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Influence of Infection Delay on the Covid-19 Pandemic with Vaccination Control: Modeling and Simulation

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Abstract

In this work, we formulate a mathematical model of the killer COVID-19 pandemic with time delay and some governmental measures that include vaccine subsidies to understand the dynamic behavior of COVID-19. For the dynamic study, a new model, SV_1V_2IR was used purposed in which infectious individuals were divided into five sub compartments. Our aim is to construct a more reliable and realistic model for a complete mathematical and computational analysis and design of different control strategies for the proposed deterministic model. We first obtain the basic reproduction number for the model is computed using a next-generation technique to predict the future dynamics of the pandemic. The local stability of the model was also investigated at each equilibrium point. The findings show that the time delay can produce a Hopf bifurcation for a SV_1V_2IR model. The obtained numerical results are discussed and predict through graphs.

Keywords: Time delayed (T.D.), Equilibrium points (E.Ps.), COVID-19 Pandemic, Stability, Hopf Bifurcation (H.B.), Numerical simulation (N.S.).

1. Introduction

Mathematical models of infectious disease transmission are increasingly being used to guide public health policy also they are used characterize the complex interactions, and enable information from diverse sources to give a clear understanding of the behavior of these diseases. Infectious disease epidemiology models are inherently multidisciplinary because the transmission of infection within a population is affected not just by the biologicl characteristics of the infectious agent and its host but also by the patterns of contact between hosts and the environment. There are many examples of the spread and control of epidemics, such as of examples Ferguson et al. (1) where studied the transmission intensity and impact of control policies on the foot and mouth epidemic in Great Britain. Donnelly et al. (2) suggested the epidemiological and genetic analysis of severe acute respiratory syndrome. Cauchemez et al. (3) studied the Middle East respiratory syndrome coronavirus: quantification of the extent of the epidemic, surveillance biases, and transmissibility.

Mohsen et al. (4) studied the global stability of COVID-19 model involving the quarantine strategy and media coverage effects. AL-Husseiny et al. (5) have discussed the effect of individuals asymptomatic (Carrier) on the dynamical behavior of a COVID-19 virus. Hattaf et al. (6) they suggested modeling the dynamics of COVID-19 with carrier effect and environmental contamination. Abdulkadhim and Al-Husseiny (7) studied the global stability and bifurcation of a COVID-19 virus modeling with possible loss of the immunity.

The dynamics of populations are significantly influenced by time delays. The dynamics of state variables in many real-world processes, notably in many biological phenomena, depend on the phenomenon's history, or on the state variables' former values, in addition to the phenomenon's current state. Time delays may have an impact on the dynamics of infectious diseases, as shown in Zuo et al. (8) formulated the relationship between media coverage and an epidemic's recruitment and spread. Aekabut et al. (9)investigated a delayed SEIR epidemic model in which the diseased and latent phases are contagious. Rasha M. Yaseen et al. (10) studied the Stability and Hopf bifurcation of an epidemiological model with the effect of delaying the awareness programms and vaccination: analysis and simulation. Zhe Yin et al. (11) investigated how an age-structured SEIRS model was affected by time delays. Mohsen et al. (12) investigated the dynamics of a curfew technique in a model of the coronavirus pandemic epidemic. Zizhen et al. (13) suggested SVIRS epidemic model includes a number of delays along with incidence and treatment rates for Holling type II. Dehingia et al. (14) investigated the dynamic behavior of a SARS-CoV-2 within-host fractional order model. Shurowq et al. (15) study of the COVID-19 epidemic and the bifurcation analysis of a mathematical model for vaccination. Naji and Hussien (16) proposed and analyzed the epidemic model type of SEIR with nonlinear incidence and treatment rates and also used time delays owing to the incubation period. Ahmed et al. (17) discussed a mathematical model for the dynamics of COVID-19 pandemic involving infected immigrants. Naji and Mohsen [18] studied the stability analysise with the bifurcation of an SVIRE epidemic model involving immigrants. Ahmed and AL- Husseiny (19) studied the dynamical behavior of an eco-epidemiological model involving disease in predators and stage structure in prey. Mohsen and Hattaf (20) studied the dynamics of a generalized fractional epidemic model of COVID-19 with carrier effect.

This study presents and evaluates a mathematical model that represents the dynamics of the COVID-19 pandemic's delayed infection and includes two stages of vaccination. This paper is organized as follows: Section 2 illustrates the innovative coronavirus mathematical modeling and the two steps of immunization with delayed infection. Local stability and H.B. are discussed in Section 3. In Section 4, a numerical simulation is utilized to analyze the effects of altering every system parameter.

2. Mathematical model: see in(21)

The system of the time delayed is:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\alpha S}{1+nV_1} - \beta_1 S(t-\tau)I(t-\tau) - \mu S \\
\frac{dV_1}{dt} &= \frac{\alpha S}{1+nV_1} - \gamma V_1 - \beta_2 V_1 I - \mu V_1 \\
\frac{dV_2}{dt} &= \gamma V_1 - \beta_3 V_2 I - \mu V_2 \\
\frac{dI}{dt} &= \beta_1 S(t-\tau)I(t-\tau) + \beta_2 V_1 I + \beta_3 V_2 I - (\mu+\mu_1)I - \theta I \\
\frac{dR}{dt} &= \theta I - \mu R
\end{aligned} \tag{1}$$

Now, see(21) we get;

$$(t) = \frac{\theta \check{I}}{\mu}, \tag{2}$$

As a result, the system below will be studied rather than system (1);

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\alpha S}{1+nV_1} - \beta_1 S(t-\tau)I(t-\tau) - \mu S \\
\frac{dV_1}{dt} &= \frac{\alpha S}{1+nV_1} - \gamma V_1 - \beta_2 V_1 I - \mu V_1 \\
\frac{dV_2}{dt} &= \gamma V_1 - \beta_3 V_2 I - \mu V_2 \\
\frac{dI}{dt} &= \beta_1 S(t-\tau)I(t-\tau) + \beta_2 V_1 I + \beta_3 V_2 I - (\mu+\mu_1)I - \theta I
\end{aligned} \tag{3}$$

3. Local stability analysis (L.S.A.) and hopf bifurcation (H.B.)

In this section, the L.S. and H.B. of system (3) are studied. The position and quantity of equilibrium points are known and don't alter with time delays.

Accordingly, from (21) system (3) have six E.Ps., say

$$E_0 = (\bar{S},0,0,0)\ when\ \alpha = 0\ ,\ \ E_1 = \left(\bar{\bar{S}}, \bar{\bar{V}}_1, 0,0\right)\ when\ \gamma = 0,$$
$$E_2 = \left(\hat{S},0,0,\hat{I}\right)\ when\ \alpha = 0\ ,\ \ E_3 = \left(\check{S}, \check{V}_1, \check{V}_2, 0\right),$$
$$E_4 = \left(\tilde{\tilde{S}}, \tilde{\tilde{V}}_1, 0, \tilde{\tilde{I}}\right)\ when\ \gamma = 0\ and\ \ E_5 = (S^*, V_1^*, V_2^*, I^*).$$

It is well known that, the Jacobian matrix(J.M) of system (3) at any E.Ps. $E = (S, V_1, V_2, I)$ is $J(E) = \left(a_{ij}\right)_{4\times 4}\ \ ; i,j = 1,2,3,4.$

where

$$\begin{aligned}
&a_{11} = -\left[\frac{\alpha}{1+nV_1} + \beta_1 I e^{-\lambda\tau} + \mu\right]\ ,\ a_{12} = \frac{n\alpha S}{(1+nV_1)^2}\ , a_{14} = -\beta_1 S e^{-\lambda\tau}\ ,\ a_{21} = \frac{\alpha}{1+nV_1}\ , \\
&a_{22} = -\left[\frac{n\alpha S}{(1+nV_1)^2} + \gamma + \beta_2 I + \mu\right]\ , a_{24} = -\beta_2 V_1\ ,\ a_{32} = \gamma\ \ , a_{33} = -[\beta_3 I + \mu], \\
&a_{34} = -\beta_3 V_2\ , a_{41} = \beta_1 I e^{-\lambda\tau}\ ,\ a_{42} = \beta_2 I\ ,\ a_{43} = \beta_3 I, a_{13} = a_{23} = a_{31} = 0, \\
&a_{44} = \beta_1 S e^{-\lambda\tau} + \beta_2 V_1 + \beta_3 V_2 - (\mu + \mu_1 + \theta)
\end{aligned} \tag{4}$$

while its associated characteristic equation(C.E.) takes the form

$$P(\lambda)+Q(\lambda)e^{-\lambda\tau}=0 \tag{5}$$

here $P(\lambda)$ and $Q(\lambda)$ are polynomials of λ. Accordingly the L.S. properties of system (3) at all feasible E.Ps. are determined by the roots of the above equation for all $\tau\geq 0$.

- For the (FEP) when $\alpha=0$, Equation (4) reduces to

$$J(E_0)=\begin{bmatrix}-\mu & 0 & 0 & -\beta_1\bar{S}e^{-\lambda\tau}\\ 0 & -(\gamma+\mu) & 0 & 0\\ 0 & \gamma & -\mu & 0\\ 0 & 0 & 0 & \beta_1\bar{S}e^{-\lambda\tau}-(\mu+\mu_1+\theta)\end{bmatrix} \tag{6}$$

The (C.E.) of $J(E_0)$ is:

$$\left(\beta_1\bar{S}e^{-\lambda\tau}-(\mu+\mu_1+\theta)-\lambda\right)(-\mu-\lambda)(-(\gamma+\mu)-\lambda)(-\mu-\lambda)=0 \tag{6a}$$

Equation (6a) represents the eigenvalues of $J(E_0)$ and has 4-roots:

$$\left.\begin{array}{l}\lambda_1=\beta_1\bar{S}e^{-\lambda\tau}-(\mu+\mu_1+\theta)\\ \lambda_2=-\mu\\ \lambda_3=-(\gamma+\mu)\\ \lambda_4=-\mu\end{array}\right\} \tag{6b}$$

Now, for $\tau=0$ we get the eigenvalues will be negative and FEP is locally asymptotically stable (L.A.S.) if

$$\beta_1\bar{S}<\mu+\mu_1+\theta \tag{7}$$

Now, for $\tau>0$ the Equation (6b), has two wholly imaginary roots, namely= $\pm i\omega$ ($\omega>0$).
By substituting $\lambda=\pm i\omega$ in Equation (6b) we get:

$$\beta_1\bar{S}(cos\omega\tau-isin\omega\tau)=\mu+\mu_1+\theta+i\omega$$

So, separating the real and imagined parts yields

$$\left.\begin{array}{l}\beta_1\bar{S}cos\omega\tau=\mu+\mu_1+\theta\\ \beta_1\bar{S}sin\omega\tau=-\omega\end{array}\right\} \tag{8}$$

Squaring each equation and then adding them, we get that

$$\omega=\mp\sqrt{\beta_1^2(\bar{S})^2-(\mu+\mu_1+\theta)^2}$$

Note that, under the condition (7),$\omega(\tau)$ with $\tau>0$ cannot be real, which contradicts with the assumption. Therefore, the C.E. (6a) can't have purely imaginary root, and FEP is L.A.S. for all $\tau\geq 0$ if the condition (7) hold.

- For the (SEP) when $\gamma=0$, Equation (4) reduces to

$$J(E_1)=\begin{bmatrix}-\left(\frac{\alpha}{1+n\bar{\bar{V}}_1}+\mu\right) & \frac{n\alpha\bar{\bar{S}}}{\left(1+n\bar{\bar{V}}_1\right)^2} & 0 & -\beta_1\bar{\bar{S}}e^{-\lambda\tau}\\ \frac{\alpha}{1+n\bar{\bar{V}}_1} & -\left(\frac{n\alpha\bar{\bar{S}}}{\left(1+n\bar{\bar{V}}_1\right)^2}+\mu\right) & 0 & -\beta_2\bar{\bar{V}}_1\\ 0 & 0 & -\mu & 0\\ 0 & 0 & 0 & \beta_1\bar{\bar{S}}e^{-\lambda\tau}+\beta_2\bar{\bar{V}}_1-(\mu+\mu_1+\theta)\end{bmatrix} \tag{9}$$

The C.E. of $J(E_1)$ is

$$[\lambda^2+A_1\lambda+A_2][-\mu-\lambda]\left[\beta_1\bar{\bar{S}}e^{-\lambda\tau}+\beta_2\bar{\bar{V}}_1-(\mu+\mu_1+\theta)-\lambda\right]=0 \tag{10a}$$

Where

$$A_1 = \frac{\alpha}{1+n\bar{\bar{V}}_1} + \frac{n\alpha\bar{\bar{S}}}{\left(1+n\bar{\bar{V}}_1\right)^2} + 2\mu$$
$$A_2 = \mu\left(\frac{n\alpha\bar{\bar{S}}}{\left(1+n\bar{\bar{V}}_1\right)^2} + \frac{\alpha}{1+n\bar{\bar{V}}_1} + \mu\right)$$

The Equation (10a) represents the eigenvalues of $J(E_1)$ and has 4-roots:

$$\left.\begin{aligned} \lambda_{1,2} &= -\frac{A_1}{2} \mp \frac{1}{2}\sqrt{A_1^2 - 4A_2} \\ \lambda_3 &= -\mu \\ \lambda_4 &= \beta_1\bar{\bar{S}}e^{-\lambda\tau} + \beta_2\bar{\bar{V}}_1 - (\mu + \mu_1 + \theta) \end{aligned}\right\} \quad (10b)$$

Now, for $\tau = 0$ we get all the above eigenvalues will be negative and the SEP is L.A.S. if

$$\beta_1\bar{\bar{S}} + \beta_2\bar{\bar{V}}_1 < \mu + \mu_1 + \theta \quad (11)$$

Now, for $\tau > 0$ the Equation (10b), has two wholly imaginary roots, namely= $\pm i\omega$ ($\omega > 0$).
By substituting $\lambda = \pm i\omega$ in Equation (10b) we get :

$$\beta_1\bar{\bar{S}}(cos\omega\tau - isin\omega\tau) = \mu + \mu_1 + \theta - \beta_2\bar{\bar{V}}_1 + i\omega$$

So, separating the real and imagined parts yields

$$\left.\begin{aligned} \beta_1\bar{\bar{S}}cos\omega\tau &= \mu + \mu_1 + \theta - \beta_2\bar{\bar{V}}_1 \\ \beta_1\bar{\bar{S}}sin\omega\tau &= -\omega \end{aligned}\right\} \quad (12)$$

Squaring each equation and then adding them, we get that

$$\omega = \mp\sqrt{\beta_1^2\left(\bar{\bar{S}}\right)^2 - \left(\mu + \mu_1 + \theta - \beta_2\bar{\bar{V}}_1\right)^2}$$

Note that, under the condition (11),$\omega(\tau)$ with $\tau > 0$ cannot be real, which contradicts with the assumption. Therefore, the C.E. (10a) can't have purely imaginary root, and SEP is L.A.S. for all $\tau \geq 0$ if the condition (11) hold.

For the (TEP) when $\alpha = 0$, Equation (4) reduces to

$$J(E_2) = \begin{bmatrix} -\left(R_1e^{-\lambda\tau} + R_2\right) & 0 & 0 & -R_3e^{-\lambda\tau} \\ 0 & -\left(\gamma + \beta_2\hat{I} + \mu\right) & 0 & 0 \\ 0 & \gamma & -\left(\beta_3\hat{I} + \mu\right) & 0 \\ R_1e^{-\lambda\tau} & \beta_2\hat{I} & \beta_3\hat{I} & R_3e^{-\lambda\tau} + R_4 \end{bmatrix} \quad (13)$$

Where, $R_1 = \beta_1\hat{I}$, $R_2 = \mu$, $R_3 = \beta_1\hat{S}$, $R_4 = -(\mu + \mu_1 + \theta)$.
The C.E. of $J(E_2)$ is given by

$$\left[\lambda^2 + B_1\lambda + B_2 + (B_3\lambda + B_4)e^{-\lambda\tau}\right]\left[-\left(\gamma + \beta_2\hat{I} + \mu\right) - \lambda\right]\left[-\left(\beta_3\hat{I} + \mu\right) - \lambda\right] = 0 \quad (14)$$

With

$$B_1 = R_2 - R_4\,,\ B_2 = -R_2R_4,$$
$$B_3 = R_1 - R_3\,,\ B_4 = -R_2R_3 - R_1R_4.$$

Now, when $= 0$, Equation (14) is;

$$[\lambda^2 + (B_1 + B_3)\lambda + B_2 + B_4]\left[-\left(\gamma + \beta_2\hat{I} + \mu\right) - \lambda\right]\left[-\left(\beta_3\hat{I} + \mu\right) - \lambda\right] = 0 \quad (15)$$

So, either

$$\left[-\left(\gamma + \beta_2\hat{I} + \mu\right) - \lambda\right]\left[-\left(\beta_3\hat{I} + \mu\right) - \lambda\right] = 0 \quad (16a)$$

Or,

$[\lambda^2 + (B_1 + B_3)\lambda + B_2 + B_4] = 0$ (16b)

From Equation (16a) we obtain that

$$\lambda_2 = -\left(\gamma + \beta_2\hat{I} + \mu\right) < 0$$

$$\lambda_3 = -\left(\beta_3\hat{I} + \mu\right) < 0$$

Which is always negative eigenvalue.

Now, it is easy to verify that $B_1 + B_3 > 0$ and $B_2 + B_4 > 0$ under the following sufficient conditions

$\beta_1\hat{S} < \beta_1\hat{I} + 2\mu + \mu_1 + \theta,$ (17a)

$\mu\beta_1\hat{S} < \left(\beta_1\hat{I} + \mu\right)(\mu + \mu_1 + \theta).$ (17b)

We see that all roots of Equation (16b), which represent the eigenvalues of (13), have negative real parts. Consequently, under the conditions (17a-17b), TEP is L.A.S. for system (3) when $\tau = 0$.

Now, for $\tau > 0$, then

Either, Equation (16a) Which is always negative eigenvalue.

Or,

$\left[\lambda^2 + B_1\lambda + B_2 + (B_3\lambda + B_4)e^{-\lambda\tau}\right] = 0$ (18)

From (18) has two wholly imaginary roots, namely= $\pm i\omega$ ($\omega > 0$).

By substituting $\lambda = \pm i\omega$ in Equation (18) and separating the real and imagined parts yields

$$\begin{aligned} B_4 cos\omega\tau + B_3\omega sin\omega\tau &= \omega^2 - B_2 \\ B_3\omega cos\omega\tau - B_4 sin\omega\tau &= -B_1\omega \end{aligned} \quad (19)$$

Through squaring and adding both equations in Equation (19), we get

$\omega^4 + h_1\omega^2 + h_2 = 0$ (20)

Further, by letting $\hbar = \omega^2$ in Equation (20)

$g(\hbar) = \hbar^2 + g_1\hbar + g_2 = 0$ (21a)

Where

$$g_1 = R_2^2 + R_4^2 - (R_1 - R_3)^2,$$

$$g_2 = R_2^2 R_4^2 - (R_2R_3 + R_1R_4)^2.$$

Straightforward computation shows that due to the following condition

$R_2R_4 < R_2R_3 + R_1R_4$ (21b)

we obtain $g_2 < 0$. So, there exists a unique positive root in accordance with Descartes' rule of signs $\hbar_0 = \omega_0^2$ providing Equation (21a). Which is, Equation (20) has a positiveω_0. As a result, Equation (18) has at least two roots $\pm\omega_0$ that are purely imaginary and which correspond to the time delay τ. Additionally, by substituting ω_0 in (19) and solving the system for τ, yields :

$\tau_k = \frac{1}{\omega_0} cos^{-1} \frac{(B_4 - B_1B_3)\omega_0^2 - B_2B_4}{B_4^2 + B_3^2\omega_0^2} + \frac{2k\pi}{\omega_0}$ (22)

Where, k = 0,1,2,…. Hence we get the corresponding $\tau_k > 0$ for which system (3) has two wholly imaginary roots $\pm\omega_0$.

Let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ be a root of Equation (18) near $\tau = \tau_k$ with $\mu(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$. Then comes the next theorem:

Theorem1. The C.E. (18), roots satisfy the following transversality requirement;

$\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_k} > 0$ (23a)

Provided that

$\omega_0^2 > B_2$ (23b)

Proof. By substituting $\lambda(\tau)$ in (18) and differentiating the resulting equation in τ,we can get

$$\left[2\lambda + B_1 + B_3 e^{-\lambda\tau} - \tau(B_3\lambda + B_4)e^{-\lambda\tau}\right]\frac{d\lambda}{d\lambda} = \lambda(B_3\lambda + B_4)e^{-\lambda\tau} \tag{24}$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + B_1}{\lambda(B_3\lambda + B_4)e^{-\lambda\tau}} + \frac{B_3}{\lambda(B_3\lambda + B_4)} - \frac{\tau}{\lambda} \tag{25}$$

Since for $= \tau_0$, and $\lambda = i\omega_0$, we have got

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{B_1 + 2i\omega_0}{B_1\omega_0^2 + i\omega_0(\omega_0^2 - B_2)} + \frac{B_3}{B_3\omega_0^2 + iB_4\omega_0} - \frac{i\tau_0}{\omega_0} \tag{26}$$

Now, since

$$sign\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0} = sign\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} \tag{27}$$

It is clear that:

$$Re\left[\frac{B_1 + 2i\omega_0}{B_1\omega_0^2 + i\omega_0(\omega_0^2 - B_2)}\right] = \frac{B_1^2 + 2(\omega_0^2 - B_2)}{B_1^2\omega_0^2 + (\omega_0^2 - B_2)^2},$$

$$Re\left[\frac{B_3}{B_3\omega_0^2 + iB_4\omega_0}\right] = \frac{B_3^2}{B_3^2 + B_4^2},$$

$$Re\left[\frac{i\tau_0}{\omega_0}\right] = zero.$$

Hence, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{B_1^2 + 2(\omega_0^2 - B_2)}{\varphi_0} + \frac{B_3^2}{\varphi_1}$$

Where

$$\varphi_0 = B_1^2\omega_0^2 + (\omega_0^2 - B_2)^2 > 0,$$

$$\varphi_1 = B_3^2 + B_4^2 > 0.$$

We obtain $\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} > 0$ under (23b).

This outcome demonstrates how the roots of C.E. (18), as τ passes through τ_0, traverse the imaginary axis from left to right. As a result, the system (3) experiences an H.B. at $\tau = \tau_0$ and loses its stability.

For the (FOEP), Equation (5) reduces to

$$J(E_3) = \begin{bmatrix} -\left(\frac{\alpha}{1 + n\breve{V}_1} + \mu\right) & \frac{n\alpha\breve{S}}{(1 + n\breve{V}_1)^2} & 0 & -\beta_1\breve{S}e^{-\lambda\tau} \\ \frac{\alpha}{1 + n\breve{V}_1} & -\left(\frac{n\alpha\breve{S}}{(1 + n\breve{V}_1)^2} + \gamma + \mu\right) & 0 & -\beta_2\breve{V}_1 \\ 0 & \gamma & -\mu & -\beta_3\breve{V}_2 \\ 0 & 0 & 0 & \beta_1\breve{S}e^{-\lambda\tau} + \beta_2\breve{V}_1 + \beta_3\breve{V}_2 - (\mu + \mu_1 + \theta) \end{bmatrix} \tag{28}$$

The C.E. of $J(E_3)$ is given by

$$[\lambda^2 + C_1\lambda + C_2][-\mu - \lambda]\left[\beta_1\breve{S}e^{-\lambda\tau} + \beta_2\breve{V}_1 + \beta_3\breve{V}_2 - (\mu + \mu_1 + \theta) - \lambda\right] = 0 \tag{29a}$$

Where

$$C_1=\frac{\alpha}{1+n\breve{V}_1}\left(1+\frac{n\breve{S}}{1+n\breve{V}_1}\right)+\gamma+2\mu$$

$$C_2=\frac{\alpha}{1+n\breve{V}_1}\left(\frac{n\mu\breve{S}}{1+n\breve{V}_1}+\gamma+\mu\right)+\mu(\gamma+\mu)$$

The eq. (29a) represents the eigenvalues of $J(E_3)$ and has 4-roots:

$$\left.\begin{aligned}&\lambda_{1,2}=-\frac{C_1}{2}\mp\frac{1}{2}\sqrt{C_1^2-4C_2}\\&\lambda_3=-\mu\\&\lambda_4=\beta_1\breve{S}e^{-\lambda\tau}+\beta_2\breve{V}_1+\beta_3\breve{V}_2-(\mu+\mu_1+\theta)\end{aligned}\right\}\quad(29b)$$

Now, for $\tau=0$ we get all the above eigenvalues will be negative and the FOEP is L.A.S. if the following condition

$$\beta_1\breve{S}+\beta_2\breve{V}_1+\beta_3\breve{V}_2<\mu+\mu_1+\theta\quad(30)$$

Now, for $\tau>0$ suppose that (29b), has two wholly imaginary roots, namely$\lambda=\pm i\omega$ ($\omega>0$).
By substituting $\lambda=\pm i\omega$ in Equation (29b) we get:

$$\beta_1\breve{S}(cos\omega\tau-isin\omega\tau)=\mu+\mu_1+\theta-\beta_2\breve{V}_1-\beta_3\breve{V}_2+i\omega$$

So, separating the real and imagined parts yields

$$\left.\begin{aligned}&\beta_1\breve{S}cos\omega\tau=\mu+\mu_1+\theta-\beta_2\breve{V}_1-\beta_3\breve{V}_2\\&\beta_1\breve{S}sin\omega\tau=-\omega\end{aligned}\right\}\quad(31)$$

Squaring each equation and then adding them, we get that

$$\omega=\mp\sqrt{\beta_1^2\left(\breve{S}\right)^2-\left(\mu+\mu_1+\theta-\beta_2\breve{V}_1-\beta_3\breve{V}_2\right)^2}$$

Note that, under the condition (30),$\omega(\tau)$ with $\tau>0$ cannot be real, which contradicts with the assumption. Therefore, the C.E. (29a) can’t have purely imaginary root, and E_1 is L.A.S. for all $\tau\geq 0$ if the condition (30) hold.

For the (FIEP) when $\gamma=0$, Equation (4) reduces to

$$J(E_4)=\left(d_{ij}\right)_{4\times 4}\ ;\ i,j=1,2,3,4$$

Here

$$d_{11}=-\left[R_1+R_2e^{-\lambda\tau}\right],d_{12}=\frac{n\alpha\tilde{\breve{S}}}{\left(1+n\tilde{\breve{V}}_1\right)^2},d_{14}=-R_3e^{-\lambda\tau},d_{21}=\frac{\alpha}{1+n\tilde{\breve{V}}_1},$$

$$d_{22}=-\left(\frac{n\alpha\tilde{\breve{S}}}{\left(1+n\tilde{\breve{V}}_1\right)^2}+\beta_2\tilde{\breve{I}}+\mu\right),d_{24}=-\beta_2\tilde{\breve{V}}_1\ ,d_{33}=-\left(\beta_3\tilde{\breve{I}}+\mu\right),\quad d_{41}=R_2e^{-\lambda\tau},$$

$$d_{42}=\beta_2\tilde{\breve{I}}\ ,d_{43}=\beta_3\tilde{\breve{I}}\ ,\ d_{44}=R_3e^{-\lambda\tau}+R_4,d_{13}=d_{23}=d_{31}=d_{32}=d_{34}=0\,.\quad(32)$$

Where, $R_1=\frac{\alpha}{1+n\tilde{\breve{V}}_1}+\mu$, $R_2=\beta_1\tilde{\breve{I}}$, $R_3=\beta_1\tilde{\breve{S}}$, $R_4=\beta_2\tilde{\breve{V}}_1-(\mu+\mu_1+\theta)$.

The C.E of $J(E_4)$ is

$$\left[-\left(\beta_3\tilde{\breve{I}}+\mu\right)-\lambda\right]\left[\lambda^3+D_1\lambda^2+D_2\lambda+D_3+(D_4\lambda^2+D_5\lambda+D_6)e^{-\lambda\tau}\right]=0\quad(33)$$

With

$D_1 = R_1 - d_{22} - R_4$, $D_2 = -(d_{22}R_1 + R_1R_4 + d_{12}d_{21} + d_{23}d_{32}) + d_{22}R_4$,
$D_3 = R_1R_4d_{22} + R_4d_{12}d_{21} - R_1d_{23}d_{32}$, $D_4 = R_2 - R_3$,
$D_5 = -(R_2d_{22} + R_1R_3 + R_2R_4) + d_{22}R_3$,
$D_6 = R_1R_3d_{22} + R_3d_{12}d_{21} + R_2R_4d_{22} + R_3d_{21}d_{32} - (R_2d_{12}d_{23} + R_2d_{12}d_{23} + R_1d_{23}d_{32} + R_2d_{232}d_{32})$.
Now, when $= 0$, Equation (33) becomes

$$\left[-\left(\beta_3\tilde{\tilde{I}} + \mu\right) - \lambda\right][\lambda^3 + (D_1 + D_4)\lambda^2 + (D_2 + D_5)\lambda + D_3 + D_6] = 0 \quad (34)$$

So, either

$$\left[-\left(\beta_3\tilde{\tilde{I}} + \mu\right) - \lambda\right] = 0 \quad (35a)$$

or,

$$[\lambda^3 + (D_1 + D_4)\lambda^2 + (D_2 + D_5)\lambda + D_3 + D_6] = 0 \quad (35b)$$

From Equation (35a) then

$$\lambda_3 = -\beta_3\tilde{\tilde{I}} - \mu < 0$$

Which is always negative eigenvalue.
Now, it is easy to verify that $D_1 + D_4 > 0$ and $D_3 + D_6 > 0$ under the following sufficient conditions

$$\beta_1\tilde{\tilde{S}} + \beta_2\tilde{\tilde{V}}_1 < \mu + \mu_1 + \theta \quad (36a)$$

$$\frac{n\alpha^2\tilde{\tilde{S}}}{\left(1+n\tilde{\tilde{V}}_1\right)^3} < \left(\frac{\alpha}{1+n\tilde{\tilde{V}}_1} + \beta_1\tilde{\tilde{I}} + \mu\right)\left(\frac{n\alpha\tilde{\tilde{S}}}{\left(1+n\tilde{\tilde{V}}_1\right)^2} + \beta_2\tilde{\tilde{I}} + \mu\right) \quad (36b)$$

We have

$$(D_1 + D_4)(D_2 + D_5) - (D_3 + D_6) > 0.$$

Under the following sufficient conditions

$$\frac{n\beta_1\alpha\tilde{\tilde{S}}}{\left(1+n\tilde{\tilde{V}}_1\right)^2} < \left(\frac{n\alpha\tilde{\tilde{S}}}{\left(1+n\tilde{\tilde{V}}_1\right)^2} + \beta_2\tilde{\tilde{I}} + \mu\right)\beta_2 \quad (36c)$$

$$\beta_2\tilde{\tilde{V}}_1\left(\beta_1\tilde{\tilde{S}} + \beta_2\tilde{\tilde{V}}_1\right) + \frac{\beta_1\alpha\tilde{\tilde{S}}}{1+n\tilde{\tilde{V}}_1} < \beta_2\tilde{\tilde{V}}_1(\mu + \mu_1 + \theta) \quad (36d)$$

So, according to Routh-Hurwitz criterion, we see that all roots of Equation (35b), which represent the eigenvalues of (32), have negative real parts. Consequently, under the conditions (36a)-(36d), E_4 is L.A.S for system (3) when $\tau = 0$.

Now, for $\tau > 0$, then
Either, Equation (35a) which is always negative eigenvalue.
or,

$$\left[\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3 + (D_4\lambda^2 + D_5\lambda + D_6)e^{-\lambda\tau}\right] = 0 \quad (37)$$

From (37) has two wholly imaginary roots, namely= $\pm i\omega$ ($\omega > 0$).
By substituting $\lambda = \pm i\omega$ in Equation (37) and separating the real and imagined parts yields

$$\begin{aligned}(D_6 - D_1\omega^2)cos\omega\tau + D_5\omega sin\omega\tau &= D_1\omega^2 - D_3 \\ D_5\omega cos\omega\tau + (D_4\omega^2 - D_6)sin\omega\tau &= \omega^3 - D_2\omega\end{aligned} \quad (38)$$

Through squaring and adding both equations in Equation (38), we get

$$\omega^6 + h_1\omega^4 + h_2\omega^2 + h_3 = 0 \quad (39)$$

Further , by letting $\upsilon = \omega^2$ in Equation (39)

$$h(\upsilon) = \upsilon^3 + h_1\upsilon^2 + h_2\upsilon + h_3 = 0 \quad (40a)$$

Where

$h_1 = D_1^2 - 2D_2 - D_4^2,$
$h_2 = 2D_4D_6 + D_2^2 - D_5^2 - 2D_1D_3,$
$h_3 = D_3^2 - D_6^2.$

Straightforward computation shows that due to the following condition

$$D_3 < D_6 \qquad (40b)$$

we obtain $h_3 < 0$. So, there exists a unique positive root in accordance with Descartes' rule of signs $v_0 = \omega_0^2$ providing Equation (40a). Which is, Equation (39) has a positive ω_0. As a result, Equation (37) has at least two roots $\pm\omega_0$ that are purely imaginary and which correspond to the time delay τ. Additionally, by substituting ω_0 in (38) and solving the system for τ, yields :

$$\tau_k = \frac{1}{\omega_0} cos^{-1} \frac{(D_5 - D_1D_4)\omega_0^4 + (D_1D_6 + D_3D_4 - D_2D_5)\omega_0^2 - D_3D_6}{D_4^2\omega_0^4 + (D_5^2 - 2D_4D_6)\omega_0^2 + D_6^2} + \frac{2k\pi}{\omega_0} \qquad (41)$$

where, k = 0,1,2,…. Hence we get the corresponding $\tau_k > 0$ for which system (3) has two wholly imaginary roots $\pm\omega_0$.

Let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ be a root of Equation (37) near $\tau = \tau_k$ with $\mu(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$. Then comes the next theorem:

Theorem2. The C.E. (37), roots satisfy the following transversality requirement;

$$\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_k} > 0 \qquad (42a)$$

Provided that

$$2D_2 > D_1^2 \qquad (42b)$$

$$D_2^2 > 2D_1D_3 \qquad (42c)$$

$$2D_4D_6 > 2D_4^2\omega_0^2 + D_5^2 \qquad (42d)$$

Proof. By substituting $\lambda(\tau)$ in Equation (37) and differentiating the resulting equation in τ,we can get

$$\left[3\lambda^2 + 2D_1\lambda + D_2 + (2D_4\lambda + D_5)e^{-\lambda\tau} - \tau(D_4\lambda^2 + D_5\lambda + D_6)e^{-\lambda\tau}\right]\frac{d\lambda}{d\lambda} = \lambda(D_4\lambda^2 + D_5\lambda + D_6)e^{-\lambda\tau} \qquad (43)$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2D_1\lambda + D_2}{\lambda(D_4\lambda^2 + D_5\lambda + D_6)e^{-\lambda\tau}} + \frac{2D_4\lambda + D_5}{\lambda(D_4\lambda^2 + D_5\lambda + D_6)} - \frac{\tau}{\lambda} \qquad (44)$$

Since for $= \tau_0$, and $\lambda = i\omega_0$, we have got

$$\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0}^{-1} = \frac{(D_2 - 3\omega_0^2) + 2iD_1\omega_0}{-\omega_0^2(\omega_0^2 - D_2) + i\omega_0(D_1\omega_0^2 - D_3)} + \frac{D_5 + 2iD_4\omega_0}{-D_5\omega_0^2 + i\omega_0(D_6 - D_4\omega_0^2)} - \frac{i\tau_0}{\omega_0}$$

Now, since

$$sign\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0} = sign\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} \qquad (45)$$

It is clear that:

$$Re\left[\frac{(D_2-3\omega_0^2)+2iD_1\omega_0}{-\omega_0^2(\omega_0^2-D_2)+i\omega_0(D_1\omega_0^2-D_3)}\right]=\frac{2D_1(D_1\omega_0^2-D_3)-(\omega_0^2-D_2)(D_2-3\omega_0^2)}{\omega_0^2(\omega_0^2-D_2)^2+(D_1\omega_0^2-D_3)^2}$$

$$Re\left[\frac{D_5+2iD_4\omega_0}{-D_5\omega_0^2+i\omega_0(D_6-D_4\omega_0^2)}\right]=\frac{2D_4(D_6-D_4\omega_0^2)-D_5^2}{D_5^2\omega_0^2+(D_6-D_4\omega_0^2)^2}$$

$$Re\left[\frac{i\tau_0}{\omega_0}\right]=zero$$

Hence, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0}=\frac{2D_1(D_1\omega_0^2-D_3)-(\omega_0^2-D_2)(D_2-3\omega_0^2)}{\Psi_0}+\frac{2D_4(D_6-D_4\omega_0^2)-D_5^2}{\Psi_1}$$

Where

$$\Psi_0=\omega_0^2(\omega_0^2-D_2)^2+(D_1\omega_0^2-D_3)^2>0\,,$$

$$\Psi_1=D_5^2\omega_0^2+(D_6-D_4\omega_0^2)^2>0\,.$$

We obtain $\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0}>0$ (42b- 42d).

This outcome demonstrates how the roots of C.E. (37), as τ passes through τ_0, traverse the imaginary axis from left to right. As a result, the system (3) experiences an H.B. at $\tau=\tau_0$ and loses its stability.

For the (SIEP), Equation (4) reduces to

$$J(E_5)=(c_{ij})_{4\times4}\quad;\ i,j=1,2,3,4$$

Here

$$c_{11}=-[R_1+R_2e^{-\lambda\tau}]\ ,c_{12}=\frac{n\alpha S^*}{(1+nV_1^*)^2}\quad,c_{14}=-R_3e^{-\lambda\tau},\ \ c_{21}=\frac{\alpha}{1+nV_1^*},$$
$$c_{22}=-\left(\frac{n\alpha S^*}{(1+nV_1^*)^2}+\gamma+\beta_2I^*+\mu\right),c_{24}=-\beta_2V_1^*,c_{32}=\gamma,\ \ c_{33}=-(\beta_3I^*+\mu)\,,$$
$$c_{34}=-\beta_3V_2^*\ \ ,c_{41}=R_2e^{-\lambda\tau},\qquad c_{42}=\beta_2I^*\ ,\quad c_{43}=\beta_3I^*,\ \ c_{44}=R_3e^{-\lambda\tau}+R_4,$$
$$c_{13}=c_{31}=c_{23}=0. \tag{46}$$

Where,

$$R_1=\frac{\alpha}{1+nV_1^*}+\mu\,,\ R_2=\beta_1I^*\ \ and\ R_3=\beta_1S^*\,,R_4=\beta_2V_1^*+\beta_3V_2^*-(\mu+\mu_1+\theta)$$

The C.E. of $J(E_5)$ is

$$[\lambda^4+C_1\lambda^3+C_2\lambda^2+C_3\lambda+C_4+(C_5\lambda^3+C_6\lambda^2+C_7\lambda+C_8)e^{-\lambda\tau}]=0 \tag{47}$$

With

$$C_1 = R_1 - (R_4 + c_{22} + c_{33}), C_2 = (c_{22} + c_{33})(R_4 - R_1) + c_{22}c_{33} - (R_1R_4 + c_{12}c_{21} + c_{24}c_{42} + c_{34}c_{43}),$$

$$C_3 = \big(c_{22}c_{33} + R_4c_{22} + R_4c_{33} - (c_{24}c_{42} + c_{34}c_{43})\big)R_1 + (c_{12}c_{21} - c_{22}c_{33})R_4 + (c_{12}c_{21} + c_{24}c_{42})c_{33} + (c_{22}c_{34} - c_{24}c_{32})c_{43},$$

$$C_4 = (c_{22}c_{34}c_{43} + c_{24}c_{33}c_{42} - R_4c_{22}c_{33} - c_{24}c_{32}c_{43})R_1 + c_{12}c_{21}(c_{34}c_{43} - c_{33}R_4),$$

$$C_5 = R_2 - R_3, C_6 = (c_{22} + c_{33})(R_3 - R_2) - (R_1R_3 + R_2R_4),$$

$$C_7 = (c_{33}(c_{22} + R_4) - c_{24}(c_{42} + c_{12}) - c_{34}c_{43})R_2 + (R_3c_{22} + R_4c_{22} + R_3c_{33})R_1 + (c_{12}c_{21} + c_{21}c_{42} - c_{22}c_{33})R_3,$$

$$C_8 = (c_{21}c_{32}c_{43} - (R_1c_{22} + c_{12}c_{21} + c_{21}c_{42})c_{33})R_3 + \big(c_{22}c_{34}c_{43} + c_{24}c_{33}c_{42} + c_{12}c_{24}c_{33} - (R_4c_{22}c_{33} + c_{24}c_{32}c_{43})\big)R_2$$

Now, when $= 0$, Equation (47) is:

$$[\lambda^4 + (C_1 + C_5)\lambda^3 + (C_2 + C_6)\lambda^2 + (C_3 + C_7)\lambda + C_4 + C_8] = 0 \quad (48)$$

all the eigenvalues of Equation (48) will be present in the left half plane and the SIEP is L.A.S. of system (3) under the following condition:

$$2(\beta_1S^* + \beta_2V_1^* + \beta_3V_2^*) < \mu + \mu_1 + \theta \quad (49)$$

Now, for $\tau > 0$, then from Equation (47) has two wholly imaginary roots, namely $\lambda = \pm i\omega$ ($\omega > 0$).

By substituting $\lambda = \pm i\omega$ in Equation (47) and separating the real and imaginary parts, which gives

$$\begin{aligned}(C_8 - C_6\omega^2)cos\omega\tau + (C_7\omega - C_5\omega^3)sin\omega\tau &= C_2\omega^2 - \omega^4 - C_4,\\ (C_7\omega - C_5\omega^3)cos\omega\tau + (C_6\omega^2 - C_8)sin\omega\tau &= C_1\omega^3 - C_3\omega.\end{aligned} \quad (50)$$

Through squaring and adding both equations in Equation (50), we get

$$\omega^8 + h_1\omega^6 + h_2\omega^4 + h_3\omega^2 + h_4 = 0 \quad (51)$$

Further, by letting $\upsilon = \omega^2$ in Equation (51)

$$h(\upsilon) = \upsilon^4 + h_1\upsilon^3 + h_2\upsilon^2 + h_3\upsilon + h_4 = 0, \quad (52a)$$

where

$$h_1 = C_1^2 - 2C_2 - C_5^2,$$
$$h_2 = 2C_5C_7 + C_2^2 + 2C_4 - 2C_1C_3 - C_6^2,$$
$$h_3 = 2C_6C_8 + C_3^2 - C_7^2 - 2C_1C_4,$$
$$h_4 = C_4^2 - C_8^2.$$

Straightforward computation shows that due to the following condition

$$C_4 < C_8 \quad (52b)$$

we obtain $h_4 < 0$. So, there exists a unique positive root in accordance with Descartes' rule of signs $\upsilon_0 = \omega_0^2$ providing Equation (52a). Which is, Equation (51) has a positive ω_0. As a result, Equation (47) has at least two roots $\pm\omega_0$ that are purely imaginary and which correspond to the time delay τ. Additionally, by substituting ω_0 in (50) and solving the system for τ, yields :

$$\tau_k = \frac{1}{\omega_0}cos^{-1}\frac{(C_6 - C_1C_5)\omega_0^6 + (C_1C_7 + C_5 + C_3 - C_6C_2 - C_8)\omega_0^4 + (C_6C_4 + C_2C_8 - C_3C_7)\omega_0^2 - C_4C_8}{C_5^2\omega_0^6 + (C_6^2 - 2C_5C_7)\omega_0^4 + (C_7^2 - 2C_5C_8)\omega_0^2 + C_8^2} + \frac{2k\pi}{\omega_0} \quad (53)$$

where, k=0,1,2,…. Hence we get the corresponding $\tau_k > 0$ for which system (3) has two wholly imaginary roots $\pm\omega_0$.
Let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ be a root of C.E.(47) near $\tau = \tau_k$ with $\mu(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$. Then comes the next theorem.

Theorem3. The C.E. (47), roots satisfy the following transversality requirement;

$$\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_k} > 0 \quad (54a)$$

Provided that

$$4\omega_0^6 + 2C_1^2\omega_0^5 + 2(C_2^2 + 2C_4)\omega_0^2 + 2C_2C_3\omega_0 > 6C_2\omega_0^4 + C_3(C_1 + 4) + 2C_2C_4\,, \quad (54b)$$

$$2C_5^2\omega_0^5 + 2C_6^2\omega_0^2 > C_5C_7\omega_0^3 + 2C_6(C_7 + C_8). \quad (54c)$$

Proof. By substituting $\lambda(\tau)$ in Equation (47) and differentiating the resulting equation in τ,we can get

$$[4\lambda^3 + 3C_1\lambda^2 + 2C_2\lambda + C_3 + (3C_5\lambda^2 + 2C_6\lambda + C_7)e^{-\lambda\tau} - \tau(C_5\lambda^3 + C_6\lambda^2 + C_7\lambda + C_8)e^{-\lambda\tau}]\frac{d\lambda}{d\lambda} = \lambda(C_5\lambda^3 + C_6\lambda^2 + C_7\lambda + C_8)e^{-\lambda\tau}. \quad (55)$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3+3C_1\lambda^2+2C_2\lambda+C_3}{\lambda(C_5\lambda^3+C_6\lambda^2+C_7\lambda+C_8)e^{-\lambda\tau}} + \frac{3C_5\lambda^2+2C_6\lambda+C_7}{\lambda(C_5\lambda^3+C_6\lambda^2+C_7\lambda+C_8)} - \frac{\tau}{\lambda} \quad (56)$$

Since for $= \tau_0$, and $\lambda = i\omega_0$, we have got

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{C_3 - 3C_1\omega_0^2 + i\omega_0(2C_2 - 4\omega_0^2)}{C_1\omega_0^4 + i\omega_0(\omega_0^4 + C_4 - C_2\omega_0^2 - C_3\omega_0)} + \frac{-3C_5\omega_0^2 + C_7 + 2iC_6\omega_0}{C_5\omega_0^4 + i\omega_0(C_7 + C_8 - C_6\omega_0^2)} - \frac{i\tau_0}{\omega_0}$$

Now, since

$$sign\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0} = sign\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} \quad (57)$$

It is clear that :

$$Re\left[\frac{C_3-3C_1\omega_0^2+i\omega_0(2C_2-4\omega_0^2)}{C_1\omega_0^4+i\omega_0(\omega_0^4+C_4-C_2\omega_0^2-C_3\omega_0)}\right] = \frac{C_1C_3\omega_0^3-3C_1^2\omega_0^5+(2C_2-4\omega_0^2)(\omega_0^4+C_4-C_2\omega_0^2-C_3\omega_0)}{\omega_0\left(C_1^2\omega_0^6+(\omega_0^4+C_4-C_2\omega_0^2-C_3\omega_0)^2\right)},$$

$$Re\left[\frac{-3C_5\omega_0^2+C_7+2iC_6\omega_0}{C_5\omega_0^4+i\omega_0(C_7+C_8-C_6\omega_0^2)}\right] = \frac{C_5C_7\omega_0^3-3C_5^2\omega_0^5+2C_6(C_7+C_8-C_6\omega_0^2)}{C_5^2\omega_0^6+(C_6+C_7-C_8\omega_0^2)^2},$$

$$Re\left[\frac{i\tau_0}{\omega_0}\right] = zero.$$

Hence, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{C_1C_3\omega_0^3-3C_1^2\omega_0^5+(2C_2-4\omega_0^2)(\omega_0^4+C_4-C_2\omega_0^2-C_3\omega_0)}{\Psi_0} + \frac{C_5C_7\omega_0^3-3C_5^2\omega_0^5+2C_6(C_7+C_8-C_6\omega_0^2)}{\Psi_1}.$$

Where

$$\Psi_0 = \omega_0(C_1^2\omega_0^6 + (\omega_0^4 + C_4 - C_2\omega_0^2 - C_3\omega_0)^2) > 0\,,$$

$$\Psi_1 = C_5^2\omega_0^6 + (C_6 + C_7 - C_8\omega_0^2)^2 > 0\,.$$

We obtain $\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} > 0$ under (54a-54b).

This outcome demonstrates how the roots of C.E. (47), as τ passes through τ_0, traverse the imaginary axis from left to right. As a result, the system (3) experiences an H.B. at $\tau = \tau_0$ and loses its stability.

4. Numerical Simulation and Discussion

numerical simulation is employed in this section to show the findings of our investigation. In this section, the following hypothetical parameters have been chosen:

$$\Lambda = 0.042\ , \beta_1 = 0.03\ , \beta_2 = 0.02\ , \beta_3 = 0.01\ , \alpha = 0.3\ ,$$
$$n = 50\ ,\ \ \theta = 0.1\ ,\ \gamma = 0.01\ ,\ \mu_1 = 0.03\ ,\ \mu = 0.00015\ , \tau = 25.2225. \tag{58}$$

Investigated is the dynamical behavior of system (3) near the SIEP point when the T.D. is increased. For the set of parameter values provided by (58), the system (3) is numerically solved, and **Figure 1** shows the trajectory of the system (3).

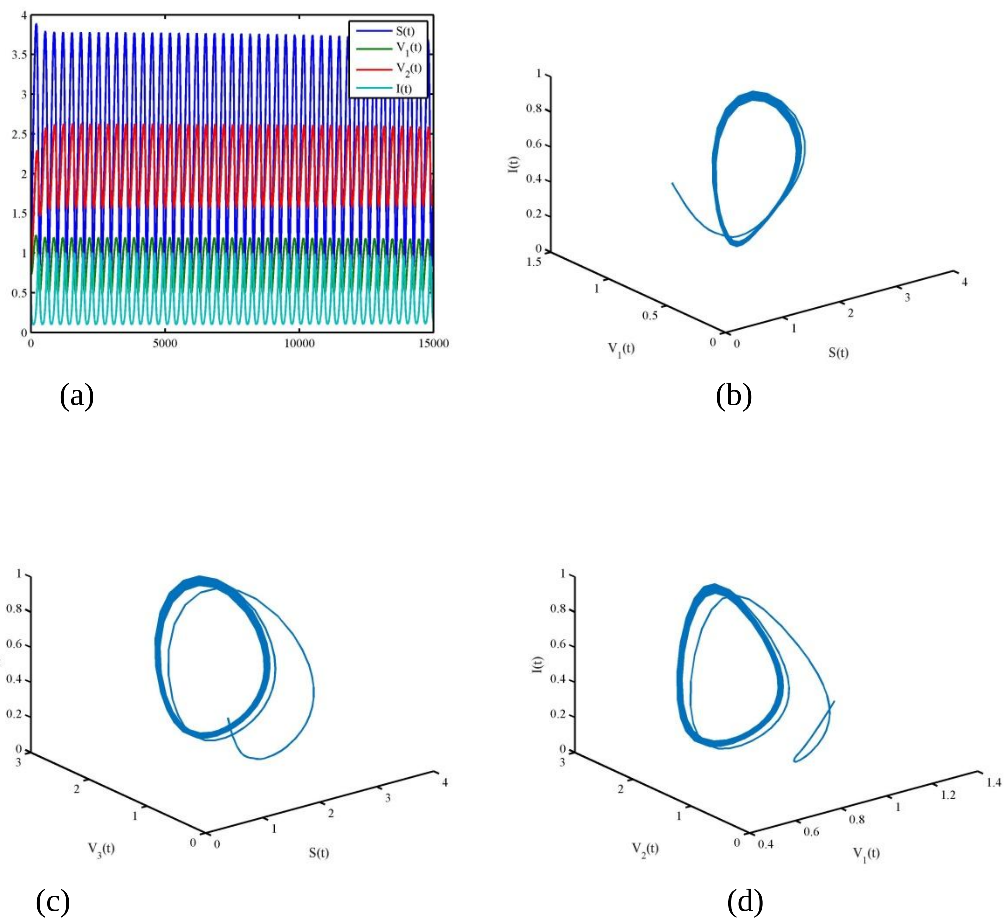


Figure 1. Periodic solution near (SIEP) of system (3) for (58) (a) Periodic solution near (SIEP). (b),(c) and (d) 3D-periodic solution.

Equation (58) is used to observe that for the provided data, with $\tau = 0$ system (3) has a locally asymptotically stable (L.A.S.) to SIEP as shown in **Figure 2**.

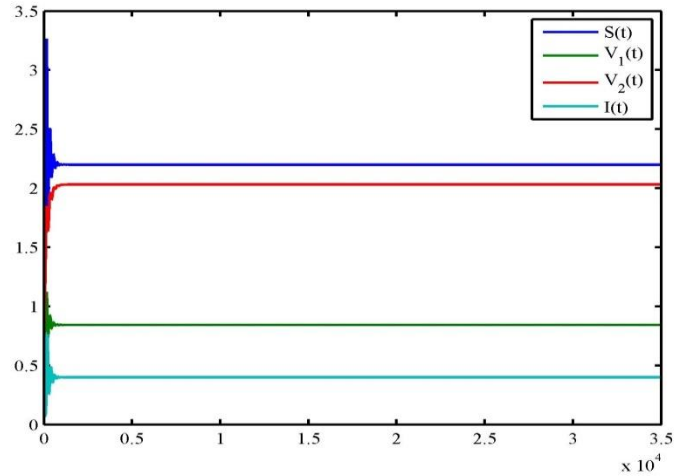


Figure 2. System (3) trajectories using the information provided by (58) with $\tau = 0$ approach to SIEP.

Equation (58) is used to observe that for the provided data, with $\alpha = 0$ and $\beta_1 = 0.0003$ system (3) has a locally asymptotically stable (L.A.S.) to FEP as shown in **Figure 3**.

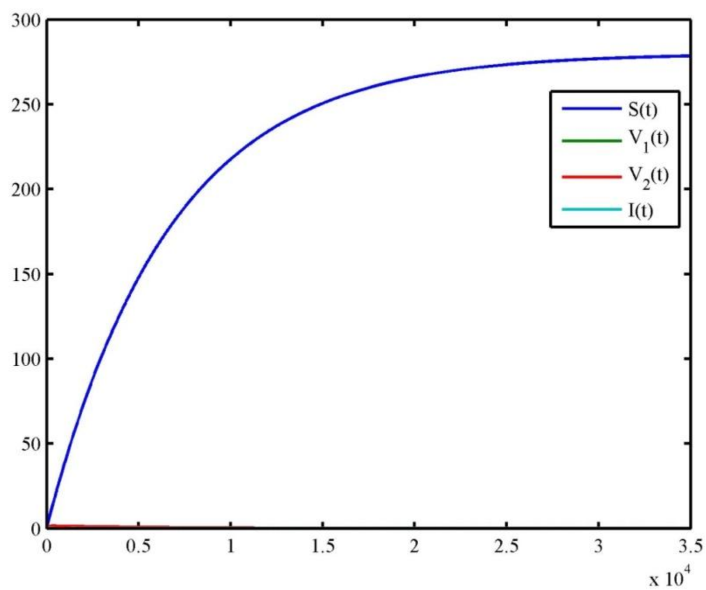


Figure 3. System (3) trajectories using the information provided by (58) with $\alpha = 0$ and $\beta_1 = 0.0003$ approach to FEP.

Equation (58) is used to observe that for the provided data, with $\gamma = 0$, $\beta_1 = 0.0003$ *and* $\beta_2 = 0.0002$ system (3) has a L.A.S. to SEP as shown in **Figure 4**.

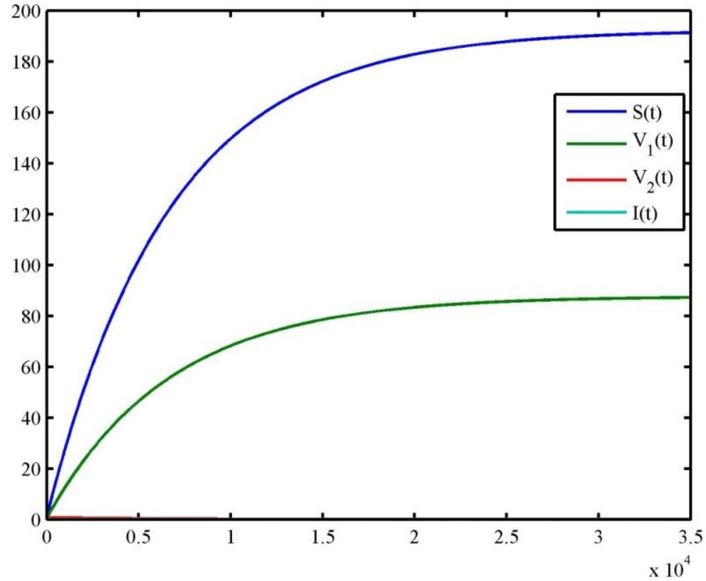


Figure 4. System (3) trajectories using the information provided by equation (58) with $\gamma = 0$, $\beta_1 = 0.0003$ *and* $\beta_2 = 0.0002$ approach to SEP.

We talk about how the time delay affects how the system behaves close to the TEP.

For $\tau = 10 < \tau_0 = 13.5$ and $\alpha = 0$ with the set of data in equation (58) TEP is still L.A.S.as shown in **Figure 5**.

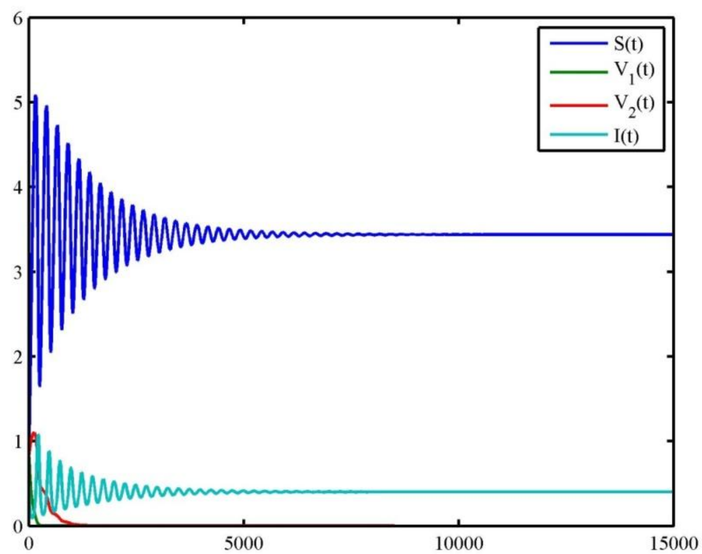
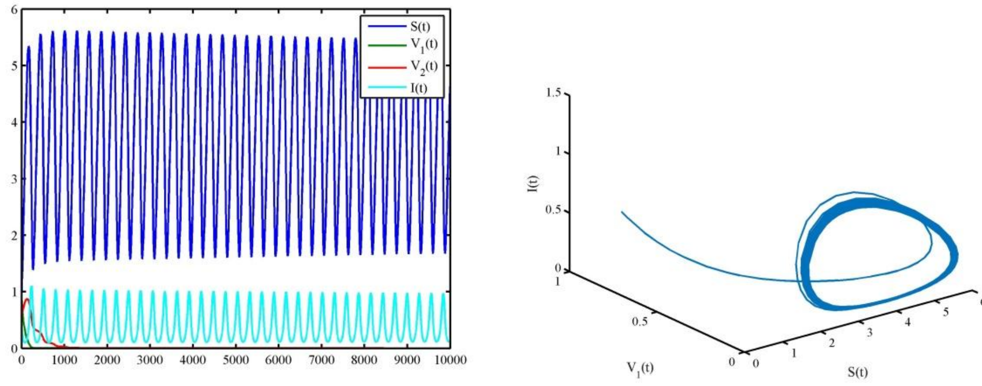


Figure 5. System (3) trajectories using the information provided by (58) with $\alpha = 0$ and $\tau = 10$ approach to TEP.

Now, for $\tau_0 = 13.5$ and $\alpha = 0$ with the set of data in equation (58) a H.B. occurs at TEP as shown in **Figure 6**.



(a) (b)

Figure 6. System (3) trajectories using the information provided by (58) with $\alpha = 0$ and $\tau = 13.5$. (a) Periodic solution near TEP. (b) 3D- periodic solution.

Equation (58) is used to observe that for the provided data, with $\beta_1 = 0.0003\ and\ \beta_2 = 0.0001$ system (3) has a L.A.S. to FOEP as shown in **Figure 7**.

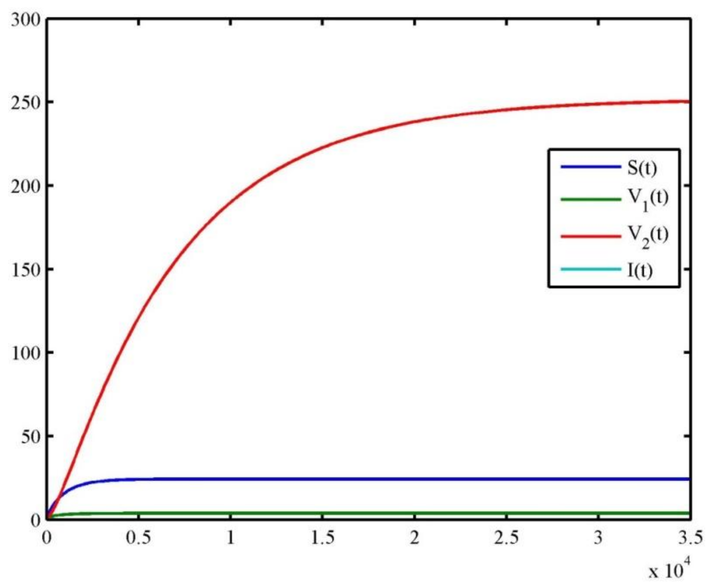


Figure 7. System (3) trajectories using the information provided by equation (58) with $\beta_1 = 0.0003\ and\ \beta_2 = 0.0001$ approach to FOEP.

We talk about how the time delay affects how the system behaves close to the FIEP.
For $\tau = 10 < \tau_0 = 19$ and $\gamma = 0$ with the set of data in (58) FIEP is still L.A.S. as shown in **Figure 8**.

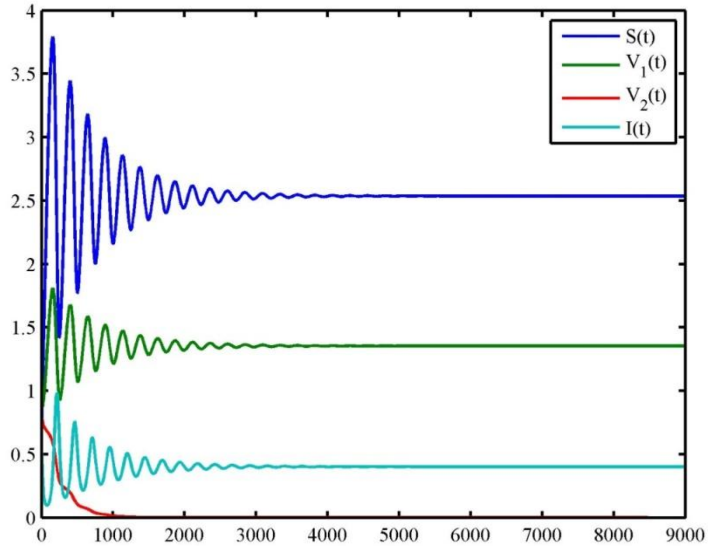
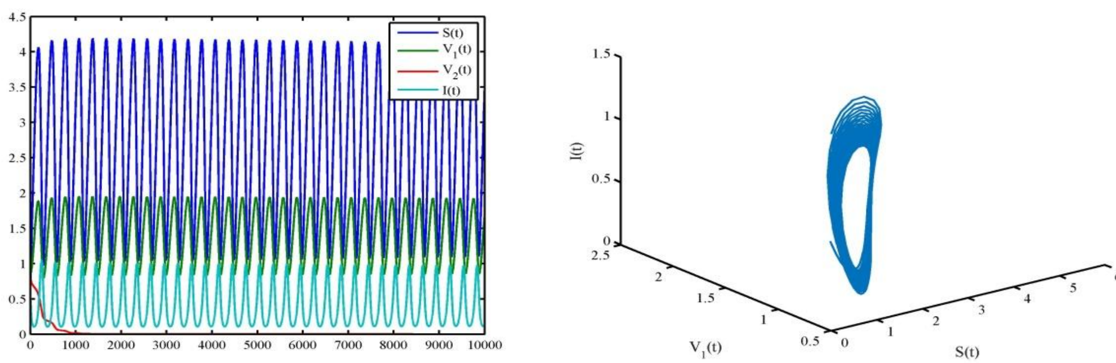


Figure 8. System (3) trajectories using the information provided by (58) with $\gamma = 0$ and $\tau = 10$ approach to FIEP.

Now, for $\tau_0 = 19$ and $\gamma = 0$ with the set of data in equation (58) a H.B. occurs at FIEP as shown in **Figure 9**.



(a) (b)

Figure 9. System (3) trajectories using the information provided by (58) with $\gamma = 0$ and $\tau = 19$. (a) Periodic solution near FIEP. (b) 3D- periodic solution.

3. Conclusion

A mathematical model was proposed and studied for the effect of two stages of the vaccine against the Coronavirus, which includes a time delay for the period of infection with the virus. The suggested system has six equilibrium points, namely FEP, SEP, TEP, FOEP, FIEP and SIEP. The FEP, SEP and FOEP are seen absolutely stable for all $\tau \geq 0$. The TEP, FIEP and SIEP is asymptotically stable for $\tau \in [0, \tau_0)$, but an H.B. occurs when $\tau = \tau_0$. The TEP is still L.A.S. For $\tau = 10 < \tau_0 = 13.5$ and $\alpha = 0$ with the set of data in equation (58) .While, for $\tau = \tau_0 = 13.5$ and $\alpha = 0$, a H.B. is demonstrated near TEP. For $\tau = 10 < \tau_0 = 19$ and $\gamma = 0$ with the set of data in equation (58), the FIEP is still L.A.S. . While, for $\tau = \tau_0 = 19$ and $\gamma = 0$**,** a H.B. is demonstrated near FIEP .

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Conflict of Interest

The authors declare that there are no competing interests regarding the publication of this paper.

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Ethical Clearance

Ethics of scientific research were carried out in accordance with international conditions.

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