$0.01 \leq d_2 < 0.869$	A_5	$d_2 = 0.869$
$0.869 \leq d_2 < 1$	A_3	

5. Conclusion

In this work, the occurrence of local bifurcation have been discussed with an appropriate conditions of food chain which contains a prey–intermediate predator-top predator model with fear and Crowley-Martin-type of functional response have been studied, transcritical and pitch fork bifurcation occurrence near $A_1.A_2.A_3$ *and* A_4, while a saddle-node bifurcation occurs

near A_5.Finally, in order to demonstrate how local bifurcation occurs in this system, numerical simulations are employed. The results of these simulations are as follows:
*System (1) has no periodic dynamics
*The parameters $r_i. d_i. i = 1.2$, $K_1. m_1.\ \beta_1$ have a significant impact in the system dynamics (1), as opposed to other parameters $K_2. m_2.\ \beta_2. l_i. i = 1.2\ and\ a_i. i = 1.2.3.4$ the solution continues to approach the equilibrium point.

Acknowledgment

Our researcher extends his Sincere thanks to the editor and members of the preparatory committee of the Ibn AL-Haitham Journal of Pure and Applied Science.

Conflict of Interest

There are no conflicts of interest.

Funding

There is no funding for the article.

References

1. Meng XY, Huo HF, Zhang XB. Stability and global Hopf bifurcation in a delayed food web consisting of a prey and two predators. Commun Nonlinear Sci Numer Simul. 2011;16:4335–4348. https://doi.org/10.1016/j.cnsns.2011.03.009
2. Arancibia-Ibarra C, Aguirre P, Flores J, van Heijster P. Bifurcation analysis of a predator-prey model with predator intraspecific interactions and ratio-dependent functional response. Appl Math Comput. 2021;402:126152. https://doi.org/10.1016/j.amc.2021.126152
3. Magudeeswaran S, Vinoth S, Sathiyanathan K, Sivabalan M. Impact of fear on delayed three species food-web model with Holling Type-II functional response. Int J Biomath. 2022;15:2250014. https://doi.org/10.1142/S1793524522500140
4. Majeed AA, Ismaeeb MH. The bifurcation analysis of prey-predator model in the presence of stage structured with harvesting and toxicity. In: Journal of Physics: Conference Series; IOP Publishing; 2019; Vol. 1362, p. 12155. https://doi.org/10.1088/1742-6596/1362/1/012155
5. Din Q. Stability, bifurcation analysis and chaos control for a predator-prey system. J Vib Control. 2019;25:612–626. https://doi.org/10.1177/1077546318790871
6. Prasad KD, Sasmal SK. Dynamics of anti-predator behavior and effect of fear on prey–predator model. J Biol Syst. 2022;30:887–912. https://doi.org/10.1142/S0218339022500322
7. Shuai Z, Peng Y, Liu X, Li Z, Guerrero JM, Shen ZJ. Parameter stability region analysis of islanded microgrid based on bifurcation theory. IEEE Trans Smart Grid. 2019;10:6580–6591.
8. Huang C, Li H, Cao J. A novel strategy of bifurcation control for a delayed fractional predator–prey model. Appl Math Comput. 2019;347:808–838. https://doi.org/10.1016/j.amc.2018.11.031
9. Pamuk S, İrem ÇAY. Stability and Hopf bifurcation analysis of a mathematical model in tumor angiogenesis. Anadolu Univ J Sci Technol A-Applied Sci Eng. 2018;19:50–57.
10. Mukherjee D, Maji C. Bifurcation analysis of a Holling Type II predator-prey model with refuge. Chin J Phys. 2020;65:153–162. https://doi.org/10.1016/j.cjph.2020.02.012
11. Majeed AA, Alabacy ZK. The persistence and bifurcation analysis of an ecological model with fear effect involving prey refuge and harvesting. In: AIP Conference Proceedings; AIP Publishing LLC; 2022; Vol. 2394, p. 70001. https://doi.org/10.1063/5.0121877
12. Zhang H, Cai Y, Fu S, Wang W. Impact of the fear effect in a prey-predator model incorporating a prey refuge. Appl Math Comput. 2019;356:328–337. https://doi.org/10.1016/j.amc.2019.03.034
13. Wang Y, Cao J, Huang C. Dynamics of a fractional three-species food chain system with mixed functional responses and fear effect. 2023. https://doi.org/10.21203/rs.3.rs-2466067/v1

14. Othman KB, Amen AI. Periodic solutions of the forest pest system via Hopf bifurcation and averaging theory. Iraqi J Sci. 2022;5496–5509. https://doi.org/10.24996/ijs.2022.63.12.35
15. Aziz MM, Mohammed SAU. Analysis of stability and chaos of discrete time system with local bifurcation. In: 2022 8th International Conference on Contemporary Information Technology and Mathematics (ICCITM); IEEE; 2022; p. 425–429.
16. Majeed NS. Local bifurcation analysis for a special type of SIR epidemic model. Int J Sci Res IJSR. 2017;6:1143–1146.
17. Majeed SN. Dynamical study of an SIR epidemic model with nonlinear incidence rate and regress of treatment. Ibn AL-Haitham Journal for Pure and Applied Sciences. 2018;510–522. https://doi.org/10.30526/2017.IHSCICONF.1810
18. Jalal AA, Amen AI, Sulaiman NA. Darboux integrability of a generalized 3D chaotic Sprott ET9 system. Baghdad Sci J. 2022;19:542. https://doi.org/10.21123/bsj.2022.19.3.0542
19. Akhtar S, Ahmed R, Batool M, Shah NA, Chung JD. Stability, bifurcation and chaos control of a discretized Leslie prey-predator model. Chaos Solitons Fractals.2021;152:111345. https://doi.org/10.1016/j.chaos.2021.111345
20. Aziz AA, Majeed AA. The fear effects on the dynamical study of an ecological model with Crowley-Martin-Type of functional-response. In: AIP Conference Proceedings; AIP Publishing; 2024; Vol. 3097. https://doi.org/10.1063/5.0209494