Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2025, 38 (1)

The Local Bifurcation of Food web Prey-Predator Model involving fear and anti-Predator behavior

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Received:12 June 2023 Accepted:27 August 2023 Published:20 January 2025
doi.org/10.30526/38.1.3598

Abstract

In this paper, the conditions under which the occurrence of the local bifurcation (such as saddle-node (SN), transcritical (TC), and pitchfork (PT)) of all stable points of a food web model have been investigated. Fear and anti-predator responses involving the Holling-type IV and Growly-Martin functional responses have been found. It has been shown that there are transcritical and pitchfork bifurcations near $H_3, H_4 and\ H_6$, as well as a saddle-node bifurcation close to the positive equilibrium point. In addition, there is a saddle-node bifurcation in close to the positive equilibrium point. These divergences have materialised into existence. In conclusion, to prove that the analytical results are correct, a numerical simulation of a set of parameters and starting conditions has been used.

Keywords: Prey- Predator, Local bifurcation, Global bifurcation, Sotomayor's theorem.

1. Introduction

In the past few decades, a prominent topic in population dynamics has been the study of dynamical behaviors for massive predator-prey models. Numerous outcomes have been reported. Any biological or environmental characteristics are liable to change over time in the actual world. Since the selection pressures on systems in a fluctuating environment differ from those in a stable environment, the impacts of a periodically altering environment are crucial for evolutionary theory(1,2). In the meantime, the gestational time delay is a frequent instance since, typically, the predator's present birth rate is determined by the amount of prey it has consumed in the past (3–5).

The bifurcation analysis has been proposed to predict the parameter stability boundary of a nonlinear system. Bifurcation occurs when a small continuous change of parameter values causes a sudden change in system behavior Compared with the small-signal analysis that analyzes the perturbation fixed equilibrium (6,7), the bifurcation analysis can perform parametric stability analysis that traces equilibrium solutions as the parameters change(8–10).

Bifurcation theory analyses quantitative changes in the phase portrait, such as the emergence and disappearance of equilibria, periodic orbits, or more complex phenomena like weird attractors. Understanding of nonlinear dynamical systems depends critically on the techniques and findings of bifurcation theory(11–13).

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Bifurcation can take many distinct forms. (saddle-node, transcritical, pitchfork, period-doubling, Neimark-sacker, and hopf bifurcations) but the most significant ones are (SN), (TC), (PT), and hopf bifurcations. If the differential equation x=f(x,μ) had taken into account the point at which this dynamic system's behavior qualitatively changed, or what is described mathematically as a bifurcation(14–16).

Many researchers have been studying bifurcation in dynamical systems accurately in the ordinary differential equations in both linear and nonlinear, autonomous and non-autonomous(17,18).

Orrell and Smith(19), studied visualizing bifurcation in high dimensional systems. Yuan and Zhao (20) studied bifurcation for a fractional order predator–prey system involving two nonidentical .

While Majeed and Ismaeeb (21) studied the bifurcation analysis of a prey-predator model in the presence of a stage structured with harvesting and toxicity

In this paper, the conditions of the local bifurcation (LB) of the food web model with fear and anti-predator model have been found.

2. Model Formulation (22):

The following section includes an ecological model with three species that had been proposed: the first and second preys with just one predator, which are indicated by the magnitude of their populations at the time. L_1(t), L_2(t)and L_3(t) respectively .

$$\begin{aligned}
\frac{dL_1}{dt} &= S_1L_1\left(1-\frac{L_1}{K_1}\right)-\frac{C_1L_1L_3}{a_1+L_1^2}\\
\frac{dL_2}{dt} &= \frac{S_2\ L_2}{1+f_1\ L_3}-CL_2^2-\frac{C_2L_2L_3}{(1+a_2L_2)(1+a_3L_3)} \qquad (1)\\
\frac{dL_3}{dt} &= \frac{g_1L_1L_3}{a_1+L_1^2}+\frac{g_2L_2L_3}{(1+a_2L_2)(1+a_3L_3)}-\mu L_1L_3-dL_3
\end{aligned}$$

The following table can be used to describe the positive system (1) parameters.

Table 1: The system's (1) parameters are as follows:

parameters	Biological meaning
$S_i.i=1.2$	First and second preys growth rates, respectively
$K_1>0$	Carrying capacity of first prey
$C_i>0, i=1,2$	Attack rate of the first prey and second prey by predator respecively
$a_1>0$	Protection rate to the prey and predator by the environment
C	The rate of internal competition for the second prey
f_1	The second prey's fear rate from predator
$g_i, i=1,2$	Conversion rate of food from first prey and second prey to predator respecively
a_2	Handling time of the second prey
a_3	The magnitude of interference among predators
μ	The rate of anti-predator behavior of first prey to predator
d	The natural death rate of predator

3. Local bifurcation analysis:

The analysis of the (LB) of model (1) has been explored in this section, with a particular emphasis on the changes that occur around each equilibrium point when the parameter values in the dynamic behavior change. With the support of Sotomayor's theorem, our goal is to give higher order conditions that guarantee the appearance of the most frequent local bifurcations.
Now, according to Jacobean matrix $J(L_1, L_2, L_3)$ of the system (1) which is given in (22) as follows:

$$J = \left[a_{ij}\right]_{3\times 3} \tag{2}$$

where:

$$a_{11} = S_1 - \frac{2S_1L_1}{K_1} - \frac{C_1L_3\left(a_1 - L_1^2\right)}{\left(a_1+L_1^2\right)^2},\ a_{12} = 0\ ,\ a_{13} = -\frac{C_1L_1}{a_1+L_1^2},$$

$$a_{21} = 0\ , a_{22} = \frac{S_2}{1+f_1L_3} - 2CL_2 - \frac{C_2L_3}{(1+a_2L_2)(1+a_3L_3)}$$

$$a_{23} = -\frac{S_2f_1L_2}{(1+f_1L_3)^2} - \frac{C_2L_3}{(1+a_2L_2)(1+a_3L_3)^2}$$

$$a_{32} = \frac{g_2L_3}{(1+a_2L_2)^2\ (1+a_3L_3)}\ .\ a_{31} = \frac{g_1L_3(a_1-L_1^2)}{\left(a_1+L_1^2\right)^2} - \mu L_3.$$

$$a_{33} = \frac{g_1L_1}{a_1+L_1^2} + \frac{g_2L_2}{(1+a_2L_2)(1+a_3L_3)^2} - \mu L_1 - d.$$

For any non- zero vector V $=(v_1, v_2, v_3)^T$:

$$D^2F_\mu(X,\mu)(V,V) = [A_{i1}]_{3\times 1}, \tag{3}$$

$$A_{11} = -\frac{2S_1}{K_1} + \frac{4a_1C_1L_1L_3}{(a_1+L_1^2)^2}v_1^2 - \frac{C_1}{(a_1+L_1^2)}\big((a_1+L_1^2) - 2L_1^2 + (a_1+2L_1)\big)v_1v_3,$$

$$A_{21} = -2C + \frac{C_2L_3}{(1+a_2L_2)^2(1+a_3L_3)}v_1^2 - \left(\frac{2S_2f_1}{(1+f_1L_3)^2} - \frac{C_2a_2L_3}{(1+a_2L_2)^2(1+a_3L_3)^2} + \frac{C_2}{(1+a_2L_2)(1+a_3L_3)^2}\right)v_2v_3 + \left(\frac{2S_2f_1^2L_3}{(1+f_1L_3)^3} - \frac{C_2(1-a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^3}\right)v_3^2,$$

$$A_{31} = \frac{2g_1L_1L_3}{a_1+L_1^2}v_1^2 + \left(\frac{g_1\left(1-2L_1^2\right)}{a_1+L_1^2} + g_1 - 2\mu\right)v_1v_3 - \frac{2a_2g_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}v_2^2 - \frac{g_2}{(1+a_2L_2)^2(1+a_3L_3)^2}(a_2L_2 - 1)v_2v_3 - \frac{2g_2a_3L_3}{(1+a_2L_2)(1+a_3L_3)^3}v_3^2.$$

and

$$D^3F_\mu(X,\mu)(V,V,V) = [B_{i1}]_{3\times 1}\ , \tag{4}$$

$$B_{11} = \frac{4C_1a_1L_3\left(1-4L_1^2\right)}{\left(a_1+L_1^2\right)^2}v_1^3 - \frac{1}{a_1+L_1^2}\Big(2L_1(C_1-2) - 2C_1L_1\big((a_1+L_1^2) - 2L_1^2\big) + 2C_1\big(1 - L_1(a_1 + 2L_1)\big) + \frac{4C_1a_1L_1}{a_1+L_1^2}\Big)v_1^2v_3,$$

$B_{21} = -\frac{2C_2a_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}v_1^2v_2 + \left(\frac{2C_2a_2^2L_3}{(1+a_2L_2)^3(1+a_3L_3)^2} + \frac{C_2a_2}{(1+a_2L_2)}\right)v_2^2v_3 + \frac{C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}v_2v_3^2 + \frac{C_2}{(1+a_2L_2)^2(1+a_3L_3)^2}v_1^2v_3 + \left(\frac{4S_2f_1L_3}{(1+f_1L_3)^3} - \frac{C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3} + \frac{2C_2a_3}{(1+a_2L_2)(1+a_3L_3)}\right)v_2v_3^2 + \left(\frac{2S_2f_1^2(1-2f_1L_3)}{(1+f_1L_3)^6} + \frac{4C_2a_3}{(1+a_2L_2)(1+a_3L_3)^3}\right)v_3^3,$

$B_{31} = \frac{2g_1L_3(a_1-L_1^2)}{(a_1+L_1^2)^2}v_1^3 - \frac{2g_1L_1(2a_1+1)}{(a_1+L_1^2)^2}v_1^2v_3 + \frac{6a_2^2g_2L_3}{(1+a_2L_2)^4(1+a_3L_3)}v_2^3 + \frac{2a_2g_2}{(1+a_2L_2)^3(1+a_3L_3)^2}(a_2L_2 - 2)v_2^2v_3 + \frac{2g_1L_1}{a_1+L_1^2}v_1^2v_3 + \frac{a_3g_2}{(1+a_2L_2)^2(1+a_3L_3)^3}(2a_2L_3 + a_2L_2 - 2)v_2v_3^2 - \frac{2g_2a_3(1-2a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^4}v_3^3.$

where $X = (L_1, L_2, L_3)$ and μ be any parameter.

Theorem (1): System (1), with the value of the parameter

$\acute{d} = d = \mu K_1 - \frac{g_1k_1}{a_1+K_1^2} - \frac{g_2S_2}{C+a_2S_2}$, has a transcritical and pitchfork bifurcation at $H_3 = (K_1, \frac{S_2}{C}, 0,)$ if the following conditions:

$$\mu K_1 > \frac{g_1k_1}{a_1+K_1^2} + \frac{g_2S_2}{C+a_2S_2}\,, \tag{5}$$

$$\frac{1}{2} > L_1^2 \tag{6}$$

$$\acute{w}_1 \neq \acute{w}_2, \tag{7}$$

$$\acute{w}_3 \neq \acute{w}_4. \tag{8}$$

where:

$\acute{w}_1 = \frac{2g_1{M_1}^2L_1L_3}{a_1+L_1^2} + \frac{g_1M_1(1-2L_1^2)}{a_1+L_1^2} + g_1M_1 + \frac{g_2M_2}{(1+a_2L_2)^2(1+a_3L_3)^2},$

$\acute{w}_2 = \frac{2a_2g_2{M_2}^2L_3}{(1+a_2L_2)^3(1+a_3L_3)} + \frac{a_2L_2g_2M_2}{(1+a_2L_2)^2(1+a_3L_3)^2} + 2\mu M_1 + \frac{2g_2a_3L_3}{(1+a_2L_2)(1+a_3L_3)^3},$

$\acute{w}_3 = \frac{2g_1{M_1}^3L_3(a_1-L_1^2)}{(a_1+L_1^2)^2} + \frac{6a_2^2g_2{M_1}^3L_3}{(1+a_2L_2)^4(1+a_3L_3)} + \frac{2g_1{M_1}^2L_1}{a_1+L_1^2} + \frac{2{a_2}^2g_2{M_2}^2L_2}{(1+a_2L_2)^3(1+a_3L_3)^2} + \frac{a_3g_2M_2}{(1+a_2L_2)^2(1+a_3L_3)^3}(2a_2L_3 + a_2L_2),$

$\acute{w}_4 = \frac{2g_1{M_1}^2L_1(2a_1+1)}{(a_1+L_1^2)^2} + \frac{4a_2g_2{M_2}^2}{(1+a_2L_2)^3(1+a_3L_3)^2} + \frac{2a_3g_2M_2}{(1+a_2L_2)^2(1+a_3L_3)^3} + \frac{2g_2a_3(1-2a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^4}.$

Proof: Using the Jacobian matrix in eq. (1.8) in (22)

$\acute{J}_3 = J_3(H_3, d) = [\acute{r}_{ij}]_{3\times3}$, where $\acute{r}_{ij} = r_{ij}$, except $\acute{r}_{33} = 0$.

The characterizing eq. of $\acute{J}_3$ then has an eigenvalue of zero(say λ_{3L_3}).

Now, let $V^1 = (v_1^{[1]}, v_2^{[1]}, v_3^{[1]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{3L_3} = 0$. Thus, $(\acute{J}_3 - \lambda_{3L_3}I)V^1 = 0$, this gives:

$v_1^{[1]} = M_1v_3^{[1]}, v_2^{[1]} = M_2\, v_3^{[1]}$, where $M_1 = -\frac{C_1K_1}{(a_1+K_1^2)S_1}$, $M_2 = -\frac{S_2f_1}{2C}$.

and $v_3^{[1]}$ any non-zero real number .

Consider $Z^{[1]} = (z_1^{[1]}, z_2^{[1]}, z_3^{[1]})^T$ be the eigenvector connected with an eigenvalue $\lambda_{3L_3} = 0$ of the matrix $\acute{J}_3^T$.Then, $\left(\acute{J}_3^T - \lambda_{3L_3} I\right) Z^{[1]} = 0$.

By solving this equation for , $Z^{[1]} = \left(0,0,z_3^{[1]}\right)^T$, where $Z_3^{[1]}$ any non-zero real number.

Now, consider that:

$$\frac{\partial f}{\partial d} = f_d(X,d) = \left(\frac{\partial f_1}{\partial d}, \frac{\partial f_2}{\partial d}, \frac{\partial f_3}{\partial d}\right)^T = (0,0,-L_3)^T.$$

So, $f_d\left(H_3, \acute{d}\right) = (0,0,0)^T$ and hence $\left(Z^{[1]}\right)^T f_d\left(H_3, \acute{d}\right) = 0$

The (SD) bifurcation requirement cannot be satisfied according to Sotomayor's theorem. As a result, the first requirement for (TC) bifurcation is realized. Now

$$Df_d(X,d) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where, $Df_d(X,d)$ represents the derivative of $f_d(X,d)$ with respect to $X = (L_1, L_2, L_3)^T$. Furthermore, it is observed that:

$$Df_d(H_3,d)V^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} M_1 v_3^{[1]} \\ M_2\, v_3^{[1]} \\ v_3^{[1]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -v_3^{[1]} \end{bmatrix}$$

$$\left(Z^{[1]}\right)^T \left[Df_d(H_3,d)V^{[1]}\right] = \left(0,0,z_3^{[1]}\right)^T \left(0,0,-v_3^{[1]}\right)$$

$$= -v_3^{[1]} z_3^{[1]} \neq 0.$$

By substituting $V^{[1]}$ in (3) we get:

$$D^2 F_\mu(H_3,d)\left(V^{[1]}, V^{[1]}\right) = \left(\acute{A}_{\iota 1}\right)_{3\times 1},$$

$$\acute{A}_{11} = -\frac{2S_1}{K_1} + \left(v_3^{[1]}\right)^2 \left[\frac{4a_1 C_1 L_1 L_3}{\left(a_1 + L_1^2\right)^2} {M_1}^2 - \frac{C_1}{\left(a_1 + L_1^2\right)} \left(\left(a_1 + L_1^2\right) - 2L_1^2 + \left(a_1 + 2L_1\right)\right) M_1\right],$$

$$\acute{A}_{21} = -2C + \left(v_3^{[1]}\right)^2 \left[\frac{C_2 L_3}{(1+a_2L_2)^2(1+a_3L_3)} {M_1}^2 - \left(\frac{2S_2 f_1}{(1+f_1L_3)^2} - \frac{C_2 a_2 L_3}{(1+a_2L_2)^2(1+a_3L_3)^2} + \frac{C_2}{(1+a_2L_2)(1+a_3L_3)^2}\right) M_2 + \left(\frac{2S_2 f_1^2 L_3}{(1+f_1L_3)^3} - \frac{C_2(1-a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^3}\right)\right],$$

$$\acute{A}_{31} = \left(v_3^{[1]}\right)^2 \left[\frac{2g_1 {M_1}^2 L_1 L_3}{a_1 + L_1^2} + \left(\frac{g_1\left(1-2L_1^2\right)}{a_1 + L_1^2} + g_1 - 2\mu\right) M_1 - \frac{2a_2 g_2 {M_2}^2 L_3}{(1+a_2L_2)^3(1+a_3L_3)} - \frac{a_2 L_2 g_2 M_2}{(1+a_2L_2)^2(1+a_3L_3)^2} + \frac{g_2 M_2}{(1+a_2L_2)^2(1+a_3L_3)^2} - \frac{2g_2 a_3 L_3}{(1+a_2L_2)(1+a_3L_3)^3}\right].$$

Hence, it was obtained

$$\left(Z^{[1]}\right)^T \left[D^2 F_\mu(H_3,d)\left(V^{[1]}, V^{[1]}\right)\right] = z_3^{[1]} \left(v_3^{[1]}\right)^2 (\acute{w}_1 - \acute{w}_2) \neq 0.$$

So, by condition (7) that system (1) has a (TC) bifurcation at H_3 with a parameter $\acute{d} = d$,If the requirement (7) is not met.

By substituting $V^{[1]}$ into (4) we get:

$D^3F_{\mu}(H_3,d)\big(V^{[1]},V^{[1]},V^{[1]}\big)=\big(\acute{B}_{i1}\big)_{3\times1}$

$\acute{B}_{11}=\left(v_3^{[1]}\right)^3[\frac{4C_1a_1L_3(1-4L_1^2)}{(a_1+L_1^2)^2}{M_1}^3-\frac{{M_1}^2}{a_1+L_1^2}\Big(2L_1(C_1-2)-2C_1L_1\big((a_1+L_1^2)-2L_1^2\big)+2C_1\big(1-L_1(a_1+2L_1)\big)+\frac{4C_1a_1L_1}{a_1+L_1^2}\Big)],$

$\acute{B}_{21}=\left(v_3^{[1]}\right)^3[-\frac{2{M_1}^2M_2C_2a_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}+\frac{2C_2{M_2}^2a_2^2L_3}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{{M_2}^2C_2a_2}{(1+a_2L_2)}+\frac{M_2C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{C_2{M_1}^2}{(1+a_2L_2)^2(1+a_3L_3)^2}+\Big(\frac{4S_2M_2f_1L_3}{(1+f_1L_3)^3}-\frac{M_2C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{2M_2C_2a_3}{(1+a_2L_2)(1+a_3L_3)}\Big)+\Big(\frac{2S_2f_1^2(1-2f_1L_3)}{(1+f_1L_3)^6}+\frac{4C_2a_3}{(1+a_2L_2)(1+a_3L_3)^3}\Big)],$

$\acute{B}_{31}=\left(v_3^{[1]}\right)^3[\frac{2g_1{M_1}^3L_3(a_1-L_1^2)}{(a_1+L_1^2)^2}-\frac{2g_1{M_1}^2L_1(2a_1+1)}{(a_1+L_1^2)^2}+\frac{6a_2^2g_2{M_1}^3L_3}{(1+a_2L_2)^4(1+a_3L_3)}+\frac{2{a_2}^2g_2{M_2}^2L_2}{(1+a_2L_2)^3(1+a_3L_3)^2}-\frac{4a_2g_2{M_2}^2}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{2g_1{M_1}^2L_1}{a_1+L_1^2}+\frac{a_3g_2M_2}{(1+a_2L_2)^2(1+a_3L_3)^3}(2a_2L_3+a_2L_2)-\frac{2a_3g_2M_2}{(1+a_2L_2)^2(1+a_3L_3)^3}-\frac{2g_2a_3(1-2a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^4}.$

$\big(Z^{[1]}\big)^T\big[D^3F_{\mu}(H_3,d)\big(V^{[1]},V^{[1]},V^{[1]}\big)\big]=z_3^{[1]}\left(v_3^{[1]}\right)^3(\acute{w}_3-\acute{w}_4)\neq0$

So, there is pitchfork bifurcation at H_3 where$\acute{d}=d$.

Theorem (2): Suppose that condition (1.9b), revers conditions (1.9c) and (1.9d) in (22) with the following conditions are satisfied:

$$L_1^2<\frac{1}{4}, \tag{8}$$

$$L_3<min\left\{\frac{1}{a_3},\frac{1}{2f_1}\right\}, \tag{9}$$

$$\dot{e}_{31}\dot{e}_{13}=\dot{e}_{32}\dot{e}_{23} \tag{10}$$

$$\dot{w}_1\neq\dot{w}_2, \tag{11}$$

$$\dot{w}_3\neq\dot{w}_4, \tag{12}$$

Where:

$\dot{w}_1=\left(v_3^{[2]}\right)^2\Big(\frac{4a_1C_1\dot{\alpha}_3{\dot{\alpha}_1}^2L_1L_3}{(a_1+L_1^2)^2}+\frac{2C_1\dot{\alpha}_1\dot{\alpha}_3L_1^2}{(a_1+L_1^2)}+\frac{{\dot{\alpha}_1}^2\dot{\alpha}_4C_2L_3}{(1+a_2L_2)^2(1+a_3L_3)}+\frac{\dot{\alpha}_2\dot{\alpha}_4C_2a_2L_3}{(1+a_2L_2)^2(1+a_3L_3)^2}+\frac{2\dot{\alpha}_4S_2f_1^2L_3}{(1+f_1L_3)^3}+\frac{2g_1{\dot{\alpha}_1}^2\dot{\alpha}_4L_1L_3}{a_1+L_1^2}+\Big(\frac{g_1(1-2L_1^2)}{a_1+L_1^2}+g_1\Big)\dot{\alpha}_1+\frac{g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^2}\Big),$

$\dot{w}_2=\frac{2S_1\dot{\alpha}_3}{K_1}+2C\dot{\alpha}_4+\left(v_3^{[2]}\right)^2\Big(\frac{C_1\dot{\alpha}_1}{(a_1+L_1^2)}\big((a_1+L_1^2)+(a_1+2L_1)\big)+\Big(\frac{2S_2f_1}{(1+f_1L_3)^2}+\frac{C_2}{(1+a_2L_2)(1+a_3L_3)^2}\Big)\dot{\alpha}_2\dot{\alpha}_4+\frac{C_2(1-a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^3}+2\mu\dot{\alpha}_1+\frac{2a_2g_2{\dot{\alpha}_2}^2L_3}{(1+a_2L_2)^3(1+a_3L_3)}+\frac{a_2L_2g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^2}+\frac{2g_2a_3L_3}{(1+a_2L_2)(1+a_3L_3)^3}\Big),$

$$\dot{w}_3=\frac{2C_1\dot{\alpha}_1{}^3\dot{\alpha}_3a_1L_3(1-4L_1^2)}{(a_1+L_1^2)^2}+\dot{\alpha}_1{}^2\dot{\alpha}_3C_1L_1+\frac{\dot{\alpha}_1{}^2\dot{\alpha}_3L_1}{a_1+L_1^2}+\frac{\dot{\alpha}_1{}^2\dot{\alpha}_3C_1L_1(a_1+2L_1)}{a_1+L_1^2}+\frac{4\dot{\alpha}_1{}^2\dot{\alpha}_3C_1a_1L_1}{(a_1+L_1^2)^2}+\frac{2C_2\dot{\alpha}_2{}^2\dot{\alpha}_4a_2^2L_3}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{\dot{\alpha}_2{}^2\dot{\alpha}_4C_2a_2}{(1+a_2L_2)}+\frac{\dot{\alpha}_2\dot{\alpha}_4C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{C_2\dot{\alpha}_1{}^2\dot{\alpha}_4}{(1+a_2L_2)^2(1+a_3L_3)^2}+\frac{4S_2\dot{\alpha}_2\dot{\alpha}_4f_1L_3}{(1+f_1L_3)^3}+\frac{2\dot{\alpha}_2\dot{\alpha}_4C_2a_3}{(1+a_2L_2)(1+a_3L_3)}+\frac{2S_2f_1^2(1-2f_1L_3)}{(1+f_1L_3)^6}+\frac{4C_2a_3}{(1+a_2L_2)(1+a_3L_3)^3}+\frac{2g_1\dot{\alpha}_1{}^3L_3(a_1-L_1^2)}{(a_1+L_1^2)^2}+\frac{6a_2^2g_2\dot{\alpha}_1{}^3L_3}{(1+a_2L_2)^4(1+a_3L_3)}+\frac{2a_2{}^2g_2\dot{\alpha}_2{}^2L_2}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{2g_1\dot{\alpha}_1{}^2L_1}{a_1+L_1^2}+\frac{a_3g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^3}(2a_2L_3+a_2L_2),$$

$$\dot{w}_4=\frac{\dot{\alpha}_1{}^2\dot{\alpha}_3C_1L_1}{a_1+L_1^2}+\frac{2\dot{\alpha}_1{}^2\dot{\alpha}_3C_1L_1^3}{a_1+L_1^2}+\frac{\dot{\alpha}_1{}^2\dot{\alpha}_3C_1}{a_1+L_1^2}+\frac{2\dot{\alpha}_1{}^2\dot{\alpha}_3\dot{\alpha}_2C_2a_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}+\frac{\dot{\alpha}_2C_2a_2\dot{\alpha}_4(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{2g_1\dot{\alpha}_1{}^2L_1(2a_1+1)}{(a_1+L_1^2)^2}+\frac{4a_2g_2\dot{\alpha}_2{}^2}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{2a_3g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{2g_2a_3(1-2a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^4}.$$

Then, system (1) with parameter value :

$\dot{d}=d=-\frac{(a_1+\dot{L}_1^2)(e_{11}e_{23}e_{32}+e_{13}e_{31}e_{22})+g_1\dot{L}_1e_{11}e_{22}-\mu\dot{L}_1e_{11}e_{22}(a_1+\dot{L}_1^2)}{e_{11}e_{22}(a_1+\dot{L}_1^2)}$, has a transcritical and pitch fork bifurcation at H_4, similary for H_5

Proof: Using the Jacobian matrix in eq. (1.9) in (22)

$\dot{J}_4=J_4(H_4,\dot{d})=[\dot{e}_{ij}]_{3\times3}$, where $\dot{e}_{ij}=e_{ij}$, except

$\dot{e}_{33}=\frac{g_1\dot{L}_1}{(a_1+\dot{L}_1^2)}-\mu\dot{L}_1-\dot{d}$.

the characterizing eq. of $\dot{J}_4$ then has an eigenvalue of zero (say λ_{4L_3}).

Now, let $V^{[2]}=(v_1^{[2]},v_2^{[2]},v_3^{[2]})^T$be the eigenvector corresponding to the eigenvalue $\lambda_{4L_3}=0$. Thus,$(\dot{J}_4-\lambda_{4L_3}I)V^{[2]}=0$, this gives:

$v_1^{[2]}=\dot{\alpha}_1v_3^{[2]},v_2^{[2]}=\dot{\alpha}_2v_3^{[2]}$.

where: $\dot{\alpha}_1=-\frac{\dot{e}_{13}}{\dot{e}_{11}}$, $\dot{\alpha}_2=-\frac{\dot{e}_{23}}{\dot{e}_{22}}$, and $v_3^{[2]}$ any non-zero real number .

Consider $Z^{[2]}=(z_1^{[2]},z_2^{[2]},z_3^{[2]})^T$ be the eigenvector connected with an eigenvalue $\lambda_{4L_3}=0$ of the matrix $\dot{J}_4^T$.Then, $(\dot{J}_4^T-\lambda_{4L_3}I)\,Z^{[2]}=0$.

By solving this equation for , $Z^{[2]}=\left(\dot{\alpha}_3z_3^{[2]},\dot{\alpha}_4z_3^{[2]},z_3^{[2]}\right)^T$, where:

$\dot{\alpha}_3=-\frac{\dot{e}_{31}}{\dot{e}_{11}}$, $\dot{\alpha}_4=-\frac{\dot{e_{32}}}{\dot{e_{22}}}$, and $z_3^{[2]}$ any non-zero real number.

Now, consider that:

$$\frac{\partial f}{\partial d}=f_d(X,d)=\left(\frac{\partial f_1}{\partial d},\frac{\partial f_2}{\partial d},\frac{\partial f_3}{\partial d}\right)^T=(0,0,-L_3)^T.$$

So, $f_d(H_4,d)=(0,0,0)^T$ and hence $\left(Z^{[2]}\right)^Tf_d(H_4,d)=0$.

The (SD) bifurcation requirement cannot be satisfied according to Sotomayor's theorem. As a result, the first requirement for (TC) bifurcation is realized. Now

$$Df_d(X,d) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where, $Df_d(X,d)$ represents the derivative of $f_d(X,d)$ with respect to $X = (L_1, L_2, L_3)^T$. Furthermore, it is observed that:

$$Df_d(H_4,d)V^{[2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 v_3^{[2]} \\ \dot{\alpha}_2 v_3^{[2]} \\ v_3^{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -v_3^{[2]} \end{bmatrix}$$

$$\left(Z^{[2]}\right)^T \left[Df_d(H_4,d)V^{[2]}\right] = \left(\dot{\alpha}_3 z_3^{[2]}, \dot{\alpha}_4 z_3^{[2]}, z_3^{[2]}\right)^T \left(0,0,-v_3^{[2]}\right)$$

$$= -v_3^{[2]} z_3^{[2]} \neq 0$$

$$D^2 F_\mu(H_4,d)\left(V^{[2]}, V^{[2]}\right) = \left[\dot{A}_{\iota 1}\right]_{3\times 1}$$

$$\dot{A}_{11} = -\frac{2S_1}{K_1} + \left(v_3^{[2]}\right)^2 \left[\frac{4a_1 C_1 \dot{\alpha}_1{}^2 L_1 L_3}{\left(a_1+L_1^2\right)^2} - \frac{C_1 \dot{\alpha}_1}{\left(a_1+L_1^2\right)}\left(\left(a_1+L_1^2\right)+\left(a_1+2L_1\right)\right) + \frac{2C_1 \dot{\alpha}_1 L_1^2}{\left(a_1+L_1^2\right)}\right],$$

$$\dot{A}_{21} = -2C + \left(v_3^{[2]}\right)^2 \left[\frac{\dot{\alpha}_1{}^2 C_2 L_3}{(1+a_2L_2)^2(1+a_3L_3)} - \left(\frac{2S_2 f_1}{(1+f_1L_3)^2} + \frac{C_2}{(1+a_2L_2)(1+a_3L_3)^2}\right)\dot{\alpha}_2 + \frac{\dot{\alpha}_2 C_2 a_2 L_3}{(1+a_2L_2)^2(1+a_3L_3)^2} + \left(\frac{2S_2 f_1^2 L_3}{(1+f_1L_3)^3} - \frac{C_2(1-a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^3}\right)\right],$$

$$\dot{A}_{31} = \left(v_3^{[2]}\right)^2 \left[\frac{2g_1 \dot{\alpha}_1{}^2 L_1 L_3}{a_1+L_1^2} + \left(\frac{g_1\left(1-2L_1^2\right)}{a_1+L_1^2} + g_1\right)\dot{\alpha}_1 - 2\mu\dot{\alpha}_1 - \frac{2a_2 g_2 \dot{\alpha}_2{}^2 L_3}{(1+a_2L_2)^3(1+a_3L_3)} - \frac{a_2 L_2 g_2 \dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^2} + \frac{g_2 \dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^2} - \frac{2g_2 a_3 L_3}{(1+a_2L_2)(1+a_3L_3)^3}\right].$$

Hence, it was obtained .

$$\left(Z^{[2]}\right)^T \left[D^2 F_\mu(H_4,d)\left(V^{[2]}, V^{[2]}\right)\right] = z_3^{[2]}(\dot{w}_1 - \dot{w}_2) \neq 0,$$

This implies that system (1) has a (TC) bifurcation at H_4 with a parameter $\dot{d} = d$, If the requirement (11) unsatisfied then.

By substituting $V^{[2]}$ in (4) we get:

$$D^3 F_\mu(H_4,d)\left(V^{[2]}, V^{[2]}, V^{[2]}\right) = \left(\dot{B}_{\iota 1}\right)_{3\times 1}$$

$$\dot{B}_{11} = 2\left(v_3^{[1]}\right)^3 \left[\frac{2C_1 \dot{\alpha}_1{}^3 a_1 L_3\left(1-4L_1^2\right)}{\left(a_1+L_1^2\right)^2} - \frac{\dot{\alpha}_1{}^2 C_1 L_1}{a_1+L_1^2} + \dot{\alpha}_1{}^2 C_1 L_1 + \frac{\dot{\alpha}_1{}^2 L_1}{a_1+L_1^2} - \frac{2\dot{\alpha}_1{}^2 C_1 L_1^3}{a_1+L_1^2} - \frac{\dot{\alpha}_1{}^2 C_1}{a_1+L_1^2} + \frac{\dot{\alpha}_1{}^2 C_1 L_1 (a_1+2L_1)}{a_1+L_1^2} + \frac{4\dot{\alpha}_1{}^2 C_1 a_1 L_1}{\left(a_1+L_1^2\right)^2}\right],$$

$$\dot{B}_{21} = \left(v_3^{[1]}\right)^3 \left[-\frac{2\dot{\alpha}_1{}^2 \dot{\alpha}_2 C_2 a_2 L_3}{(1+a_2L_2)^3(1+a_3L_3)} + \frac{2C_2 \dot{\alpha}_2{}^2 a_2^2 L_3}{(1+a_2L_2)^3(1+a_3L_3)^2} + \frac{\dot{\alpha}_2{}^2 C_2 a_2}{(1+a_2L_2)} + \frac{\dot{\alpha}_2 C_2 a_2 (1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3} + \frac{C_2 \dot{\alpha}_1{}^2}{(1+a_2L_2)^2(1+a_3L_3)^2} + \left(\frac{4S_2 \dot{\alpha}_2 f_1 L_3}{(1+f_1L_3)^3} - \frac{\dot{\alpha}_2 C_2 a_2 (1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3} + \frac{2\dot{\alpha}_2 C_2 a_3}{(1+a_2L_2)(1+a_3L_3)}\right) + \left(\frac{2S_2 f_1^2 (1-2f_1L_3)}{(1+f_1L_3)^6} + \frac{4C_2 a_3}{(1+a_2L_2)(1+a_3L_3)^3}\right)\right],$$

$\dot{B}_{31} = \left(v_3^{[1]}\right)^3 [\frac{2g_1\dot{\alpha}_1{}^3 L_3(a_1-L_1^2)}{(a_1+L_1^2)^2} - \frac{2g_1\dot{\alpha}_1{}^2 L_1(2a_1+1)}{(a_1+L_1^2)^2} + \frac{6a_2^2 g_2\dot{\alpha}_1{}^3 L_3}{(1+a_2L_2)^4(1+a_3L_3)} + \frac{2a_2{}^2 g_2\dot{\alpha}_2{}^2 L_2}{(1+a_2L_2)^3(1+a_3L_3)^2} - \frac{4a_2g_2\dot{\alpha}_2{}^2}{(1+a_2L_2)^3(1+a_3L_3)^2} + \frac{2g_1\dot{\alpha}_1{}^2 L_1}{a_1+L_1^2} + \frac{a_3g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^3}(2a_2L_3 + a_2L_2) - \frac{2a_3g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^3} - \frac{2g_2a_3(1-2a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^4}$.

$\left(Z^{[2]}\right)^T \left[D^3F_\mu(H_4,d)\left(V^{[2]},V^{[2]},V^{[2]}\right)\right] = \left(v_3^{[2]}\right)^3 z_3^{[4]}(\dot{w}_3 - \dot{w}_4) \neq 0$

So, Pitch fork bifurcation can be found at H_4 where $\dot{d} = d$**.**

Theorem (3): system (1) with parameter value :

$\bar{S}_1 = S_1 = \frac{C_1\bar{L}_3}{a_1}$, has a transcritical and pitchfork bifurcation at H_6.if the next conditions hold:

$\bar{w}_1 \neq \bar{w}_2$ (13)

$\bar{w}_3 \neq \bar{w}_4$ (14)

Where:

$\bar{w}_1 = \left(v_1^{[3]}\right)^2 [\frac{4a_1C_1L_1L_3}{(a_1+L_1^2)^2} + \frac{2C_1L_1^2}{(a_1+L_1^2)}]$,

$\bar{w}_1 = \frac{2S_1}{K_1} + \left(v_1^{[3]}\right)^2 \frac{C_1}{(a_1+L_1^2)}\left((a_1 + L_1^2) + (a_1 + 2L_1)\right)$,

$\bar{w}_3 = \frac{2C_1\sigma_1{}^3 a_1L_3(1-4L_1^2)}{(a_1+L_1^2)^2} + \dot{\alpha}_1{}^2 C_1L_1 + \frac{\sigma_1{}^2 L_1}{a_1+L_1^2} + \frac{\sigma_1{}^2 C_1L_1(a_1+2L_1)}{a_1+L_1^2} + \frac{4\sigma_1{}^2 C_1a_1L_1}{(a_1+L_1^2)^2}$,

$\bar{w}_4 = \frac{\sigma_1{}^2 C_1L_1}{a_1+L_1^2} + \frac{2\sigma_1{}^2 C_1L_1^3}{a_1+L_1^2} - \frac{\sigma_1{}^2 C_1}{a_1+L_1^2}$.

Proof: Using the Jacobian matrix is presented in equation (1.10) in (22),

$\bar{J}_6 = J_6(H_6,\bar{S}_1) = \left[\bar{t}_{ij}\right]_{3\times3}$, where $\bar{t}_{ij} = t_{ij}$, except $\bar{t}_{11} = 0$.

Then the characterizing equation of $\bar{J}_6$ had a zero eigenvalue (say λ_{6L_1}).

Now, let $V^{[3]} = (v_1^{[3]}, v_2^{[3]}, v_3^{[3]})^T$be the eigenvector corresponding to the eigenvalue $\lambda_{6L_1} = 0$. Thus,$\left(\bar{J}_6 - \lambda_{6L_1}I\right)V^{[3]} = 0$, this gives:

$v_2^{[3]} = \sigma_1 v_1^{[3]}$, $v_3^{[3]} = \sigma_2 v_1^{[3]}$,

Where:

$\sigma_1 = -\frac{t_{31}t_{23}}{t_{32}t_{23}+t_{22}t_{33}}$, $\sigma_2 = \frac{t_{31}t_{22}}{t_{32}t_{23}+t_{22}t_{33}}$ and $v_1^{[6]}$ any non-zero real number.

Let $Z^{[3]} = (z_1^{[3]}, z_2^{[3]}, z_3^{[3]})^T$ be the eigenvector connected with an eigenvalue $\lambda_{6L_1} = 0$ of the matrix $\bar{J}_6^T$.Then,$\left(\bar{J}_6^T - \lambda_{6L_1}I\right) Z^{[3]} = 0$.

By solving this equation for , $Z^{[3]} = \left(z_1^{[3]}, 0,0\right)^T$,

Where $z_1^{[3]}$ any non-zero real number.

Now, consider that:

$$\frac{\partial f}{\partial S_1} = f_{S_1}(X, S_1) = \left(\frac{\partial f_1}{\partial S_1}, \frac{\partial f_2}{\partial S_1}, \frac{\partial f_3}{\partial S_1}\right)^T = \left(L_1(1 - \frac{1}{K_1}), 0, 0\right)^T.$$

So, $f_{S_1}(H_6, \bar{S}_1) = (0,0,0)^T$, and hence $\left(Z^{[3]}\right)^T f_{S_1}(H_6, \bar{S}_1) = 0$.

The (SD) bifurcation requirement cannot be satisfied according to Sotomayor's theorem. As a result, the first requirement for (TC) bifurcation is realized. Now

$$Df_{\bar{S}_1}(X, \bar{S}_1) = \begin{bmatrix} 1 - \frac{1}{K_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where, $Df_{\bar{S}_1}(X, \bar{S}_1)$ represents the derivative of $f_{\bar{S}_1}(X, \bar{S}_1)$ with respect to $X = (L_1, L_2, L_3)^T$. Furthermore, it is observed that:

$$Df_{\bar{S}_1}(H_6, \bar{S}_1)V^{[3]} = \begin{bmatrix} 1 - \frac{1}{K_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{[3]} \\ \sigma_1 v_1^{[3]} \\ \sigma_2 v_1^{[3]} \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{1}{K_1}\right) v_1^{[3]} \\ 0 \\ 0 \end{bmatrix}$$

$$\left(Z^{[3]}\right)^T \left[Df_{\bar{S}_1}(H_6, \bar{S}_1)V^{[3]}\right] = \left(z_1^{[3]}, 0, 0\right)^T \left(\left(1 - \frac{1}{K_1}\right) v_1^{[3]}, 0, 0\right)$$

$$= \left(1 - \frac{1}{K_1}\right) v_1^{[3]} z_1^{[3]} \neq 0$$

By substituting $V^{[6]}$ in (3) we get:

$D^2 F_\mu(H_6, S_1)\left(V^{[3]}, V^{[3]}\right) = [\bar{A}_{i1}]_{3\times 1}$,

$\bar{A}_{11} = -\frac{2S_1}{K_1} + \left(v_1^{[3]}\right)^2 [\frac{4a_1 C_1 L_1 L_3}{\left(a_1 + L_1^2\right)^2} - \frac{C_1}{\left(a_1 + L_1^2\right)}\left((a_1 + L_1^2) + (a_1 + 2L_1)\right) + \frac{2C_1 L_1^2}{\left(a_1 + L_1^2\right)}]$,

$\bar{A}_{21} = -2C + \left(v_3^{[1]}\right)^2 [\frac{C_2 L_3}{(1+a_2 L_2)^2 (1+a_3 L_3)} - \left(\frac{2S_2 f_1}{(1+f_1 L_3)^2} + \frac{C_2}{(1+a_2 L_2)(1+a_3 L_3)^2}\right)\sigma_1 + \frac{\sigma_1 C_2 a_2 L_3}{(1+a_2 L_2)^2 (1+a_3 L_3)^2} + \left(\frac{2S_2 f_1^2 {\sigma_2}^2 L_3}{(1+f_1 L_3)^3} - \frac{{\sigma_2}^2 C_2 (1-a_3 L_3)}{(1+a_2 L_2)(1+a_3 L_3)^3}\right)]$,

$\bar{A}_{31} = \left(v_1^{[3]}\right)^2 [\frac{2g_1 L_1 L_3}{a_1 + L_1^2} + \left(\frac{g_1\left(1 - 2L_1^2\right)}{a_1 + L_1^2} + g_1\right)\sigma_2 - 2\mu\sigma_1 - \frac{2a_2 g_2 \sigma_1 L_3}{(1+a_2 L_2)^3 (1+a_3 L_3)} - \frac{a_2 L_2 g_2 \sigma_1 \sigma_2}{(1+a_2 L_2)^2 (1+a_3 L_3)^2} + \frac{g_2 \sigma_1 \sigma_2}{(1+a_2 L_2)^2 (1+a_3 L_3)^2} - \frac{2g_2 a_3 \sigma_1 L_3}{(1+a_2 L_2)(1+a_3 L_3)^3}]$.

Hence, it was obtained

$$\left(Z^{[3]}\right)^T \left[D^2 F_\mu(H_6, S_1)\left(V^{[3]}, V^{[3]}\right)\right] = z_1^{[3]}(\bar{w}_1 - \bar{w}_2) \neq 0,$$

So, system (1) has (TC) bifurcation at H_6 with a parameter $\bar{S}_1 = S_1$. If the requirement (13) not satisfied then .

By substituting $V^{[3]}$ in (4) we get:

$D^3F_\mu(H_6,S_1)\left(V^{[3]},V^{[3]},V^{[3]}\right)=(\bar{B}_{i1})_{3\times1}$

$\bar{B}_{11}=2\left(v_1^{[3]}\right)^3\left[\frac{2C_1{\sigma_1}^3a_1L_3(1-4L_1^2)}{(a_1+L_1^2)^2}-\frac{{\sigma_1}^2C_1L_1}{a_1+L_1^2}+{\dot{\alpha}_1}^2C_1L_1+\frac{{\sigma_1}^2L_1}{a_1+L_1^2}-\frac{2{\sigma_1}^2C_1L_1^3}{a_1+L_1^2}-\frac{{\sigma_1}^2C_1}{a_1+L_1^2}+\frac{{\sigma_1}^2C_1L_1(a_1+2L_1)}{a_1+L_1^2}+\frac{4{\sigma_1}^2C_1a_1L_1}{(a_1+L_1^2)^2}\right],$

$\bar{B}_{21}=\left(v_1^{[3]}\right)^3\left[-\frac{2{\dot{\alpha}_1}^2\dot{\alpha}_2C_2a_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}+\frac{2C_2{\dot{\alpha}_2}^2a_2^2L_3}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{{\dot{\alpha}_2}^2C_2a_2}{(1+a_2L_2)}+\frac{\dot{\alpha}_2C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{C_2{\dot{\alpha}_1}^2}{(1+a_2L_2)^2(1+a_3L_3)^2}+\left(\frac{4S_2\dot{\alpha}_2f_1L_3}{(1+f_1L_3)^3}-\frac{\dot{\alpha}_2C_2a_2(1-a_3L_3)}{(1+a_2L_2)^2(1+a_3L_3)^3}+\frac{2\dot{\alpha}_2C_2a_3}{(1+a_2L_2)(1+a_3L_3)}\right)+\left(\frac{2S_2f_1^2(1-2f_1L_3)}{(1+f_1L_3)^6}+\frac{4C_2a_3}{(1+a_2L_2)(1+a_3L_3)^3}\right)\right],$

$\bar{B}_{31}=\left(v_1^{[3]}\right)^3\left[\frac{2g_1{\dot{\alpha}_1}^3L_3(a_1-L_1^2)}{(a_1+L_1^2)^2}-\frac{2g_1{\dot{\alpha}_1}^2L_1(2a_1+1)}{(a_1+L_1^2)^2}+\frac{6a_2^2g_2{\dot{\alpha}_1}^3L_3}{(1+a_2L_2)^4(1+a_3L_3)}+\frac{2{a_2}^2g_2{\dot{\alpha}_2}^2L_2}{(1+a_2L_2)^3(1+a_3L_3)^2}-\frac{4a_2g_2{\dot{\alpha}_2}^2}{(1+a_2L_2)^3(1+a_3L_3)^2}+\frac{2g_1{\dot{\alpha}_1}^2L_1}{a_1+L_1^2}+\frac{a_3g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^3}(2a_2L_3+a_2L_2)-\frac{2a_3g_2\dot{\alpha}_2}{(1+a_2L_2)^2(1+a_3L_3)^3}-\frac{2g_2a_3(1-2a_3L_3)}{(1+a_2L_2)(1+a_3L_3)^4}\right].$

$\left(Z^{[3]}\right)^T\left[D^3F_\mu(H_6,S_1)\left(V^{[3]},V^{[3]},V^{[3]}\right)\right]=2\left(v_1^{[3]}\right)^3z_1^{[3]}(\bar{w}_3-\bar{w}_4)\neq0$

So, Pitchfork bifurcation can be found at H_6 where $\bar{S}_1=S_1$**.**

Theorem (4): If conditions (1.11c) and (1.11d) in (22) are reversing if the following requirements are fulfilled:

$$\tilde{h}_{31}\tilde{h}_{31}=\tilde{h}_{32}\tilde{h}_{23}\ . \tag{15}$$

$$\tilde{h}_{13}=\tilde{h}_{22} \tag{16}$$

$$\tilde{G}_1>\tilde{G}_2 \tag{17}$$

$$\tilde{w}_1\neq\tilde{w}_2, \tag{18}$$

where:

$\tilde{w}_1=\frac{4a_1C_1L_1L_3}{(a_1+L_1^2)^2}\tilde{\rho}_1+\frac{C_2L_3}{(1+a_2L_2)^2(1+a_3L_3)}{\tilde{\rho}_1}^2+\frac{2S_2f_1^2L_3}{(1+f_1L_3)^3}+\frac{2g_1L_1L_3}{a_1+L_1^2}{\tilde{\rho}_1}^2+\left(\frac{g_1(1-2L_1^2)}{a_1+L_1^2}+g_1-2\mu\right)\tilde{\rho}_1,$

$\tilde{w}_2=\frac{C_1}{(a_1+L_1^2)}\left((a_1+L_1^2)-2L_1^2+(a_1+2L_1)\right)\tilde{\rho}_1+\left(\frac{2S_2f_1}{(1+f_1L_3)^2}+\frac{C_2a_2L_3}{(1+a_2L_2)^2(1+a_3L_3)^2}+\frac{C_2}{(1+a_2L_2)(1+a_3L_3)^2}\right)\tilde{\rho}_1+\frac{2a_2g_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}{\tilde{\rho}_2}^2+\frac{g_2}{(1+a_2L_2)^2(1+a_3L_3)^2}(a_2L_2-1)\tilde{\rho}_2+\frac{2g_2a_3L_3}{(1+a_2L_2)(1+a_3L_3)^3}.$

Then, system (1) with parameter value:

$\tilde{d}=d=\frac{\tilde{G}_1-\tilde{G}_2}{(h_{11}h_{22})(a_1+\tilde{L}_1^2)(1+a_2\tilde{L}_2)(1+a_3\tilde{L}_3)^2}$, where:

$\tilde{G}_1=-\left(a_1+\tilde{L}_1^2\right)\left(1+a_2\tilde{L}_2\right)\left(1+a_3\tilde{L}_3\right)^2(h_{11}h_{23}h_{32}+h_{13}h_{31}h_{22})+(h_{11}h_{22})\left(1+a_2\tilde{L}_2\right)\left(1+a_3\tilde{L}_3\right)^2g_1\tilde{L}_1+(h_{11}h_{22})\left(a_1+\tilde{L}_1^2\right)g_2\tilde{L}_2,$

$\tilde{G}_2=-(h_{11}h_{22})\left(a_1+\tilde{L}_1^2\right)(1+a_2\tilde{L}_2)(1+a_3\tilde{L}_3)^2\mu\tilde{L}_1.$

has a saddle-node bifurcation at $H_7 = (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$.

Proof: Using the Jacobian matrix given by eq. (1.11) in (22)

$\tilde{J}_7 = J_7(H_7, \tilde{d}) = [\tilde{h}_{ij}]_{3\times3}$, where $\tilde{h}_{ij} = h_{ij}$,except

$$\tilde{h}_{33} = \frac{g_1\tilde{L}_1}{a_1 + \tilde{L}_1^2} + \frac{g_2\tilde{L}_2}{(1 + a_2\tilde{L}_2)(1 + a_3\tilde{L}_3)^2} - \mu\tilde{L}_1 - d$$

the characterizing eq. $\tilde{J}_7$ had a zero eigenvalue (say $\lambda_{7L_3} = 0$) if and only if $\bar{\bar{\beta}}_3 = 0$.

Now, let $V^{[7]} = (v_1^{[7]}, v_2^{[7]}, v_3^{[7]})^T$be the eigenvector corresponding to the eigenvalue $\lambda_{7L_3} = 0$. Thus,$(\tilde{J}_7 - \lambda_{7L_3} I)V^{[7]} = 0$, this gives:

$$v_1^{[7]} = \tilde{\rho}_1 v_3^{[7]}, v_2^{[7]} = \tilde{\rho}_2 v_3^{[7]} \ , v_3^{[7]}$$

and $v_3^{[7]}$ any non-zero real number

Where:

$\tilde{\rho}_1 = -\frac{h_{13}}{h_{11}}$, $\tilde{\rho}_2 = -\frac{h_{23}}{h_{22}}$.

Let $Z^{[7]} = (z_1^{[7]}, z_2^{[7]}, z_3^{[7]})^T$ be the eigenvector connected with to the eigenvalue $\lambda_{7L_3} = 0$ of the matrix $\tilde{J}_7^T$.Then, $(\tilde{J}_7^T - \lambda_{7L_3} I)\, Z^{[7]} = 0$.

By solving this equation for , $Z^{[7]} = \left(\tilde{\rho}_3 z_3^{[7]}, \tilde{\rho}_4 z_3^{[7]}, z_3^{[7]}\right)^T$,

and $z_3^{[7]}$ any non-zero real number.

Where:

$\tilde{\rho}_3 = -\frac{\widetilde{h_{31}}}{\widetilde{h_{11}}}$, $\tilde{\rho}_4 = -\frac{\tilde{h}_{32}}{\widetilde{h_{22}}}$.

Now, consider that:

$$\frac{\partial f}{\partial d} = f_d(X, d) = \left(\frac{\partial f_1}{\partial d}, \frac{\partial f_2}{\partial d}, \frac{\partial f_3}{\partial d}\right)^T = (0,0,-L_3)^T.$$

So, $f_d(H_7, d) = (0,0,-\tilde{L}_3)^T$and hence

$(Z^{[7]})^T f_d(H_7, d) = -\tilde{L}_3 z_3^{[7]} \neq 0$.

By substituting $V^{[7]}$ in (3) we get:

$D^2F_\mu(H_7, d)(V^{[7]}, V^{[7]}) = (A_{ij})$

$A_{11} = -\frac{2S_1}{K_1} + \left(v_3^{[7]}\right)^2 [\frac{4a_1C_1L_1L_3}{(a_1+L_1^2)^2}\tilde{\rho}_1 - \frac{C_1}{(a_1+L_1^2)}\left((a_1 + L_1^2) - 2L_1^2 + (a_1 + 2L_1)\right)\tilde{\rho}_1]$,

$A_{21} = -2C + \left(v_3^{[7]}\right)^2 [\frac{C_2L_3}{(1+a_2L_2)^2(1+a_3L_3)}\tilde{\rho}_1{}^2 - \left(\frac{2S_2f_1}{(1+f_1L_3)^2} + \frac{C_2a_2L_3}{(1+a_2L_2)^2(1+a_3L_3)^2} + \frac{C_2}{(1+a_2L_2)(1+a_3L_3)^2}\right)\tilde{\rho}_1 + \frac{2S_2f_1^2L_3}{(1+f_1L_3)^3}]$,

$$A_{31} = \left(v_3^{[7]}\right)^2 \left[\frac{2g_1L_1L_3}{a_1+L_1^2}\tilde{\rho}_1{}^2 + \left(\frac{g_1(1-2L_1^2)}{a_1+L_1^2}+g_1-2\mu\right)\tilde{\rho}_1 - \frac{2a_2g_2L_3}{(1+a_2L_2)^3(1+a_3L_3)}\tilde{\rho}_2{}^2 - \frac{g_2}{(1+a_2L_2)^2(1+a_3L_3)^2}(a_2L_2-1)\tilde{\rho}_2 - \frac{2g_2a_3L_3}{(1+a_2L_2)(1+a_3L_3)^3}\right]$$

Hence, it was obtained by conditions

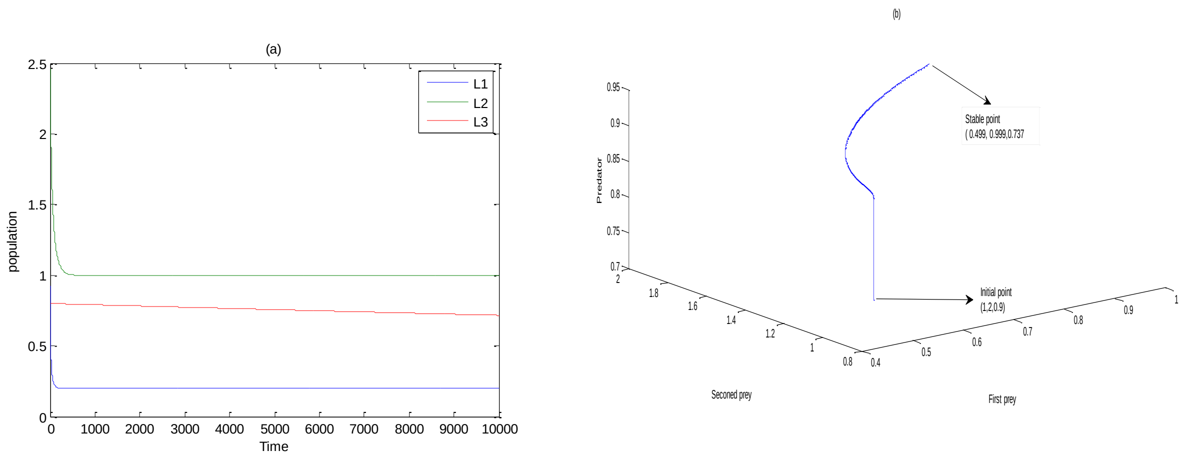
$$\left(Z^{[7]}\right)^T\left[D^2F_\mu(H_7,d)\left(V^{[7]},V^{[7]}\right)\right] = \left(v_3^{[7]}\right)^2 z_3^{[7]}(\widetilde{w}_1-\widetilde{w}_2) \neq 0.$$

This implies that system (1) has no (PT) bifurcation at H_7 where $\tilde{d}=d$,and has a (SN) bifurcation at H_7 with a parameter $\tilde{d}=d$.

4. **Numerical simulation:**

In the following section the system's (1) dynamic behaviour has been investigated . To validate the findings of the study and the influence of the parameters on the dynamic model, calculations can be performed for a number of sets of parameters with various initial points. **Figure(1)(a-b**) It appears that the system (1) has a positive global equilibrium point at the fictitious set of parameters (19).

$$S_1=1, K_1=0\cdot5\ \ , C_1=0\cdot0001, a_1=1.5, S_2=0.5, f_1=0\cdot001, C=0.5$$
$$, C_2=0\cdot0001, a_2=1.5, a_3=1.5, g_1=0\cdot00001, g_2=0\cdot00001,$$



$$\mu=0\cdot002, d=0\cdot001 \qquad (19)$$

Figure 1(a − b).Time series of the system's solution (1) start with initial point (1 , 2 , 0.9). (a) time series of the solution approaches to H_7 =(0.499,0.999,0.737), (b) Solution of system (1).

In order to investigate the effects of parameters, on the dynamical behavior of the system (1), the system (1) has been numerically solved using the data provided in (19) Results can be achieved by changing one parameter at a time.

summarizes how additional parameters affect dynamics.

Table 2. the system's (1) point of bifurcation .

Range of parameter	The stable point	The bifurcation point
$0.1 \le S_1 < 1$	H_7	
$0.3 < K_1 < 1.99$	H_7	
$0.0001 \le C_1 < 1.79$	H_7	$C_1 = 1.79$
$1.79 \le C_1 \le 2$	H_6	
$1.5 \le a_i < 3$ i=1,2,3	H_7	

$0.5 \le S_2 < 2$	H_7	
$0.001 \le d < 0.095$	H_7	d=0.095
$0.095 \le d \le 0.9$	H_3	
$0.002 \le \mu < 0.179$	H_7	$\mu = 0.077$
$0.0179 \le \mu < 1$	H_3	
$0.0001 \le C_2 < 0.34$	H_7	$C_2 < 0.34$
$0.34 \le C_2 < 1$	H_4	
$0.001 \le f_1 < 0.99$	H_7	
$0.5 \le C < 1$	H_7	
$0.0001 \le g_1 < 0.00001$	H_7	

The effect of changing the parameter C_1 in the vicinity $0.0001 \le C_1 < 1.79$ the solution gets closer to H_7, showing in **Figure 2a** , increasing the range further $1.79 \le C_1 < 2$ the solution gets closer to H_6, showing in **Figure 2b** .

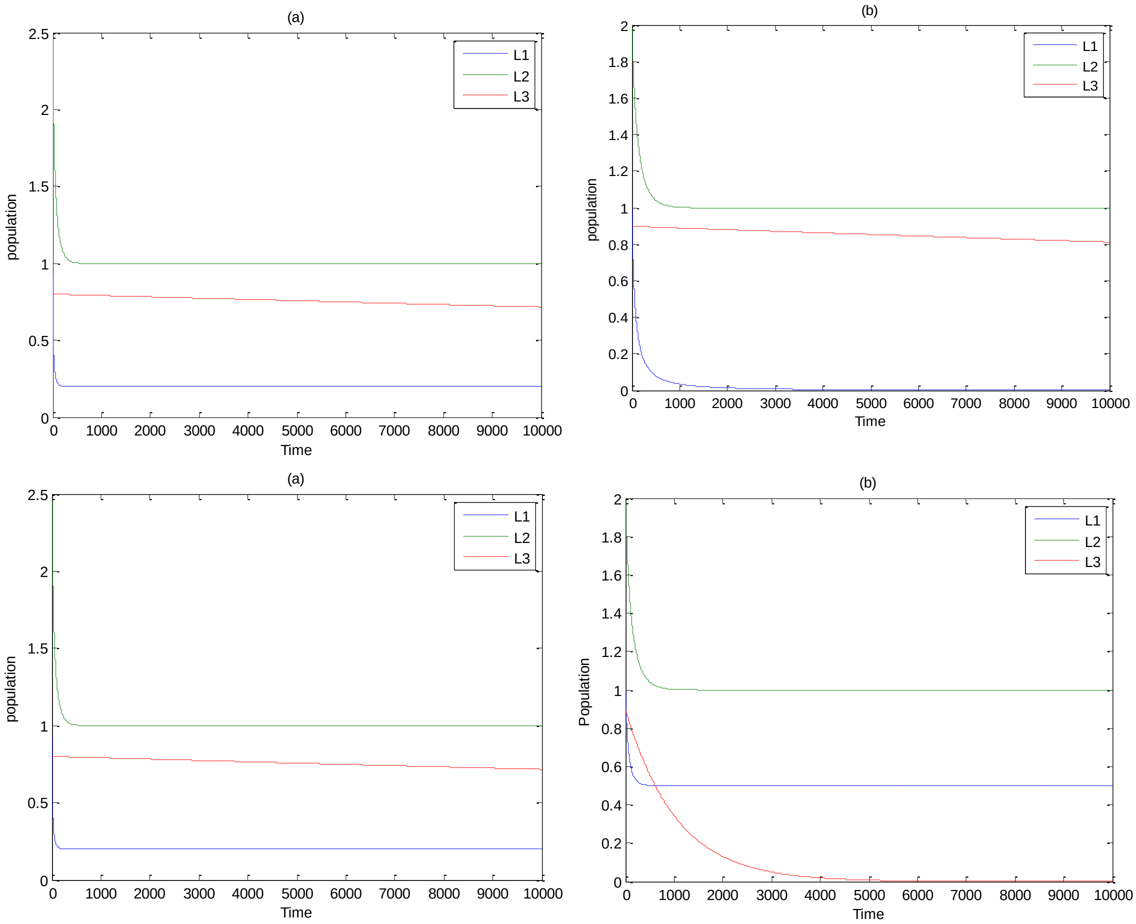


Figure 2 (a-b). (a) Time series of the system's solution (1) with $C_1 = 0.0001$,which approaches to $H_7 = (0.499, 0.999, 0.737)$, and (b) time series of the solution of system (1) with $C_1 = 1.79$,which approaches to $H_6 = (0, 0.999, 0.812)$.

For the parameter d in the vicinity $0.001 \le d < 0.095$ the solution gets closer to H_7, showing in **Figure3a**, increasing the range further $0.095 \le d < 0.9$ the solution gets closer to H_3, showing in **Figure 3b**.

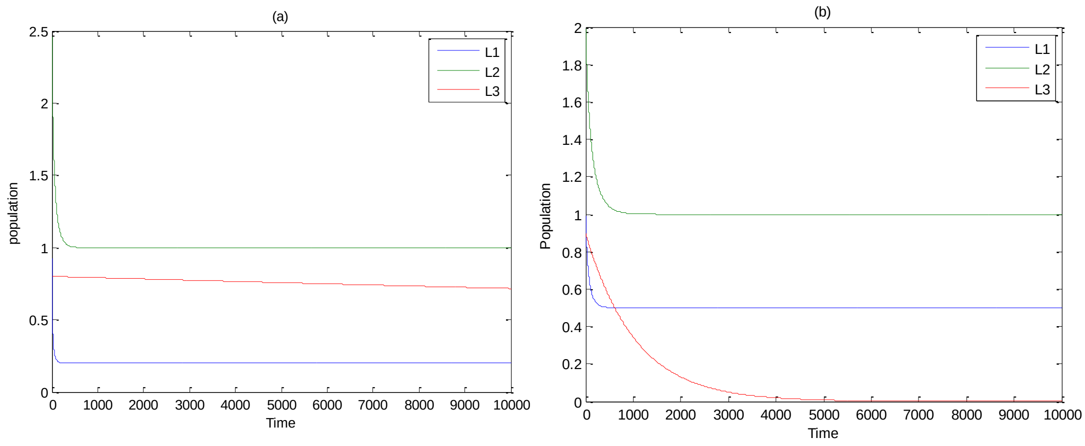


Figure3. (a-b). (a) Time series of the system's solution (1.1) with $\mathrm{d} = 0.001$,which approaches to $\mathrm{H}_7 = (0.499,0.999,0.737)$, and (b) time series of the solution of system (1.1) with $\mathrm{d} = 0.095$,which approaches to $\mathrm{H}_3 = (0.5,1,0)$.

For the parameter μ in the vicinity $0.002 \leq \mu < 0.179$ the solution gets closer to H_7, showing in **Figure 4a**, increasing the range further $0.179 \leq \mu < 1$ the solution gets closer to H_3, showing in **Figure 4b**.

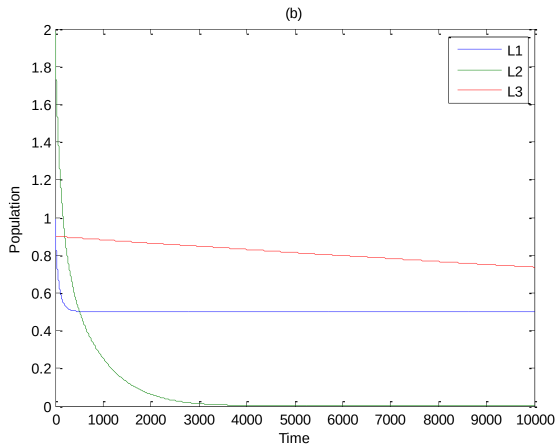


Figure 4. (a-b). (a) Time series of the system's solution (1.1) with $\mu = 0.002$,which approaches to $\mathrm{H}_7 = (0.499,0.999,0.737)$, and (b) time series of the solution of system (1.1) with $\mu = 0.179$,which approaches to $\mathrm{H}_3 = (0.5,1,0)$.

For the parameter C_2 in the vicinity $0.0001 \leq C_2 < 0.34$ the solution gets closer to H_7, showing in **Figure 5a**, increasing the range further $0.34 \leq C_2 < 1$ the solution gets closer to H_4, showing in **Figurer 5b**.

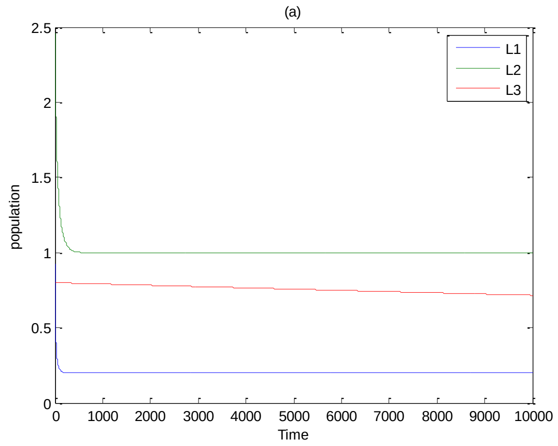


Figure 5. (a-b). (a) Time series of the system's solution (1.1) with $C_2 = 0.0001$,which approaches to $H_7 = (0.499,0.999,0.737)$, and (b) time series of the solution of system (1.1) with $C_2 = 0.34$,which approaches to $H_4 = (0.499,0,0.736)$.

5. Conclusion

In this article, we present the mathematical model of three differential equations for organisms that characterize the effect of anti-predation behavior in the mathematical model containing fear have been analyzed, which are proposed .(LB) have been examined by changing a model's parameter in order to examine bifurcation curves' ability to predict dynamic behavior as well as its occurrence states of (SN) bifurcation occurring at point H_7 and (TC) and (PF) bifurcation occurring at points $H_3, H_4,\ H_6$The bifurcation that occurs at point is determined . Numerical simulation is used. With data given in equation (19). Which are summarized as follow:

1) A periodic dynamics for the system (1) does not exist .
2) The parameters $C_1, d, C_2, and\ \mu$, have an important effect in the dynamics of the system (1). While the others parameters not effect on the bifurcation of system 1.

Acknowledgments

We would like to express our deep appreciation to the reviewers of this work and the publishers of the "Ibn Al-Haitham for pure and Science Journal" for this splendid opportunity.

Conflict of Interest

The authors declare that there are no competing interests regarding the publication of this paper.

Funding

This work is not supported by any the Foundation.

Ethical Clearance

Ethics of scientific research were carried out in accordance with international conditions.

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