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Classification of Subsets of the Projective Line of Order Thirty-Two and its Partitioning into Distinct Subsets

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Abstract

The aim of this paper is to find the inequivalent k-sets in the finite projective line of order thirty-two, $PG(1,32)$. The number of projectively distinct 4-set is five and all of them are of type N(neither harmonic nor equianharmonic). The k-sets, $k = 4, \dots, 11$ have been done, where the number of projectively distinct are 5,11,53,148,481,1240,2964,6049, respectively. The k-sets $k = 12, \dots, 17$ classified depending on the projectively distinct 11-sets whose have non-trivial subgroups only, where the numbers of projectively distinct are 493,5077,2583,288,2412,697. The stabilizer group of each k-sets is computed. The kind of groups that computed for the k-sets are I, Z_2, Z_3, V_4, S_3, $Z_2 \times Z_2 \times Z_2$, $Z_2 \times Z_2 \times Z_2 \times Z_2$ and the large group is the dihedral group of order eleven appears when k is equal to eleven. Also, the projective line $PG(1,32)$ is partitioned into three distinct 11-sets such that two of them are projectively equivalent, and into eight 4-sets of types N_1, N_2, N_3,. N_4, N_5, and into eight 4-sets four of them of type N_3, N_4.

Keywords: Cross-ratio, Finite field, Partition of sets, Projective line.

1. Introduction

Let $F_q = \{\infty, 0, 1, \omega, \omega^2, \dots \omega^{q-2}\}$ be a finite field generated by ω. In $PG(1,q)$, a k-set can be formed by adding one point from the other $q - k + 2$ points to any $(k-1)$-set. From the Fundamental Theorem of Projective Geometry, any three points on a line are projectively equivalent. See [1. Ch. 6]. The points of $PG(1,q)$ are $P(x_0, x_1)$, x_0 and $x_1 \in F_q$ but not both zero. Each point $P(x_0, x_1)$, with $x_1 \neq 0$ is determined by the non-homogeneous coordinate x_0/x_1; the coordinate for point $P(1,0)$ is ∞. Then, the point of $PG(1,q)$ can be represented by the set $F_q \cup \{\infty\}$. A projectivity $\varphi = M(A)$ of $PG(1,q)$ is given by $Y = XA$, where $X = (x_0, x_1)$, $Y = (y_0, y_1)$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $s = y_0/y_1$ and $t = x_0/x_1$; then $s =$

$(at + c/bt + d)$. If $Q_i = P_iA$ for i =2,3,4 and P_i , Q_i have the respective coordinates t_i and s_i, then φ is given by

$$\frac{(s-s_3)(s_2-s_4)}{(s-s_4)(s_2-s_3)} = \frac{(t-t_3)(t_2-t_4)}{(t-t_4)(t_2-t_3)}.$$

In (1), the classification of the projective lines over Galois field of order $q = 2,3,4,5,7,8,9$ are given. In (2)the author did in his thesis, a classification of $PG(1,11)$, and in (3) the author did in his thesis a classification of $PG(1,13)$. In (4), the authors classified the k-sets in projective line of order twenty-seven with partition of the space into five 4-sets, one of type E (equianharmonic) and four of type N (neither harmonic nor equianharmonic), and the full classification of $PG(1, q)$, $q = 19,23,25$ with its application into error-correcting codes have been done as mentioned in the sources of (4). In (5) the authors studied the geometry of line in $PG(1,17)$ rise up to error-correcting code. In (6) the author gave the full classification of inequivalent k-sets in $PG(1,16)$, and for some k in $PG(1,29)$ and in $PG(1,31)$ as in (7) and sources therein. The study of finite dimensional finite projective space has been done by many authors for specific field F_q as appears in the sources (8-28).

Definition 1. (1)**:** The cross-ratio of four ordered distinct points P_1, P_2, P_3, P_4 with coordinates t_1, t_2, t_3, t_4 is

$$\lambda = \{P_1, P_2; P_3, P_4\} = \{t_1, t_2; t_3, t_4\} = \frac{(t_1-t_3)(t_2-t_4)}{(t_1-t_4)(t_2-t_3)}.$$

The cross-ratio has property that

$\lambda=\{t_1, t_2; t_3, t_4\} = \{t_2, t_1; t_4, t_3\} = \{t_3, t_4; t_1, t_2\} = \{t_4, t_3; t_2, t_1\}$.

So $\{P_1, P_2; P_3, P_4\}$ is invariant under a projective group of order four (Klein Group) V_4.

Thus, under all permutations of $\{P_1, P_2; P_3, P_4\}$, the cross-ratio take just the six values

$\lambda,\ 1/\lambda,\ 1-\lambda,\ 1/(1-\lambda),\ (\lambda-1)/\lambda,\ \lambda/(\lambda-1)$.

Also, $\{t_1, t_2; t_3, t_4\}$ takes the values ∞, 0 or 1 if and only if two of the t_i are equal(1).

Definition 2. (1): The 4-set is called harmonic, denoted by H, if the cross-ratio are $-1, 2, 1/2$, equianharmonic, denoted by E, if $\lambda = 1/(1-\lambda)$ or $\lambda = (\lambda-1)/\lambda$ and neither harmonic nor equianharmonic, denoted by N, if the cross-ratio another value.

Clear the characteristic of F_q is 2, so there are no harmonic 4-set. When $p = 3$, then $\lambda = -1 = 2 = 1/2$. The cross-ratio of type E exist if $q \equiv 1$ or 0 (mod 3).

Definition 3. Let ρ_1 and ρ_2 be two projective spaces of n-dimension. A projectivity $\varphi: \rho_1 \longrightarrow \rho_2$ is a bijection given by a non-singular matrix A such that $P(X') = P(X)\varphi$ if and only if $tX' = XA$, where $t \in F_q \setminus \{0\}$. Write $\varphi = M(A)$, then $\varphi = M(\lambda A)$ for any $\lambda \in F_q \setminus \{0\}$.

To determine a projectivity (non-singular 2×2 matrix) on the projective line it enough to have three distinct points.

2. Materials and Methods

2.1.The Projective Line of Order 32

In $PG(1,32)$, the projective line over Galois field of order 32, there are 33 points. The points of $PG(1,32)$ are $F_{32} \cup \{\infty\}$ =$\{\infty, 0, 1, \omega, \ldots, \omega^{30}\}$.

The polynomial function $f(x) = x^2 + \omega^6 x + \omega$ is primitive over F_{32}, then 33 points of $PG(1,32)$can be generated by non-singular matrix; $A = C(f) = [[0 \quad 1], [\omega \quad \omega^6]]$, such that $P(i) = (1,0)A^i, i = 0, \ldots, 32$ as in **Table 1.**

Table 1. The points of $PG(1,32)$.

$P(0) = [1,0]$	$P(1) = [0,1]$	$P(2) = [\omega^{26},1]$
$P(3) = [\omega^{18},1]$	$P(4) = [\omega^{3},1]$	$P(5) = [1,1]$
$P(6) = [\omega^{5},1]$	$P(7) = [\omega^{9},1]$	$P(8) = [\omega^{28},1]$
$P(9) = [\omega^{19},1]$	$P(10) = [\omega^{12},1]$	$P(11) = [\omega^{30},1]$
$P(12) = [\omega^{11},1]$	$P(13) = [\omega^{24},1]$	$P(14) = [\omega^{25},1]$
$P(15) = [\omega^{15},1]$	$P(16) = [\omega^{10},1]$	$P(17) = [\omega^{16},1]$
$P(18) = [\omega^{22},1]$	$P(19) = [\omega^{17},1]$	$P(20) = [\omega^{7},1]$
$P(21) = [\omega^{8},1]$	$P(22) = [\omega^{21},1]$	$P(23) = [\omega^{2},1]$
$P(24) = [\omega^{20},1]$	$P(25) = [\omega^{13},1]$	$P(26) = [\omega^{4},1]$
$P(27) = [\omega^{23},1]$	$P(28) = [\omega^{27},1]$	$P(29) = [\omega,1]$
$P(30) = [\omega^{29},1]$	$P(31) = [\omega^{14},1]$	$P(32) = [\omega^{6},1]$

3. Results and Discussion

This section includes the classification's results of the projective line $PG(1,32)$ into k-sets, where $k = 4, \dots, 17$.

3.1. The 4-sets

Let ξ be all different 3-sets in $PG(1,32)$. Then the order of ξ is $|\xi| = 33 \cdot 32 \cdot 31 = 32736$. But as mentioned in Section 3, any three distinct points on a line are projectively equivalent, so we can fixed the 3-set, $\mathcal{O} = \{\infty, 0,1\}$ to construct $(3 + i)$-set, $i = 0,1, \dots, \frac{q-5}{2}, q > 5$ if q odd and $i = 0,1, \dots, \frac{q-4}{2}, q > 4$ if q even.

A 4-set is constructed by adding to $\mathcal{O} = \{\infty, 0,1\}$ one point from the complement of $\mathcal{O}$.

Let $\mathcal{S}$ be the set of all different 4-set in $PG(1,32)$. Then $\mathcal{S}$ has order $|\mathcal{S}| = \binom{33}{4} = 40920$.

A 4-set of type H and E when $q = 2^5$ does not exist but the 4-set of type N has been divided into 5 classes.

$N_1 \ni \{\infty, 0,1, a\}, a = \{\omega, \omega^{13}, \omega^{14}, \omega^{17}, \omega^{18}, \omega^{30}\}$;

$N_2 \ni \{\infty, 0,1, b\}, b = \{\omega^{2}, \omega^{3}, \omega^{5}, \omega^{26}, \omega^{28}, \omega^{29}\}$;

$N_3 \ni \{\infty, 0,1, c\}, c = \{\omega^{4}, \omega^{6}, \omega^{10}, \omega^{21}, \omega^{25}, \omega^{27}\}$;

$N_4 \ni \{\infty, 0,1, d\}, d = \{\omega^{7}, \omega^{9}, \omega^{15}, \omega^{16}, \omega^{22}, \omega^{24}\}$;

$N_5 \ni \{\infty, 0,1, e\}, e = \{\omega^{8}, \omega^{11}, \omega^{12}, \omega^{19}, \omega^{20}, \omega^{23}\}$.

Since any two 4-sets with same cross-ratio are projectively equivalent, so each class $N_i, i = 1, \dots 5$ is projectively unique. Then among the 40920 of 4-sets there are only five projectively distinct 4-sets, which are given in **Table 2** with its stabilizer group type denoted by SG.

Table 2. The Inequivalent 4-set

Symbol	**4-set**	**.**
$\mathcal{T}_1$	$\{\infty, 0,1, \omega\}$	$V_4 =< \omega/t, t + \omega/t + 1 >$
$\mathcal{T}_2$	$\{\infty, 0,1, \omega^2\}$	$V_4 =< \omega^2/t, t + \omega^2/t + 1 >$
$\mathcal{T}_3$	$\{\infty, 0,1, \omega^4\}$	$V_4 =< \omega^4/t, t + \omega^4/t + 1 >$
$\mathcal{T}_4$	$\{\infty, 0,1, \omega^7\}$	$V_4 =< \omega^7/t, t + \omega^7/t + 1 >$
$\mathcal{T}_5$	$\{\infty, 0,1, \omega^8\}$	$V_4 =< \omega^8/t, t + \omega^8/t + 1 >$

Remark 4.

(i) To reduce the number of constructed $(3+i)$-sets, we will use idea of group action to partition $PG(1,32)$ into distinct orbits and take the first point from each orbit to do the extension of $(3+i)$-sets.

(ii) The GAP program (29) is used to find the action groups, to find the stabilizer group of each $(3+i)$-set, and to run the algorithm (see (2)) which is used find the non-equivalents $(3+i)$-sets.

(iii) To know the kind of stabilizer group of order between 4 and 32 from its structure the reference (30) is used.

3.2. The 5-sets

The projective group $G_{\mathcal{T}_i}$ acts on $\mathcal{T}_i^c$ from the right and splitting it into 5 orbits, four of them of order four and one of them singleton set. Then 5-set constructed by adding one point from each different orbit as in **Table 3**.

Table 3. Partition of $\mathcal{T}_i^c$ by the projectivities of 4-set

$\mathcal{T}_i$	Partition of $\mathcal{T}_i^c$
$\mathcal{T}_1$	$\{\omega^2,\omega^{30},\omega^{14},\omega^{18}\},\{\omega^3,\omega^{29},\omega^8,\omega^{24}\},\{\omega^4,\omega^{28},\omega^{20},\omega^{12}\},\{\omega^5,\omega^{27},\omega^9,\omega^{23}\},$ $\{\omega^6,\omega^{26},\omega^7,\omega^{25}\},\{\omega^{10},\omega^{22},\omega^{13},\omega^{19}\},\{\omega^{11},\omega^{21},\omega^{17},\omega^{15}\},\{\omega^{16}\}$
$\mathcal{T}_2$	$\{\omega^5\},\{\omega^3,\omega^{30},\omega^{22},\omega^{11}\},\{\omega^4,\omega^{29},\omega^{28},\omega^5\},\{\omega^6,\omega^{27},\omega^{16},\omega^{17}\},$ $\{\omega^7,\omega^{26},\omega^{13},\omega^{20}\},\{\omega^8,\omega^{25},\omega^9,\omega^{24}\},\{\omega^{10},\omega^{23},\omega^{18},\omega^{15}\},\{\omega^{12},\omega^{21},\omega^{14},\omega^{19}\}$
$\mathcal{T}_3$	$\{\omega,\omega^3,\omega^{12},\omega^{23}\},\{\omega^2\},\{\omega^5,\omega^{30},\omega^{20},\omega^{15}\},\{\omega^6,\omega^{29},\omega^{13},\omega^{22}\},$ $\{\omega^7,\omega^{28},\omega^{11},\omega^{24}\},\{\omega^8,\omega^{27},\omega^{25},\omega^{10}\},\{\omega^9,\omega^{26},\omega^{21},\omega^{14}\},\{\omega^{16},\omega^{19},\omega^{18},\omega^{17}\}$
$\mathcal{T}_4$	$\{\omega,\omega^6,\omega^{10},\omega^{28}\},\{\omega^2,\omega^5,\omega^{30},\omega^8\},\{\omega^3,\omega^4,\omega^{15},\omega^{23}\},\{\omega^9,\omega^{29},\omega^{27},\omega^{11}\},$ $\{\omega^{12},\omega^{26},\omega^{17},\omega^{21}\},\{\omega^{13},\omega^{25},\omega^{20},\omega^{18}\},\{\omega^{14},\omega^{24},\omega^{16},\omega^{22}\},\{\omega^{19}\}$
$\mathcal{T}_5$	$\{\omega,\omega^7,\omega^5,\omega^3\},\{\omega^2,\omega^6,\omega^{24},\omega^{15}\},\{\omega^4\},\{\omega^9,\omega^{30},\omega^{10},\omega^{29}\},$ $\{\omega^{11},\omega^{28},\omega^{18},\omega^{21}\},\{\omega^{12},\omega^{27},\omega^{26},\omega^{13}\},\{\omega^{14},\omega^{25},\omega^{22},\omega^{17}\},\{\omega^{16},\omega^{23},\omega^{19},\omega^{20}\}$

During the research, the sequence of $i_1, i_2, \ldots, i_n$ refer to type of $(n-1)$-sets in n-set.

Theorem 5: In $PG(1,32)$, there are 11 projectively inequivalent 5-sets, summarized in **Table 4**.

Table 4. Inequivalent 5-set

Symbol	5-set	Type of 5-set	SG
f_1	$\{\infty,0,1,\omega,\omega^2\}$	1 1 1 1 2	$V_4 =< \omega^2/t\,, t+\omega^2/t+1 >$
f_2	$\{\infty,0,1,\omega,\omega^3\}$	1 2 2 4 5	I
f_3	$\{\infty,0,1,\omega,\omega^4\}$	1 2 3 5 5	I
f_4	$\{\infty,0,1,\omega,\omega^5\}$	1 2 3 4 5	I
f_5	$\{\infty,0,1,\omega,\omega^6\}$	1 2 3 3 4	I
f_6	$\{\infty,0,1,\omega,\omega^{10}\}$	1 1 3 4 5	I
f_7	$\{\infty,0,1,\omega,\omega^{16}\}$	1 4 4 4 4	$V_4 = < \omega/t\,, t+\omega/t+1 >$
f_8	$\{\infty,0,1,\omega^2,\omega^4\}$	2 2 2 2 3	$V_4 =< \omega^4/t\,, t+\omega^4/t+1 >$
f_9	$\{\infty,0,1,\omega^2,\omega^8\}$	2 3 4 4 5	I
f_{10}	$\{\infty,0,1,\omega^4,\omega^8\}$	3 3 3 3 5	$V_4 = < \omega^8/t\,, t+\omega^8/t+1 >$
f_{11}	$\{\infty,0,1,\omega^7,\omega^{19}\}$	4 5 5 5 5	$V_4 =< \omega^7/t\,, t+\omega^7/t+1 >$

Table 5. Inequivalent 6-sets

Symbol	6-set	Type of 5-set	SG
h_1	$f_1 \cup \{\omega^3\}$	1 1 2 2 6 6	$Z_2 =< \omega^3/t >$
h_2	$f_1 \cup \{\omega^4\}$	1 2 3 3 4 8	I
h_3	$f_1 \cup \{\omega^6\}$	1 4 5 5 6 7	I
h_4	$f_1 \cup \{\omega^7\}$	1 2 3 5 5 6	I
h_5	$f_1 \cup \{\omega^8\}$	1 2 2 4 5 9	I
h_6	$f_1 \cup \{\omega^{10}\}$	1 1 4 4 6 6	$Z_2 =< t\omega^{19} + \omega^{29}/t\omega^{17} + \omega^{19} >$
h_7	$f_1 \cup \{\omega^{12}\}$	1 1 3 3 6 6	$Z_2 =< t + \omega^2/t + 1 >$
h_8	$f_2 \cup \{\omega^4\}$	2 2 3 3 11 11	$Z_2 =< \omega^4/t >$
h_9	$f_2 \cup \{\omega^5\}$	2 2 4 4 8 8	$Z_2 =< t\omega^{11} + \omega^{14}/t\omega^6 + \omega^{11} >$
h_{10}	$f_2 \cup \{\omega^6\}$	2 2 5 5 8 8	$Z_2 =< t\omega^6 + \omega^{12}/t\omega^3 + \omega^6 >$
h_{11}	$f_2 \cup \{\omega^7\}$	2 2 5 5 9 9	$Z_2 =< t\omega^{13} + \omega^{14}/t\omega^3 + \omega^6 >$
h_{12}	$f_2 \cup \{\omega^8\}$	2 2 2 2 2 2	$S_3 =< t + \omega/t + 1 + \omega^8, t\omega^3 + \omega^6/t >$
h_{13}	$f_2 \cup \{\omega^9\}$	2 2 4 4 9 9	$Z_2 =< t + \omega^9/t + 1 >$
h_{14}	$f_2 \cup \{\omega^{10}\}$	2 3 4 5 6 9	I
h_{15}	$f_2 \cup \{\omega^{11}\}$	2 3 4 6 6 7	I
h_{16}	$f_2 \cup \{\omega^{12}\}$	2 2 3 3 9 9	$Z_2 =< t\omega^{20} + \omega^{23}/t\omega^{19} + \omega^{20} >$
h_{17}	$f_2 \cup \{\omega^{13}\}$	2 3 4 5 6 9	I
h_{18}	$f_2 \cup \{\omega^{15}\}$	2 2 3 3 6 6	$Z_2 =< \omega^3(t + 1)/t + \omega^3 >$
h_{19}	$f_2 \cup \{\omega^{16}\}$	2 2 4 7 9 11	I
h_{20}	$f_2 \cup \{\omega^{19}\}$	2 2 4 4 6 6	$Z_2 =< t + \omega^3/t + 1 >$
h_{21}	$f_2 \cup \{\omega^{20}\}$	2 3 3 4 9 10	I
h_{22}	$f_2 \cup \{\omega^{21}\}$	2 2 5 6 7 9	I
h_{23}	$f_2 \cup \{\omega^{22}\}$	2 3 6 6 9 11	I
h_{24}	$f_2 \cup \{\omega^{23}\}$	2 2 3 3 4 4	$Z_2 =< t\omega^{11} + \omega^{12}/t\omega^8 + \omega^{11} >$
h_{25}	$f_2 \cup \{\omega^{24}\}$	2 2 4 4 5 5	$Z_2 =< \omega(t + 1)/t + \omega >$
h_{26}	$f_2 \cup \{\omega^{25}\}$	2 3 4 5 6 9	I
h_{27}	$f_2 \cup \{\omega^{28}\}$	2 3 3 5 8 9	I
h_{28}	$f_3 \cup \{\omega^{29}\}$	2 2 8 8 9 9	$Z_2 =< \omega/t >$
h_{29}	$f_3 \cup \{\omega^5\}$	3 3 4 4 11 11	$Z_2 =< \omega^5/t >$
h_{30}	$f_3 \cup \{\omega^6\}$	3 4 5 5 6 8	I
h_{31}	$f_3 \cup \{\omega^{10}\}$	3 4 6 6 10 11	I
h_{32}	$f_3 \cup \{\omega^{12}\}$	3 3 3 3 3 3	$S_3 =< t + \omega^{11}/t + 1, t\omega^{12} + \omega^{24}/t >$
h_{33}	$f_3 \cup \{\omega^{13}\}$	3 5 6 9 9 10	I
h_{34}	$f_3 \cup \{\omega^{15}\}$	3 3 4 4 6 6	$Z_2 =< t\omega^{14} + \omega^{18}/t\omega^{30} + \omega^{14} >$
h_{35}	$f_3 \cup \{\omega^{22}\}$	3 3 5 5 6 6	$Z_2 =< t\omega^{30} + \omega^{21}/t\omega^{26} + \omega^{30} >$
h_{36}	$f_3 \cup \{\omega^{23}\}$	3 3 4 4 9 9	$Z_2 =< \omega^4(t + 1)/t + \omega^4 >$
h_{37}	$f_3 \cup \{\omega^{25}\}$	3 3 5 5 10 10	$Z_2 =< t\omega^{30} + \omega^{24}/t\omega^{29} + \omega^{30} >$
h_{38}	$f_3 \cup \{\omega^{26}\}$	3 3 4 4 5 5	$Z_2 =< t + \omega^{26}/t + 1 >$
h_{39}	$f_3 \cup \{\omega^{28}\}$	3 3 9 9 11 11	$Z_2 =< \omega/t >$
h_{40}	$f_4 \cup \{\omega^6\}$	4 4 5 5 9 9	$Z_2 =< \omega^6/t >$
h_{41}	$f_4 \cup \{\omega^9\}$	4 4 5 5 10 10	$Z_2 =< t + \omega/t + 1 >$
h_{42}	$f_4 \cup \{\omega^{15}\}$	4 5 6 9 9 11	I
h_{43}	$f_4 \cup \{\omega^{16}\}$	4 4 7 7 9 9	$Z_2 =< t\omega^{25} + \omega^{30}/t\omega^{24} + \omega^{25} >$
h_{44}	$f_4 \cup \{\omega^{21}\}$	4 4 6 6 9 9	$Z_2 =< t\omega^{14} + \omega^{15}/t\omega^9 + \omega^{14} >$
h_{45}	$f_4 \cup \{\omega^{26}\}$	4 5 8 9 9 10	I
h_{46}	$f_4 \cup \{\omega^{27}\}$	4 4 5 5 6 6	$Z_2 =< \omega/t >$
h_{47}	$f_5 \cup \{\omega^7\}$	5 5 5 5 5 5	$S_3 =< \omega^7/t, t\omega^{13} + \omega^{13}/t\omega^6 + \omega^7 >$
h_{48}	$f_5 \cup \{\omega^{10}\}$	5 5 6 6 10 10	$Z_2 =< t\omega^{17} + \omega^{23}/t\omega^{16} + \omega^{17} >$
h_{49}	$f_5 \cup \{\omega^{16}\}$	5 5 7 7 9 9	$Z_2 =< t + \omega^6/t + 1 >$
h_{50}	$f_5 \cup \{\omega^{21}\}$	5 5 6 6 9 9	$Z_2 =< \omega^6(t + 1)/t + \omega^6 >$
h_{51}	$f_6 \cup \{\omega^{11}\}$	6 6 6 6 6 6	$S_3 =< \omega^{11}/t, t\omega + \omega^{11}/t + 1 >$
h_{52}	$f_6 \cup \{\omega^{19}\}$	6 6 7 7 9 9	$Z_2 =< \omega(t + 1)/t + \omega >$
h_{53}	$f_9 \cup \{\omega^{24}\}$	9 9 9 9 9 9	$S_3 =< t + \omega^8/t + 1, t\omega^{24} + \omega^{17}/t >$

Proof: To say two 5-sets $A = \{\infty, 0,1, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$ projectively equivalent we have to find a 2×2 matrix transform one of them to the other. So to find this matrix (if exists), we will construct a 2×2 matrix, say T, transform the three points $\infty, 0,1$ in A to order three points

in B, say b_1, b_2, b_3. Now if $\{a_4, a_5\}T = \{b_4, b_5\}$, then we say that A and B are projectively equivalent.

Since each 4-set in **Table 2** gives 8 orbits as in **Table 3**, so we have eight 5-sets from each 4-set; that is, we have forty 5-sets.

$\mathcal{U}_1 = \mathcal{T}_1 \cup \{\omega^2\}$, $\mathcal{U}_2 = \mathcal{T}_1 \cup \{\omega^3\}$, $\mathcal{U}_3 = \mathcal{T}_1 \cup \{\omega^4\}$, $\mathcal{U}_4 = \mathcal{T}_1 \cup \{\omega^5\}$,
$\mathcal{U}_5 = \mathcal{T}_1 \cup \{\omega^6\}$, $\mathcal{U}_6 = \mathcal{T}_1 \cup \{\omega^{10}\}$, $\mathcal{U}_7 = \mathcal{T}_1 \cup \{\omega^{11}\}$, $\mathcal{U}_8 = \mathcal{T}_1 \cup \{\omega^{16}\}$,
$\mathcal{U}_9 = \mathcal{T}_2 \cup \{\omega^5\}$, $\mathcal{U}_{10} = \mathcal{T}_2 \cup \{\omega^3\}$, $\mathcal{U}_{11} = \mathcal{T}_2 \cup \{\omega^4\}$, $\mathcal{U}_{12} = \mathcal{T}_2 \cup \{\omega^6\}$,
$\mathcal{U}_{13} = \mathcal{T}_2 \cup \{\omega^7\}$, $\mathcal{U}_{14} = \mathcal{T}_2 \cup \{\omega^8\}$, $\mathcal{U}_{15} = \mathcal{T}_2 \cup \{\omega^{10}\}$, $\mathcal{U}_{16} = \mathcal{T}_2 \cup \{\omega^{12}\}$,
$\mathcal{U}_{17} = \mathcal{T}_3 \cup \{\omega\}$, $\mathcal{U}_{18} = \mathcal{T}_3 \cup \{\omega^2\}$, $\mathcal{U}_{19} = \mathcal{T}_3 \cup \{\omega^5\}$, $\mathcal{U}_{20} = \mathcal{T}_3 \cup \{\omega^6\}$,
$\mathcal{U}_{21} = \mathcal{T}_3 \cup \{\omega^7\}$, $\mathcal{U}_{22} = \mathcal{T}_3 \cup \{\omega^8\}$, $\mathcal{U}_{23} = \mathcal{T}_3 \cup \{\omega^9\}$, $\mathcal{U}_{24} = \mathcal{T}_3 \cup \{\omega^{16}\}$,
$\mathcal{U}_{25} = \mathcal{T}_4 \cup \{\omega\}$, $\mathcal{U}_{26} = \mathcal{T}_4 \cup \{\omega^2\}$, $\mathcal{U}_{27} = \mathcal{T}_4 \cup \{\omega^3\}$, $\mathcal{U}_{28} = \mathcal{T}_4 \cup \{\omega^9\}$,
$\mathcal{U}_{29} = \mathcal{T}_4 \cup \{\omega^{12}\}$, $\mathcal{U}_{30} = \mathcal{T}_4 \cup \{\omega^{13}\}$, $\mathcal{U}_{31} = \mathcal{T}_4 \cup \{\omega^{14}\}$, $\mathcal{U}_{32} = \mathcal{T}_4 \cup \{\omega^{19}\}$,
$\mathcal{U}_{33} = \mathcal{T}_5 \cup \{\omega\}$, $\mathcal{U}_{34} = \mathcal{T}_5 \cup \{\omega^2\}$, $\mathcal{U}_{35} = \mathcal{T}_5 \cup \{\omega^4\}$, $\mathcal{U}_{36} = \mathcal{T}_5 \cup \{\omega^9\}$,
$\mathcal{U}_{37} = \mathcal{T}_5 \cup \{\omega^{11}\}$, $\mathcal{U}_{38} = \mathcal{T}_5 \cup \{\omega^{12}\}$, $\mathcal{U}_{39} = \mathcal{T}_5 \cup \{\omega^{14}\}$, $\mathcal{U}_{40} = \mathcal{T}_5 \cup \{\omega^{16}\}$.

$f_1 = \mathcal{U}_1$ projectively equivalents to $\mathcal{U}_9$;
$f_2 = \mathcal{U}_2$ projectively equivalents to $\mathcal{U}_{10}, \mathcal{U}_{13}, \mathcal{U}_{26}, \mathcal{U}_{33}$;
$f_3 = \mathcal{U}_3$ projectively equivalents to $\mathcal{U}_{16}, \mathcal{U}_{17}, \mathcal{U}_{37}, \mathcal{U}_{38}$;
$f_4 = \mathcal{U}_4$ projectively equivalents to $\mathcal{U}_{15}, \mathcal{U}_{19}, \mathcal{U}_{29}, \mathcal{U}_{36}$;
$f_5 = \mathcal{U}_5$ projectively equivalents to $\mathcal{U}_{12}, \mathcal{U}_{20}, \mathcal{U}_{23}, \mathcal{U}_{25}$;
$f_6 = \mathcal{U}_6$ projectively equivalents to $\mathcal{U}_7, \mathcal{U}_{24}, \mathcal{U}_{30}, \mathcal{U}_{39}$;
$f_7 = \mathcal{U}_8$ projectively equivalents to $\mathcal{U}_{31}$;
$f_8 = \mathcal{U}_{11}$ projectively equivalents to $\mathcal{U}_{18}$;
$f_9 = \mathcal{U}_{14}$ projectively equivalents to $\mathcal{U}_{21}, \mathcal{U}_{27}, \mathcal{U}_{28}, \mathcal{U}_{34}$;
$f_{10} = \mathcal{U}_{22}$ projectively equivalents to $\mathcal{U}_{35}$;

To find the stabilizer group of each 5-set, the same technique explained above has been used with replacement of the set B by A.

The same procedure will be used in Theorems (6), (7), (8) and (10).

3.3. The 6-set

The 6-sets are constructed by adding one point from each orbit to the corresponding 5-sets. The projective group G_{f_i} splits $f_i^c, i = 1, \ldots, 11$ into a number of orbits.

Theorem (6): In $PG(1,32)$, there are 53 projectively inequivalent 6-sets , summarized in the **Table 5.**

3.4. The 7-sets until 17-sets

The 7-sets are constructed by adding one point from each orbit to the corresponding 6-sets. The projective group G_{h_i} splits $h_i^c, i = 1, \ldots, 53$, into a number of orbits.

Theorem 7: In $PG(1,32)$, there are 148 projectively inequivalent 7-sets , summarized in **Table 6.**

Table 6. Inequivalent 7-sets

Symbol	7-set	Type of 6-set	SG
e_1	$h_1 \cup \{\omega^4\}$	1 1 2 2 8 31 31	$Z_2 =< \omega^4/t >$
e_2	$h_1 \cup \{\omega^5\}$	1 2 2 9 14 14 15	I
e_3	$h_1 \cup \{\omega^6\}$	1 2 3 10 22 27 35	I
e_4	$h_1 \cup \{\omega^7\}$	1 3 4 11 19 23 52	I
e_5	$h_1 \cup \{\omega^8\}$	1 4 5 12 15 18 22	I
e_6	$h_1 \cup \{\omega^9\}$	1 5 5 13 14 14 20	I
e_7	$h_1 \cup \{\omega^{10}\}$	1 4 5 6 14 25 46	I
e_8	$h_1 \cup \{\omega^{11}\}$	1 1 6 6 15 15 24	$Z_2 =< \omega^2(t+1)/t+\omega^2 >$
e_9	$h_1 \cup \{\omega^{12}\}$	1 1 7 7 16 23 23	$Z_2 =< t\omega^{20}+\omega^{23}/t\omega^{19}+\omega^{20} >$
e_{10}	$h_1 \cup \{\omega^{13}\}$	1 3 4 7 14 22 50	I
e_{11}	$h_1 \cup \{\omega^{14}\}$	1 4 5 7 21 33 48	I
e_{12}	$h_1 \cup \{\omega^{15}\}$	1 6 7 18 20 34 51	I
e_{13}	$h_1 \cup \{\omega^{16}\}$	1 3 5 6 19 42 44	I
e_{14}	$h_1 \cup \{\omega^{17}\}$	1 3 3 4 4 30 30	$Z_2 =< \omega^3/t >$
e_{15}	$h_2 \cup \{\omega^5\}$	2 2 8 8 9 29 29	$Z_2 =< \omega^2(t+1)/t+\omega^2 >$
e_{16}	$h_2 \cup \{\omega^6\}$	2 3 9 10 15 20 30	I
e_{17}	$h_2 \cup \{\omega^7\}$	2 4 10 14 18 27 28	I
e_{18}	$h_2 \cup \{\omega^8\}$	2 5 11 21 24 36 45	I
e_{19}	$h_2 \cup \{\omega^9\}$	2 5 12 14 14 16 27	I
e_{20}	$h_2 \cup \{\omega^{10}\}$	2 4 6 13 31 42 45	I
e_{21}	$h_2 \cup \{\omega^{12}\}$	2 2 7 7 15 15 32	$Z_2 =< t+\omega^4/t+1 >$
e_{22}	$h_2 \cup \{\omega^{13}\}$	2 4 10 16 33 37 45	I
e_{23}	$h_2 \cup \{\omega^{14}\}$	2 4 7 14 24 27 44	I
e_{24}	$h_2 \cup \{\omega^{15}\}$	2 5 6 9 14 30 34	I
e_{25}	$h_2 \cup \{\omega^{16}\}$	2 3 14 18 30 38 46	I
e_{26}	$h_2 \cup \{\omega^{17}\}$	2 3 19 23 29 30 40	I
e_{27}	$h_2 \cup \{\omega^{18}\}$	2 4 6 7 30 35 38	I
e_{28}	$h_2 \cup \{\omega^{19}\}$	2 5 7 14 18 30 36	I
e_{29}	$h_2 \cup \{\omega^{20}\}$	2 4 9 20 24 25 30	I
e_{30}	$h_2 \cup \{\omega^{21}\}$	2 5 7 14 21 27 33	I
e_{31}	$h_2 \cup \{\omega^{23}\}$	2 2 6 6 23 23 36	$Z_2 =< \omega^4(t+1)/t+\omega^4 >$
e_{32}	$h_2 \cup \{\omega^{24}\}$	2 5 13 16 24 27 28	I
e_{33}	$h_2 \cup \{\omega^{25}\}$	2 5 21 25 37 41 45	I
e_{34}	$h_2 \cup \{\omega^{26}\}$	2 4 14 27 32 34 38	I
e_{35}	$h_2 \cup \{\omega^{27}\}$	2 3 21 22 34 43 45	I
e_{36}	$h_2 \cup \{\omega^{28}\}$	2 2 19 19 28 39 39	$Z_2 =< t+\omega^2/t+1 >$
e_{37}	$h_2 \cup \{\omega^{29}\}$	2 2 10 21 21 27 27	$Z_2 =< \omega^2/t >$
e_{38}	$h_3 \cup \{\omega^7\}$	3 4 14 22 25 40 47	I
e_{39}	$h_3 \cup \{\omega^8\}$	3 5 11 22 25 43 49	I
e_{40}	$h_3 \cup \{\omega^9\}$	3 5 13 19 25 42 50	I
e_{41}	$h_3 \cup \{\omega^{10}\}$	3 3 6 41 41 48 48	$Z_2 =< t\omega^{19}+\omega^{29}/t\omega^{17}+\omega^{19} >$
e_{42}	$h_3 \cup \{\omega^{12}\}$	3 3 7 31 31 37 37	$Z_2 =< t+\omega^{12}/t+1 >$
e_{43}	$h_3 \cup \{\omega^{13}\}$	3 4 14 22 36 46 49	I
e_{44}	$h_3 \cup \{\omega^{14}\}$	3 4 7 15 34 35 46	I
e_{45}	$h_3 \cup \{\omega^{15}\}$	3 5 6 14 15 40 43	I
e_{46}	$h_3 \cup \{\omega^{16}\}$	3 3 42 42 47 49 49	$Z_2 =< t+\omega^2/t+1 >$
e_{47}	$h_3 \cup \{\omega^{17}\}$	3 3 33 33 38 43 43	$Z_2 =< \omega^2(t+1)/t+\omega^2 >$
e_{48}	$h_3 \cup \{\omega^{18}\}$	3 5 6 20 22 44 46	I
e_{49}	$h_3 \cup \{\omega^{19}\}$	3 3 6 7 14 14 52	I
e_{50}	$h_3 \cup \{\omega^{20}\}$	3 4 14 15 27 30 49	I
e_{51}	$h_3 \cup \{\omega^{24}\}$	3 5 14 15 38 49 50	I
e_{52}	$h_3 \cup \{\omega^{25}\}$	3 5 19 24 31 33 41	I
e_{53}	$h_3 \cup \{\omega^{26}\}$	3 4 4 14 15 35 51	I
e_{54}	$h_3 \cup \{\omega^{27}\}$	3 3 45 45 48 52 52	$Z_2 =< \omega^2/t >$
e_{55}	$h_4 \cup \{\omega^8\}$	4 5 10 11 27 30 47	I
e_{56}	$h_4 \cup \{\omega^9\}$	4 5 11 14 16 38 40	I
e_{57}	$h_4 \cup \{\omega^{13}\}$	4 4 18 37 37 48 48	$Z_2 =< t+\omega^2/t+1 >$
e_{58}	$h_4 \cup \{\omega^{15}\}$	4 4 6 14 21 33 50	I
e_{59}	$h_4 \cup \{\omega^{19}\}$	4 5 7 8 23 39 42	I
e_{60}	$h_4 \cup \{\omega^{24}\}$	4 5 8 14 19 22 42	I
e_{61}	$h_4 \cup \{\omega^{25}\}$	4 5 14 23 23 29 35	I

Symbol	7-set	Type of 6-set	SG
e_{62}	$h_4 \cup \{\omega^{26}\}$	4 4 20 31 31 41 41	$Z_2 =< \omega^2/t >$
e_{63}	$h_5 \cup \{\omega^{9}\}$	5 5 9 9 10 10 40	$Z_2 =< t + \omega^2/t + 1 >$
e_{64}	$h_5 \cup \{\omega^{24}\}$	5 5 28 28 45 45 53	$Z_2 =< \omega^2(t+1)/t + \omega^2 >$
e_{65}	$h_5 \cup \{\omega^{25}\}$	5 5 19 19 22 22 52	$Z_2 =< \omega^2/t >$
e_{66}	$h_6 \cup \{\omega^{18}\}$	6 6 7 7 29 31 31	$Z_2 =< t + \omega^2/t + 1 >$
e_{67}	$h_8 \cup \{\omega^{6}\}$	8 10 23 27 28 30 42	I
e_{68}	$h_8 \cup \{\omega^{7}\}$	8 8 11 27 27 39 39	$Z_2 =< t\omega^{13} + \omega^{14}/t\omega^6 + \omega^{13} >$
e_{69}	$h_8 \cup \{\omega^{8}\}$	8 8 12 19 19 21 21	$Z_2 =< \omega^8(t+1)/t + \omega^8 >$
e_{70}	$h_8 \cup \{\omega^{9}\}$	8 13 14 19 22 23 38	I
e_{71}	$h_8 \cup \{\omega^{10}\}$	8 14 14 29 31 31 37	I
e_{72}	$h_8 \cup \{\omega^{11}\}$	8 14 15 16 19 25 42	I
e_{73}	$h_8 \cup \{\omega^{12}\}$	8 16 24 29 32 36 39	I
e_{74}	$h_8 \cup \{\omega^{13}\}$	8 14 22 31 33 35 39	I
e_{75}	$h_8 \cup \{\omega^{15}\}$	8 18 21 23 24 31 34	I
e_{76}	$h_8 \cup \{\omega^{16}\}$	8 15 18 19 20 23 29	I
e_{77}	$h_9 \cup \{\omega^{7}\}$	9 10 11 12 13 25 28	I
e_{78}	$h_9 \cup \{\omega^{9}\}$	9 13 14 21 30 41 45	I
e_{79}	$h_9 \cup \{\omega^{10}\}$	9 10 14 24 27 30 38	I
e_{80}	$h_9 \cup \{\omega^{11}\}$	9 15 19 27 31 45 46	I
e_{81}	$h_9 \cup \{\omega^{12}\}$	9 16 21 27 28 36 45	I
e_{82}	$h_9 \cup \{\omega^{15}\}$	9 14 18 23 27 27 42	I
e_{83}	$h_9 \cup \{\omega^{16}\}$	9 9 19 19 28 23 43	$Z_2 =< t\omega^{25} + \omega^{30}/t\omega^{24} + \omega^{25} >$
e_{84}	$h_9 \cup \{\omega^{21}\}$	9 9 22 22 44 45 45	$Z_2 =< t\omega^{14} + \omega^{15}/t\omega^9 + \omega^{14} >$
e_{85}	$h_{10} \cup \{\omega^{10}\}$	10 14 14 41 45 45 48	I
e_{86}	$h_{10} \cup \{\omega^{13}\}$	10 14 14 25 30 30 46	I
e_{87}	$h_{10} \cup \{\omega^{16}\}$	10 10 19 19 45 45 49	$Z_2 =< t + \omega^6/t + 1 >$
e_{88}	$h_{10} \cup \{\omega^{21}\}$	10 10 22 22 28 28 50	$Z_2 =< \omega^6(t+1)/t + \omega^6 >$
e_{89}	$h_{11} \cup \{\omega^{9}\}$	11 13 19 22 40 42 49	I
e_{90}	$h_{11} \cup \{\omega^{10}\}$	11 14 20 23 42 46 50	I
e_{91}	$h_{11} \cup \{\omega^{11}\}$	11 14 15 22 33 33 41	I
e_{92}	$h_{11} \cup \{\omega^{13}\}$	11 14 27 33 37 45 43	I
e_{93}	$h_{11} \cup \{\omega^{15}\}$	11 14 14 16 18 35 50	I
e_{94}	$h_{11} \cup \{\omega^{20}\}$	11 14 21 28 30 44 45	I
e_{95}	$h_{11} \cup \{\omega^{21}\}$	11 14 14 19 22 42 53	I
e_{96}	$h_{12} \cup \{\omega^{19}\}$	12 14 19 20 22 23 24	I
e_{97}	$h_{13} \cup \{\omega^{10}\}$	13 14 14 24 34 40 46	I
e_{98}	$h_{13} \cup \{\omega^{11}\}$	13 14 15 20 22 43 52	I
e_{99}	$h_{13} \cup \{\omega^{12}\}$	13 14 16 18 21 33 44	I
e_{100}	$h_{13} \cup \{\omega^{16}\}$	13 15 19 23 36 43 44	I
e_{101}	$h_{13} \cup \{\omega^{20}\}$	13 21 21 29 29 39 39	$Z_2 =< t + \omega^9/t + 1 >$
e_{102}	$h_{13} \cup \{\omega^{25}\}$	13 14 27 33 36 45 53	I
e_{103}	$h_{14} \cup \{\omega^{13}\}$	14 14 30 33 45 47 50	I
e_{104}	$h_{14} \cup \{\omega^{15}\}$	14 14 18 21 23 31 32	I
e_{105}	$h_{14} \cup \{\omega^{20}\}$	14 21 33 36 37 40 41	I
e_{106}	$h_{14} \cup \{\omega^{21}\}$	14 15 21 22 23 31 48	I
e_{107}	$h_{14} \cup \{\omega^{22}\}$	14 23 29 34 39 42 44	I
e_{108}	$h_{14} \cup \{\omega^{23}\}$	14 15 18 22 24 25 35	I
e_{109}	$h_{14} \cup \{\omega^{24}\}$	14 14 20 25 36 38 44	I
e_{110}	$h_{14} \cup \{\omega^{27}\}$	14 14 19 19 39 43 49	I
e_{111}	$h_{14} \cup \{\omega^{28}\}$	14 27 30 34 35 36 50	I
e_{112}	$h_{14} \cup \{\omega^{29}\}$	14 23 28 30 33 42 45	I
e_{113}	$h_{15} \cup \{\omega^{12}\}$	15 16 19 20 21 31 34	I
e_{114}	$h_{15} \cup \{\omega^{13}\}$	14 14 15 15 34 44 52	I
e_{115}	$h_{15} \cup \{\omega^{16}\}$	15 19 21 31 36 43 52	I
e_{116}	$h_{15} \cup \{\omega^{21}\}$	15 21 22 33 37 48 49	I
e_{117}	$h_{15} \cup \{\omega^{22}\}$	15 15 23 23 31 31 39	$Z_2 =< t\omega^{14} + \omega^{25}/t\omega^{23} + \omega^{14} >$
e_{118}	$h_{15} \cup \{\omega^{23}\}$	14 15 19 24 29 42 43	I
e_{119}	$h_{15} \cup \{\omega^{26}\}$	15 22 23 42 50 51 52	I
e_{120}	$h_{15} \cup \{\omega^{29}\}$	15 22 27 28 30 49 52	I
e_{121}	$h_{16} \cup \{\omega^{20}\}$	16 21 23 33 39 42 53	I
e_{122}	$h_{16} \cup \{\omega^{21}\}$	14 16 19 22 23 49 52	I
e_{123}	$h_{16} \cup \{\omega^{25}\}$	14 16 22 22 27 30 43	I
e_{124}	$h_{17} \cup \{\omega^{20}\}$	14 21 29 31 41 42 46	I

Symbol	7-set	Type of 6-set	SG
e_{125}	$h_{17} \cup \{\omega^{23}\}$	14 14 21 21 24 37 38	I
e_{126}	$h_{17} \cup \{\omega^{24}\}$	14 23 25 31 33 34 48	I
e_{127}	$h_{17} \cup \{\omega^{25}\}$	14 14 36 39 40 42 42	I
e_{128}	$h_{17} \cup \{\omega^{29}\}$	14 20 27 28 33 40 45	I
e_{129}	$h_{18} \cup \{\omega^{16}\}$	18 19 22 31 33 39 52	I
e_{130}	$h_{19} \cup \{\omega^{24}\}$	19 19 25 27 27 29 29	$Z_2 =< \omega(t+1)/t+\omega >$
e_{131}	$h_{20} \cup \{\omega^{20}\}$	14 20 21 31 35 42 48	I
e_{132}	$h_{21} \cup \{\omega^{24}\}$	21 25 27 38 40 41 45	I
e_{133}	$h_{21} \cup \{\omega^{28}\}$	21 27 30 32 33 35 37	I
e_{134}	$h_{23} \cup \{\omega^{25}\}$	14 23 31 33 44 46 51	I
e_{135}	$h_{23} \cup \{\omega^{28}\}$	23 27 31 37 39 45 50	I
e_{136}	$h_{29} \cup \{\omega^{10}\}$	29 30 31 38 39 42 45	I
e_{137}	$h_{29} \cup \{\omega^{13}\}$	29 31 33 34 36 38 42	I
e_{138}	$h_{30} \cup \{\omega^{10}\}$	30 30 31 31 48 48 51	$Z_2 =< t\omega^{16}/t\omega^{10}+\omega^{16} >$
e_{139}	$h_{30} \cup \{\omega^{13}\}$	30 33 40 45 46 48 50	I
e_{140}	$h_{30} \cup \{\omega^{22}\}$	30 31 35 40 42 44 45	I
e_{141}	$h_{30} \cup \{\omega^{25}\}$	30 30 34 37 37 41 41	$Z_2 =< t\omega^{29}/t\omega^{25}+\omega^{29} >$
e_{142}	$h_{31} \cup \{\omega^{13}\}$	31 33 41 42 44 48 50	I
e_{143}	$h_{33} \cup \{\omega^{22}\}$	33 33 35 49 49 52 52	$Z_2 =< t\omega^{30}+\omega^{21}/t\omega^{26}+\omega^{30} >$
e_{144}	$h_{33} \cup \{\omega^{25}\}$	33 33 37 42 42 45 45	$Z_2 =< t\omega^{30}+\omega^{24}/t\omega^{29}+\omega^{30} >$
e_{145}	$h_{35} \cup \{\omega^{25}\}$	35 37 38 41 46 47 48	I
e_{146}	$h_{40} \cup \{\omega^{16}\}$	40 43 44 49 50 52 53	I
e_{147}	$h_{41} \cup \{\omega^{16}\}$	41 43 43 45 45 49 49	$Z_2 =< t+\omega/t+1 >$
e_{148}	$h_{42} \cup \{\omega^{16}\}$	42 42 43 43 46 52 52	$Z_2 =< t\omega^{9}+\omega^{24}/t\omega^{24}+\omega^{9} >$

By using the same technique, the inequivalent k-sets, $k = 8,9,10,11$ have been found with their stabilizer groups.

Theorem 8: In $PG(1,32)$

(i) There are 481 inequivalent 8-sets and their stabilizer given in **Table 7**.

(ii) There are 1240 inequivalent 9-sets and their stabilizer given in **Table 8**.

(iii) There are 2964 inequivalent 10-sets and their stabilizer given in **Table 9**.

(iv) There are 6049 inequivalent 11-sets and their stabilizer given in **Table 10**.

Table 7. Inequivalent 8-set

NO.	SG.
371	I
105	Z_2
5	$Z_2 \times Z_2 \times Z_2$

Table 8. Inequivalent 9-set

NO.	SG.
1125	I
100	Z_2
5	Z_3
5	S_3
5	$Z_2 \times Z_2 \times Z_2$

Table 9. Inequivalent 10-set

No.	SG.
2691	I
273	Z_2

Table 10. Inequivalent 11-set

No.	SG.
5776	I
272	Z_2
1	D_{11}

Example 9: The 8-set, 9-set and 11-set with large stabilizer group are

(i) There are five 8-sets with stabilizer group of type $Z_2 \times Z_2 \times Z_2$

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^{15}, \omega^{19}\}$, $Z_2 \times Z_2 \times Z_2 = < \frac{\omega^3}{t}, \frac{t+\omega^3}{t+1}, \frac{\omega^{16}t+\omega^{17}}{\omega^{14}t+\omega^{16}} >$.

(ii) There are five 9-sets with stabilizer group of type $Z_2 \times Z_2 \times Z_2$

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^{12}, \omega^{23}\}$, $Z_2 \times Z_2 \times Z_2 = < \frac{\omega^4}{t}, \frac{t+\omega^4}{t+1}, \frac{\omega^{11}t+\omega^{12}}{\omega^8 t+\omega^{11}} >$.

(iii) There is a unique 11-set with stabilizer group of type D_{11}

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^7, \omega^{13}, \omega^{22}, \omega^{28}\}$, $D_{11} = < t + \omega^7, \frac{\omega^{10}}{\omega^6 t+\omega^{13}} >$.

Since the numbers of k-sets, $k = 12, \ldots, 17$ are very large, so we consider only sets that have non-trivial different stabilizers.

Theorem 10: In $PG(1,32)$, there are more than

(i) 493 inequivalent 12-sets.

(ii) 5077 inequivalent 13-sets.

(iii) 2583 inequivalent 14-sets.

(iv) 288 inequivalent 15-sets.

(v) 2412 inequivalent 16-sets.

(vi) 697 inequivalent 17-sets.

The stabilizer groups of k-sets, $k = 12, \ldots, 17$ given in **Table 11**.

Table 11. The stabilizer group

***k*-sets**	**SG**
12-set	438: Z_2 10: Z_3 35: V_4 10: S_3
13-set	4240: I 448: Z_2 35: V_4
14-set	2583: I
15-set	282: Z_2 2: Z_3 4: S_3
16-set	2397: I 15: Z_2
17-set	667: Z_2 29: V_4 1: $Z_2 \times Z_2 \times Z_2 \times Z_2$

Example 11: The 12-set,13-set, 15-set and 17-set with large stabilizer groups as follows

(i) There are ten 12-sets with stabilizer group of type S_3

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^6, \omega^8, \omega^9, \omega^{11}, \omega^{16}\}$, $S_3 = < \frac{\omega^6 t+\omega^{14}}{\omega^5 t+\omega^6}, \frac{\omega^{22}}{\omega^{10}t+\omega^{16}} >$.

(ii) There are 35 13-sets with stabilizer group of type V_4

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^{10}, \omega^{12}, \omega^{24}, \omega^{26}, \omega^{18}\}$, $V_4 = < \frac{\omega^5}{t}, \frac{t+\omega^5}{t+1} >$.

(iii) There are four 15-sets with stabilizer group of type S_3

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^8, \omega^9, \omega^{20}, \omega^{11}, \omega^{24}, \omega^{16}\}$, $S_3 = < \frac{\omega^{14}t+\omega^{23}}{\omega^6 t+\omega^{14}}, \frac{\omega^6 t+\omega^{12}}{t} >$.

(iv) There is a unique 17-set with stabilizer group of type $Z_2 \times Z_2 \times Z_2 \times Z_2$

$\{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7, \omega^{14}, \omega^{24}, \omega^8, \omega^{15}, \omega^{17}, \omega^{22}, \omega^{25}\}$,

$Z_2 \times Z_2 \times Z_2 \times Z_2 = < \frac{\omega^8}{t}, \frac{t+\omega^8}{t+1}, \frac{\omega^{11}t+\omega^{18}}{\omega^{10}t+\omega^{11}}, \frac{\omega^9 t+\omega^{15}}{\omega^7 t+\omega^9} >$.

3.5. Partition of $PG(1{,}32)$

(i) The projective line $PG(1{,}32)$, can be partitioned depending on each projectively distinct 11-set into three 11-sets for example:

The complement of 11-set $\mathcal{K}_{11} = e_1 \cup \{\omega^5, \omega^6, \omega^7, \omega^8\}$, which has stabilizer group Z_2, is $\mathcal{K}_{11}^C = \{\omega^9, \omega^{10}, \dots, \omega^{30}\}$. $\mathcal{K}_{11}^C$ can be partitioned into two 11-sets as follows:

Let $\mathrm{M}_1 = \{\omega^9, \omega^{10}, \omega^{11}, \omega^{12}, \omega^{14}, \omega^{17}, \omega^{20}, \omega^{21}, \omega^{22}, \omega^{25}, \omega^{26}\}$, and $\mathrm{M}_1 = \{\omega^{13}, \omega^{15},$ $\omega^{16}, \omega^{18}, \omega^{19}, \omega^{23}, \omega^{24}, \omega^{27}, \omega^{28}, \omega^{29}, \omega^{30}\}$. $\mathcal{K}_{11}^C$ is projectively equivalent to M_1 by the matrix $[\,[\omega^{23}, \omega^3\,], [\,\omega^8, \omega^{14}]\,]$. Then the triple $\{\mathcal{K}_{11}, \mathrm{M}_1; \mathrm{M}_2\}$ formed a partition of $PG(1{,}32)$ by 11-sets.

(ii) The projective line $PG(1{,}32)$, can be partitioned depending on the five projectively distinct 4-sets into eight 4-sets plus say $\{\infty\}$ point for example:

(a) Partition by eight 4-sets of type N_1:

There are 8184 of 4-sets of type N_1.

$\{0, 1, \omega, \omega^2\}, \lambda = \omega^{17}$; $\{\omega^3, \omega^4, \omega^5, \omega^{13}\}, \lambda = \omega^{30}$; $\{\omega^6, \omega^7, \omega^8, \omega^{16}\}, \lambda = \omega^{30}$; $\{\omega^9, \omega^{10}, \omega^{11}, \omega^{19}\}, \lambda = \omega^{30}$; $\{\omega^{12}, \omega^{14}, \omega^{15}, \omega^{20}\}, \lambda = \omega^{18}$; $\{\omega^{17}, \omega^{18}, \omega^{21}, \omega^{23}\}, \lambda = \omega^{18}$; $\{\omega^{22}, \omega^{25}, \omega^{27}, \omega^{28}\}, \lambda = \omega^{30}$; $\{\omega^{24}, \omega^{26}, \omega^{29}, \omega^{30}\}, \lambda = \omega^{18}$.

(b) Partition by eight 4-sets of type N_2:

There are 8184 of 4-sets of type N_2.

$\{0, 1, \omega, \omega^7\}, \lambda = \omega^{29}$; $\{\omega^2, \omega^3, \omega^4, \omega^{13}\}, \lambda = \omega^3$; $\{\omega^5, \omega^6, \omega^8, \omega^{12}\}, \lambda = \omega^{29}$; $\{\omega^9, \omega^{10}, \omega^{11}, \omega^{14}\}, \lambda = \omega^{26}$; $\{\omega^{15}, \omega^{16}, \omega^{17}, \omega^{20}\}, \lambda = \omega^{26}$; $\{\omega^{18}, \omega^{19}, \omega^{21}, \omega^{25}\}, \lambda = \omega^{29}$; $\{\omega^{22}, \omega^{26}, \omega^{27}, \omega^{30}\}, \lambda = \omega^5$; $\{\omega^{23}, \omega^{24}, \omega^{28}, \omega^{29}\}, \lambda = \omega^{29}$.

(c) Partition by eight 4-sets of type N_3:

There are 8184 of 4-sets of type N_3.

$\{0, 1, \omega, \omega^6\}, \lambda = \omega^4$; $\{\omega^2, \omega^3, \omega^4, \omega^5\}, \lambda = \omega^{25}$; $\{\omega^7, \omega^8, \omega^9, \omega^{10}\}, \lambda = \omega^{25}$; $\{\omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}\}, \lambda = \omega^{25}$; $\{\omega^{15}, \omega^{16}, \omega^{17}, \omega^{18}\}, \lambda = \omega^{25}$; $\{\omega^{19}, \omega^{20}, \omega^{21}, \omega^{22}\}, \lambda = \omega^{25}$; $\{\omega^{23}, \omega^{24}, \omega^{25}, \omega^{27}\}, \lambda = \omega^6$; $\{\omega^{26}, \omega^{28}, \omega^{29}, \omega^{30}\}, \lambda = \omega^{25}$.

(d) Partition by eight 4-sets of type N_4:

There are 8184 of 4-sets of type N_4.

$\{0, 1, \omega, \omega^3\}, \lambda = \omega^9$; $\{\omega^2, \omega^4, \omega^5, \omega^7\}, \lambda = \omega^7$; $\{\omega^6, \omega^8, \omega^9, \omega^{11}\}, \lambda = \omega^7$; $\{\omega^{10}, \omega^{12}, \omega^{13}, \omega^{15}\}, \lambda = \omega^7$; $\{\omega^{14}, \omega^{16}, \omega^{17}, \omega^{19}\}, \lambda = \omega^7$; $\{\omega^{18}, \omega^{20}, \omega^{21}, \omega^{23}\}, \lambda = \omega^7$; $\{\omega^{22}, \omega^{24}, \omega^{26}, \omega^{29}\}, \lambda = \omega^{16}$; $\{\omega^{25}, \omega^{27}, \omega^{28}, \omega^{30}\}, \lambda = \omega^7$.

(e) Partition by eight 4-sets of type N_5:

There are 8184 of 4-sets of type N_5.

$\{0, 1, \omega, \omega^4\}, \lambda = \omega^{20}$; $\{\omega^2, \omega^3, \omega^5, \omega^6\}, \lambda = \omega^{12}$; $\{\omega^7, \omega^8, \omega^9, \omega^{14}\}, \lambda = \omega^{23}$; $\{\omega^{10}, \omega^{11}, \omega^{12}, \omega^{17}\}, \lambda = \omega^{23}$; $\{\omega^{13}, \omega^{15}, \omega^{16}, \omega^{24}\}, \lambda = \omega^8$; $\{\omega^{18}, \omega^{19}, \omega^{20}, \omega^{25}\}, \lambda = \omega^{23}$; $\{\omega^{21}, \omega^{22}, \omega^{23}, \omega^{28}\}, \lambda = \omega^{23}$; $\{\omega^{26}, \omega^{27}, \omega^{29}, \omega^{30}\}, \lambda = \omega^{12}$.

(f) Partition by four 4-sets of type N_5 and:

There are 8184 of 4-sets of type N_5.

Type N_1: $\{0, 1, \omega, \omega^2\}, \lambda = \omega^{17}$;

Type N_2: $\{\omega^3, \omega^4, \omega^5, \omega^8\}, \lambda = \omega^{26}$;

Type N_3: $\{\omega^6, \omega^7, \omega^9, \omega^{20}\}, \lambda = \omega^{25}$;

Type N_4: $\{\omega^{10}, \omega^{11}, \omega^{12}, \omega^{16}\}, \lambda = \omega^{24}$;
Type N_5: $\{\omega^{13}, \omega^{14}, \omega^{15}, \omega^{21}\}, \lambda = \omega^{20}$;
Type N_5: $\{\omega^{17}, \omega^{18}, \omega^{19}, \omega^{24}\}, \lambda = \omega^{23}$;
Type N_5: $\{\omega^{22}, \omega^{26}, \omega^{28}, \omega^{30}\}, \lambda = \omega^{12}$;
Type N_5: $\{\omega^{23}, \omega^{25}, \omega^{27}, \omega^{29}\}, \lambda = \omega^{19}$.

4. Conclusions

In this paper, we introduce and proved there are 5, 11, 53, 148, 481, 1240, 2963, 6049, 493, 5077, 2583, 288, 2412, 697, projectively inequivalent k-sets, $k = 4, \dots, 17$, respectively. The Kind of stabilizer groups which appeared were I, Z_2, Z_3, V_4, S_3, $Z_2 \times Z_2 \times Z_2$, D_{11}. Order of the projective line $PG(1,32)$, which is 33, divisible by 3 and 11 only. So we are able to partition the line into three distinct 11-sets. Also, if we exclude the point ∞ from $PG(1,32)$ we are able to partition the line into eight 4-sets of type N_1, N_2, N_3, N_4, N_5.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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