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The Time Delay Effect and Harvesting on the Predator-Prey: Analysis And Simulation

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Abstract
Mathematical modeling based on time-delay differential equations is an important tool to understand the effects of delays in biological systems and to analyze how they influence the dynamics of the asymptotic behavior of these systems. The prey-predator model described in this paper includes diseases in the prey species, harvests in each population, and time lags in predation and gestation of the predator. The solutions of the model are positive and bounded for all times within a realistic region. The existence of all fixed points has been proven. When a time lag is present, the essential conditions for the local stability of the positive equilibrium and the occurrence of Hopf bifurcations can be determined by analyzing the associated characteristic equation. The characteristics of the Hopf bifurcation are determined by applying normal form theory and the central manifold theorem. Finally, we use numerical simulations to validate our analytical results. A Hopf bifurcation in the system occurs when the delay exceeds a certain threshold.
Keywords: harvest, predator–prey, time delay, stability, Hopf bifurcation.

1. Introduction
The dynamics of interacting populations are often studied using mathematical models. Mathematical models have become essential tools to understand how diseases spread and are controlled. These models, referred to as "epidemiological models," are employed to study disease transmission and management in human or animal populations. On the other hand, the term "ecological model" pertains to mathematical models depicting the dynamic interactions among species in ecological systems. In 1925 and 1926, (1, 2) independently developed mathematical models within ecology, elucidating interactions among biological species. Eco-epidemiological models encompass mathematical representations of dynamic behaviors within ecological systems, including disease dynamics. Anderson and May (3) studied the dynamics of an eco-epidemiology model that included interactions between infected prey populations and

predators in 1986. The researchers had studied the dynamics of the Eco-epidemiological models independently for many years, for example (4-6). It is well known that prey-predator models, which may involve a variety of natural factors, can be used to describe the most significant relationships between the individuals of species in the environment; modeling predator-prey interactions is becoming the most crucial topic for ecologists and applied mathematics research. Predator-prey models may often be divided into three types based on the infectious diseases that affect the population. The first type is that models only include infected prey (7-9). In 2012, a prey-predator model of the Lotka-Volterra type without harvesting was considered by Johri et al.(10). They assumed that there is a disease only in prey and that the susceptible prey's conversion rate is the same as that of the infected prey. The eco-epidemiological model with two prey populations where one prey species has an infectious disease is suggested and studied (11). The second type only includes the diseased in the predator (12-14) the prey-predator model with disease in the predator and the functional response studied and proposed (14). The third type of prey-predator model is a disease in both populations (15 - 17). Kant and Kumar (16) investigated a prey-predator system where the prey migrates, and both populations encounter disease infection. The prey-predator model and assumed there is a disease in both populations have been studied (17). Harvesting strongly influences a model's behavior. It refers to reducing a population through hunting or individual capture. It may have a detrimental impact on harvested population density. In general, there are two kinds: linear and nonlinear. However, from a biological and economic viewpoint, nonlinear harvesting functions are better suited for use in reality. Research studies have used linear harvesting functions, constant-yield prey, or predator harvesting. In 2001, a mathematical model that included immature and mature supposes was studied by Song and Chen (18). The prey-predator system considers harvesting for the prey and predator species studied (19).The modeling of prey-predator involving the harvested with incorporating a prey refuge suggested and formulated (20). However, it is essential to consider the effect of past life history when analyzing the system's stability. Additionally, because time delays occur in so many biological situations (maturation, gestation, capture, or other factors.) in prey-predator systems and ignoring them means ignoring reality, delay differential equations are widely used in the literature on ecological interactions between predator and prey. On the other hand, several studies have been proposed to show the effect of the time delay (21-27). The influence of a delayed incubation period on disease transmission using a nonlinear incidence rate in prey-predator systems was examined (21). Al-Jubouri and Naji (23) proposed a mathematical model incorporating a time delay in the disease transmission process. The effect of time delay on the dynamics of the prey-predator model with prey harvesting was studied (25). In 2011, Naji and Ibrahim (28) formulated the following mathematical model.

$$\begin{aligned}
\frac{dS}{dt} &= rS\left(1-\frac{S+I}{K}\right) - CSI - E_1S \\
\frac{dI}{dt} &= CSI - \frac{\alpha IZ}{\gamma+I} - \lambda I - E_2I \qquad (1) \\
\frac{dZ}{dt} &= -\theta Z + \frac{hIZ}{\gamma+I} - E_3z
\end{aligned}$$

where $S(t)$, $I(t)\ and\ Z(t)$ represent the number of susceptible prey, infected prey and predator respectively; r intrinsic growth rate; K carrying capacity of the prey in absence the predator and harvesting; C infection rate; α maximum attack rate; λ death rate of $I(t)$; γ half saturation level coefficient; θ death rate of $Z(t)$; h growth rate of the $Z(t)$ due to predation of the $I(t)$; E_1 , $E_2\ and\ E_3$ are the harvesting efforts for $S(t)$, $I(t)\ and\ Z(t)$ respectively. The time delay impact is concentrated. The organization of this paper is as follows: First, The given system (1) is modified. Second, it illustrates the positivity and bound of solutions of the system (2). Third, we essentially investigate the stability and existence of Hopf bifurcation. Fourth, investigates the properties of the Hopf bifurcation. Fifth, the major theoretical results and a discussion are shown by numerical simulation. Finally, the conclusion.

Now, the improved system (1) can be expressed as follows.

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - CSI - E_1 S$$
$$\frac{dI}{dt} = CSI - \frac{\alpha I(t-\tau)Z(t-\tau)}{\gamma + I(t-\tau)} - \lambda I - E_2 I \qquad (2)$$
$$\frac{dZ}{dt} = -\theta Z + \frac{hI(t-\tau)Z(t-\tau)}{\gamma + I(t-\tau)} - E_3 z$$

Here, τ represents time delay. The system described above has the same biological interpretation for its parameters as those identified in the system (1).

2. Materials and Methods

2.1. Positive and Boundedness

Before embarking on the study, it is essential to verify the biological integrity of the proposed model. Accordingly, within this section, we introduce the following theorem, which investigates the system's positivity and boundedness.

Theorem 1. The solutions of system **(2)** are positive and bounded.

Proof: First, it is proven that for $t \geq 0$ all solutions to system **(2)** are positive.

$$\frac{dS}{dt} \geq -S\left(\frac{r(S+I)}{K} + cI + E_1\right)$$

Consequently, it is calculated to obtain it.

$$S(t) \geq S(0)exp - \left\{\int_0^t \left(\frac{r\left(S(\sigma)+I(\sigma)\right)}{K} + cI(\sigma) + E_1\right) d(\sigma)\right\}$$

Because $S(0) > 0$. We get $S(t) > 0$ for any $S(0) > 0$.

The proof of $I(t) > 0$ and $Z(t) > 0$ for all $t \geq 0$ can be done in the same way.

Next, the following is the proof that the solutions of system (2) are bounded for all $t \geq 0$.

Define $\mathcal{P}(t) = S(t) + I(t) + Z(t)$

As a result, the following is obtained:

$$\frac{d\mathcal{P}}{dt} + n\mathcal{P} \leq rS,$$

where $n = min\,\{E_1, \lambda + E_2\,, \theta + E_3\}$.

Since $\frac{dS}{dt} \leq rS\left(1 - \frac{S+I}{K}\right) - E_1 S$, we obtain according the comparison theorem (29)

$$\lim_{t\to\infty} sup\, S(t) \leq \frac{K(r - E_1)}{r}$$

Thus, for $t \geq 0$, we have $S(t) \leq \frac{K(r-E_1)}{r}$.

which implies that:

$$\frac{d\mathcal{P}}{dt} + n\mathcal{P} \leq K(r - E_1),$$

Then, after applying the Gronwall lemma to the above inequality, we have for $t \to \infty$

$$\mathcal{P}(t) \leq \frac{K(r-E_1)}{n}$$

This implies that the solutions are bounded.

2.2. Local Stability and Hopf bifurcation

In this subsection, we will determine the local stability of each equilibrium point within system (2). It is well-known that an equilibrium point's location and number unchanged with time delay.. As a result, system **(2)** has four equilibrium points. For further details, see (28)

- The first equilibrium point, namely the vanishing equilibrium point denoted $F_0 = (0,0,0)$ always exists.
- The second equilibrium point, namely the axial equilibrium point denoted $F_1 = \left(\frac{K(r-E_1)}{r}, 0,0\right)$ always exists.
- The third equilibrium point, namely the planer equilibrium point denoted $F_2 = (\bar{S}\ , \bar{I}, 0)$, where

$\bar{S}\ = E_2 + \lambda/C$ (3)

$\bar{I}\ = CK(r - E_1) - r(\lambda + E_2)/(r + CK)C$ (4)

It is clear that F_2 is exists if the following condition satisfied

$r(\lambda + E_2) < cK(r - E_1)$ (5)

- The fourth equilibrium point, namely the positive equilibrium point denoted $F_3 = (S^*\ , I^*, Z^*)$,

where

$$S^* = \frac{hK(r-E_1)-(\theta+E_3)[K(r-E_1)-\gamma(r+cK)]}{r(h-(\theta+E_3)} \quad (6)$$

$$I^* = \frac{\gamma(E_3+\theta)}{h-(\theta+E_3)}; h - (\theta + E_3) \neq 0 \quad (7)$$

$$Z^* = \frac{(\gamma+I^*)(CS^*-(\lambda+E_3))}{\alpha} \quad (8)$$

It is clear that F_3 is exists if the following conditions are hold

$$\gamma < \frac{K(r-E_1)(h-(\theta+E_3))}{(r+CK)(\theta+E_3)} \quad (9)$$

$$\theta + E_3 < h \quad (10)$$

$$\frac{\lambda+E_2}{C} < S^* \quad (11)$$

Now ,the linearization method is used to determine the stability of the above equilibrium points.

The general Jacobian matrix (Jm) for system (2) at any equilibrium point $F = (S, I, Z)$ is

$$JF = \left[\ a_{ij}\right], ij = 0,1,2,3 \quad (12)$$

where

$$a_{11} = r - \left(\frac{r}{K}(2S + I) + CI + E_1\right)\ ; a_{12} = -\left(\frac{r}{K} + C\right)S\ \ ; a_{13} = 0\ ;$$

$$a_{21} = CI\ ; a_{22} = CS - \left(\frac{\alpha\gamma Z e^{-\mu\tau}}{(\gamma+I)^2} - (\lambda + E_2\right) ; a_{23} = \frac{-\alpha I e^{-\mu\tau}}{\gamma+I};$$

$$a_{31} = 0; a_{32} = \frac{h\gamma Z e^{-\mu\tau}}{(\gamma+I)^2}; a_{33} = \frac{hIe^{-\mu\tau}}{\gamma+I} - (\theta + E_3).$$

Then , the characteristic equation corresponding to the matrix above can be expressed as:

$$q_1(\mu) + q_2(\mu)e^{-\mu\tau} = 0 \quad (13)$$

where $q_1(\mu)$ and $q_2(\mu)$ are the polynomial of μ

The (Jm) for system (2) at F_0 is as follows:

$$JF_0 = \begin{bmatrix} r - E_1 & 0 & 0 \\ 0 & -(\lambda + E_2) & 0 \\ 0 & 0 & -(\theta + E_3) \end{bmatrix} \quad (14)$$

The eigenvalues of JF_0 are

$\mu_{01} = r - E_1$, $\mu_{02} = -(\lambda + E_2) < 0$ and $\mu_{03} = -(\theta + E_3) < 0$.

The necessary condition of the coexistence of all species is $r - E_1 > 0$ see in (24)

Obtained that $\mu_{01} > 0$, Thus F_0 is unstable saddle point for all $\tau \geq 0$

The (Jm) for system (2) at F_1 is as follows:

$$JF_1 = \begin{bmatrix} -(r-E_1) & -\frac{(r-E_1)(r+CK)}{r} & 0 \\ 0 & \frac{CK(r-E_1)}{r}-(\lambda+E_2) & 0 \\ 0 & 0 & -(\theta+E_3) \end{bmatrix} \quad (15)$$

The eigenvalues of JF_1 are

$\mu_{11} = -(r-E_1) < 0$, $\mu_{12} = \frac{CK(r-E_1)}{r} - (\lambda+E_2)$ and $\mu_{13} = -(\theta+E_3) < 0$.

It is clear that F_1 is asymptotically stable for all $\tau \geq 0$ Provided that the following condition is met.

$$CK(r-E_1) < r(\lambda+E_2) \quad (16)$$

The (Jm) for system (2) at $F_2$is as follows:

$$JF_2 = \begin{bmatrix} -\frac{r}{CK}(\lambda+E_2) & -\frac{(\lambda+E_2)(r+CK)}{CK} & 0 \\ \frac{\eta}{r+CK} & 0 & -\frac{\alpha\eta e^{-\mu\tau}}{(\gamma+\bar{I})(r+CK)} \\ 0 & 0 & \frac{h\,\bar{I}e^{-\mu\tau}}{\gamma+\bar{I}}-(\theta+E_3); \end{bmatrix} = b_{ij};\ i,j = 1,..,3 \quad (17)$$

where $\eta = CK(r-E_1) - r(\lambda+E_2)$

Clearly, the roots of the following equation represent two eigenvalues of JF_2

$$\mu^2 - (b_{11}+b_{22})\mu + b_{11}b_{22} - b_{12}b_{21} = 0 \quad (18)$$

The above equation have negative real part for all $\tau \geq 0$ under the existence condition (5) of F_2.

While other eigenvalue of JF_2 is given by the root of

$$\mu_{31} + (\theta+E_3) - \frac{h\,\bar{I}e^{-\mu\tau}}{\gamma+\bar{I}} = 0 \quad (19)$$

Thus,

1- if $\tau = 0$ equation (19) has eigenvalue $\mu_{31} = \frac{h\,\bar{I}}{\gamma+\bar{I}} - (\theta+E_3)$ which is negative under the condition

$$\frac{h\,\bar{I}}{\gamma+\bar{I}} < (\theta+E_3) \quad (20)$$

Hence the F_2 is locally asymptotically stable under the conditions (5) and (20) when $\tau = 0$

2- for $\tau > 0$, if equation (19) has the roots which a pair of purely imaginary must intersect the imaginary axis, now let $\mu = i\varpi$ $(\varpi > 0)$ be the root of equation (19)

By substituting $\mu = i\varpi$ in equation (19), we obtain

$$\theta + E_3 = \frac{h\,\bar{I}}{\gamma+\bar{I}}\cos\bar{\omega}\tau$$
$$-\bar{\omega} = \frac{h\,\bar{I}}{\gamma+\bar{I}}\sin\bar{\omega}\tau \quad (21)$$

Squaring and adding both sides of equation (21) ,we obtain the following result:

$$\varpi = \pm\sqrt{\left(\frac{h\,\bar{I}}{\gamma+\bar{I}}\right)^2 - (\theta+E_3)^2} \quad (22)$$

Note that, from the condition (20), we have $\varpi(\tau)$ when $\tau > 0$ it cannot be real, which is in opposition to the assumption. As a result, the root of the characteristic equation (19) cannot be purely imaginary, and is asymptotically stable for all. For all $\tau \geq 0$, the equilibrium point F_2 demonstrates asymptotic stability.

The (Jm) for system (2) at F_3 is as follows:

$$JF_3 = \begin{bmatrix} r - R_1 & -R_2S^* & 0 \\ CI^* & R_5 - \alpha R_3 e^{-\mu\tau} & -\alpha R_4 e^{-\mu\tau} \\ 0 & hR_3 e^{-\mu\tau} & hR_4 e^{-\mu\tau} - R_6 \end{bmatrix} = c_{ij};\ i,j = 1,..3 \quad (23)$$

where

$R_1 = \frac{r}{K}(2S^* + I^*) + CI^* + E_1 > 0$; $R_2 = \frac{r}{K} + C > 0$; $R_3 = \frac{\gamma Z^*}{(\gamma + I^*)^2} > 0$;

$R_4 = \frac{I^*}{\gamma + I^*}$; $R_5 = CS^* - (\lambda + E_3) > 0$; $R_6 = (\theta + E_3) > 0$.

The characteristic equation of JF_3 is

$$\mu^3 + \mathcal{A}_1\mu^2 + \mathcal{A}_2\mu + \mathcal{A}_3 + (\mathcal{A}_4\mu^2 + \mathcal{A}_5\mu + \mathcal{A}_6)e^{-\mu\tau} = 0 \quad (24)$$

Where

$\mathcal{A}_1 = R_1 + R_6 - (r + R_5)$

$\mathcal{A}_2 = (r - R_1)(R_5 - R_6) + R_2C\,S^*I^* - R_5R_6$

$\mathcal{A}_3 = (r - R_1)R_5R_6 + R_2R_6C\,S^*I^*$

$\mathcal{A}_4 = \alpha R_3 - hR_2$

$\mathcal{A}_5 = hR_5R_6 + \alpha R_3R_6 + (r - R_1)(hR_4 - \alpha R_3)$

$\mathcal{A}_6 = -hR_4(R_2C\,S^*I^* + R_5(r - R_1)) - \alpha R_3R_6e^{-\mu\tau}(r - R_1)$

Thus,

1. If $\tau = 0$, then equation (24) becomes:

$$\mu^3 + \mu^2(\mathcal{A}_1 + \mathcal{A}_4) + \mu(\mathcal{A}_2 + \mathcal{A}_5) + (\mathcal{A}_3 + \mathcal{A}_6) = 0$$
(25)

According to the Hurwitz criterion, equation (25) have three negative roots if the following conditions are satisfied.

$r < R_1$ (26)

$R_5 < \alpha R_3$ (27)

$hR_4 < R_6$ (28)

$R_8 < R_7$ (29)

Here

$R_7 = (c_{11} + c_{22} + c_{33})[-c_{11}(c_{22} + c_{33}) - c_{22}c_{33} + c_{23}c_{32} + c_{12}c_{21}]$

$R_8 = -[c_{11}(c_{22}c_{33} - c_{23}c_{32}) - c_{33}c_{12}c_{21}]$

and

$c_{ij};\ i,j = 1,..3$ define in equation (23)

Hence the F_3 is locally asymptotically stable under the conditions (26-29)

2- When $\tau > 0$, Assume that the root of equation (24) is purely imaginary, namely $\mu = \pm i\omega$ $(\omega > 0)$ if in addition to condition (26, 27) and the following condition hold

$\mathcal{A}_6 > \mathcal{A}_3$ (30)

Let $\mu = i\omega$ is the root of equation (24) and by separating equation (24) to the real and imaginary part, yields

$$\begin{aligned} (\mathcal{A}_4\omega^2 - \mathcal{A}_6)\sin\omega\tau + \mathcal{A}_5\omega\cos\omega\tau &= \omega^3 - \mathcal{A}_2\omega \\ \mathcal{A}_5\omega\sin\omega\tau + (\mathcal{A}_6 - \mathcal{A}_4\omega^2)\cos\omega\tau &= \mathcal{A}_1\omega^2 - \mathcal{A}_3 \end{aligned} \quad (31)$$

The result of squaring and summing the above equations yields:

$$\omega^6 + \mathcal{C}_1\omega^4 + \mathcal{C}_2\omega^2 + \mathcal{C}_3 = 0 \quad (32)$$

Where

$\mathcal{C}_1 = \mathcal{A}_1^2 - \mathcal{A}_4^2 - 2\mathcal{A}_2$;

$\mathcal{C}_2 = \mathcal{A}_2^2 - \mathcal{A}_5^2 - 2\mathcal{A}_1\mathcal{A}_3 + 2\mathcal{A}_4\mathcal{A}_6$;

$\mathcal{C}_3 = \mathcal{A}_3^2 - \mathcal{A}_6^2$

Put $\mathcal{H} = \omega^2$, then equation (32) becomes

$$\mathcal{H}^3 + \mathcal{C}_1\mathcal{H}^2 + \mathcal{C}_2\mathcal{H} + \mathcal{C}_3 = 0 \qquad (33)$$

Under the conditions (27,28) and condition (30) we have $\mathcal{C}_3 < 0$. The equation (33) have a positive root which is unique say ω_0 by using Descartes' rule of sign. Hence ω_0 is also the positive root of equation (32). Hence, there exists at least a pair of imaginary roots, denoted as $\pm i\omega_0$, that satisfies equation (24). From equation (31) after substituting ω_0 , we obtain :

$$\cos\ \omega_0\tau = \frac{\mathscr{b}_1}{\mathscr{b}_2}$$

Here

$$\mathscr{b}_1 = (\mathcal{A}_5 - \mathcal{A}_1\mathcal{A}_4)\omega_0^4 + (\mathcal{A}_1\mathcal{A}_6 + \mathcal{A}_3\mathcal{A}_4 - \mathcal{A}_2\mathcal{A}_5)\omega_0^2 - \mathcal{A}_3\mathcal{A}_6$$

$$\mathscr{b}_2 = \mathcal{A}_4^2\omega_0^4 + (\mathcal{A}_5^2 - 2\mathcal{A}_4\mathcal{A}_6)\omega_0^2 + \mathcal{A}_6^2$$

Then, τ_m corresponding to ω_0 as below

$$\tau_m = \frac{1}{\omega_0}(\cos^{-1}(\frac{\ell_1}{\ell_2}) + 2\pi m)\ ; m = 0,1,2,\ldots \qquad (34)$$

Define $\tau_0 = \min_{m\geq 0} \tau_m$

Then, we obtain the following theorem

Theorem 2. The system (2) asymptotic stability at F_3 within the $\tau \in [0, \tau_0)$ and a Hopf bifurcation occurs at $\tau = \tau_0$ when specific conditions are met.

$$3(\omega_0^2)^2 + 2\mathcal{C}_1\omega_0^2 + \mathcal{C}_2 \neq 0 \qquad (35)$$

Proof.

for $\tau \in [0, \tau_0)$ F_3 is asymptotically stable as shown in conditions (26) - (29).

However, when $\tau = \tau_0$, we can demonstrate the presence of a Hopf bifurcation by establishing that F_3 is conditionally stable, specifically, by confirming that equation (24) exhibits purely imaginary roots $\pm i\,\omega_0$ at $\tau = \tau_0$, This condition can be expressed $\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} \neq 0$

If we assume that equation (24) has the eigenvalue that is $\mu(\tau) = \eth(\tau) + i\omega(\tau)$ such that $\eth(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0 > 0$. τ_0 define in equation (34).

When we differentiate equation (24) with respect to τ, and apply the chain rule. This results in the following expression:

$$[3\mu^2 + 2\mathcal{A}_1\mu + \mathcal{A}_2 + (2\mathcal{A}_4\mu + \mathcal{A}_5)e^{-\mu\tau} - \tau(\mathcal{A}_4\mu^2 + \mathcal{A}_5\mu + \mathcal{A}_6)e^{-\mu\tau}]\frac{d\mu}{d\tau}$$
$$= \mu(\mathcal{A}_4\mu^2 + \mathcal{A}_5\mu + \mathcal{A}_6)e^{-\mu\tau} \qquad (36)$$

From equation (24), we have

$$\left[\frac{d\mu}{d\tau}\right]^{-1} = \frac{(3\mu^2 + 2\mathcal{A}_1\mu + \mathcal{A}_2)e^{\mu\tau}}{\mu(\mathcal{A}_4\mu^2 + \mathcal{A}_5\mu + \mathcal{A}_6)} + \frac{2\mathcal{A}_4\mu + \mathcal{A}_5}{\mu(\mathcal{A}_4\mu^2 + \mathcal{A}_5\mu + \mathcal{A}_6)} - \frac{\tau}{\mu} \qquad (37)$$

Since for $\tau = \tau_0,$ *and* $\mu = i\omega_0$ we get

$$\left[\frac{d\mu}{d\tau}\right]^{-1} = \frac{((\mathcal{A}_2 - 3\omega_0^2) + 2i\,\mathcal{A}_1\omega_0)\,(\cos\omega_0\tau + i\sin\omega_0\tau)}{-\mathcal{A}_5\omega_0^2 + i\omega_0(\mathcal{A}_6 - \mathcal{A}_4\omega_0^2)} + \frac{\mathcal{A}_5 + 2i\,\mathcal{A}_4\omega_0}{-\mathcal{A}_5\omega_0^2 + i\omega_0(\mathcal{A}_6 - \mathcal{A}_4\omega_0^2)} - \frac{\tau_0}{i\omega_0}$$

Now since

$$sign\left[\frac{d(Re\mu)}{d\tau}\right]_{\tau=\tau_0} = sign\left[Re(\frac{d\mu}{d\tau})^{-1}\right]_{\mu=i\omega_0} . \qquad (38)$$

It is clear that :

$$\left[\frac{d\mu}{d\tau}\right]^{-1} = \frac{((\mathcal{A}_2 - 3\omega_0^2) + 2i\,\mathcal{A}_1\omega_0)\,(\cos\omega_0\tau + i\sin\omega_0\tau)}{-\mathcal{A}_5\omega_0^2 + i\omega_0(\mathcal{A}_6 - \mathcal{A}_4\omega_0^2)} + \frac{\mathcal{A}_5 + 2i\,\mathcal{A}_4\omega_0}{-\mathcal{A}_5\omega_0^2 + i\omega_0(\mathcal{A}_6 - \mathcal{A}_4\omega_0^2)} - \frac{\tau_0}{i\omega_0}$$

Hence, we have

$$Re\left[\frac{d\mu}{d\tau}\right]^{-1}_{\tau=\tau_0} = Re\,\frac{((\mathcal{A}_2-3\omega_0^2)+2i\,\mathcal{A}_1\omega_0)\,(\cos\omega_0\tau+i\sin\omega_0\tau)}{-\mathcal{A}_5\omega_0^2+i\omega_0(\mathcal{A}_6-\mathcal{A}_4\omega_0^2)} + Re\,\frac{\mathcal{A}_5+2i\,\mathcal{A}_4\omega_0}{-\mathcal{A}_5\omega_0^2+i\omega_0(\mathcal{A}_6-\mathcal{A}_4\omega_0^2)}$$

$$Re\left[\frac{d\mu}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{\omega_0^2\left[3\omega_0^4 + (2\mathcal{A}_1^2 - 4\mathcal{A}_2 - 2\mathcal{A}_4^2)\omega_0^2 + \left(\mathcal{A}_2^2 - 2\mathcal{A}_1\mathcal{A}_3 - \mathcal{A}_5^2 + 2\mathcal{A}_4\mathcal{A}_6\right)\right]}{(\mathcal{A}_5\omega_0^2)^2 + (\mathcal{A}_6 - \mathcal{A}_4\omega_0^2)^2}$$

$$= \frac{f'(\omega_0^2)}{(\mathcal{A}_5\omega_0^2)^2 + (\mathcal{A}_6 - \mathcal{A}_4\omega_0^2)^2}$$

Here $f'(\omega_0^2) = 3(\omega_0^2)^2 + 2\mathcal{C}_1\omega_0^2 + \mathcal{C}_2 \neq 0$ due to condition (35). So, we have

$$Sign\left\{\frac{d}{d\tau}(Re\mu)|_{\tau=\tau_0}\right\} = Sign\left\{Re\left(\frac{d\mu}{d\tau}\right)^{-1}_{\tau=\tau_0}\right\} = Sign\{f'(\omega_0^2)\}$$

Assuming that $\frac{d}{d\tau}(Re\mu)|_{\tau=\tau_0} < 0$, it is implies that the roots of the characteristic has roots with positive real parts at $\tau = \tau_0$. This contradicts the local stability of the positive equilibrium point .Therefore , we can deduce that $\left[\frac{d(Re\mu)}{d\tau}\right]_{\tau=\tau_0} > 0$ under condition (35).Consequently , the transversality condition is satisfied , leading to a Hopf bifurcation happens at $\tau = \tau_0$, $and\ \ \mu = i\omega_0$.

2.3. The Direction and Stability of the Hopf Bifurcation.

In this section, we investigate the orientation of the Hopf bifurcation near F_3 at $\tau = \tau_0$and establish the prerequisites for the stability of the resulting periodic solution in the system (2). We accomplish this by applying Hassard's center manifold theorem and normal form theory (30).

Theorem 3.

(i) If $\mathcal{M}_2 > 0$, then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\tau > \tau_0$, and If $\mathcal{M}_2 < 0$,then the Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for $\tau < \tau_0$.

(ii) If $\mathcal{U}_2 < 0$, then the bifurcating periodic solution are stable, and if $\mathcal{U}_2 > 0$, then the bifurcating periodic solution are unstable

(iii) If $\mathcal{T}_2 > 0$, the period of the bifurcating cyclic solutions increases, and if $\mathcal{T}_2 < 0$, the period decreases.

where $\mathcal{M}_2$, $\mathcal{U}_2$and $\mathcal{T}_2$ are given

$$\left.\begin{aligned} C_1(0) &= \frac{i}{2\omega_0\tau_0}\left(\mathcal{G}_{11}\ \mathcal{G}_{20} - 2|\mathcal{G}_{11}|^2 - \frac{|\mathcal{G}_{02}|^2}{3}\right) + \frac{\mathcal{G}_{21}}{2},\\ \mathcal{M}_2 &= -\frac{Re\{C_1(0)\}}{Re\left\{\frac{d\mu}{d\tau}(\tau_0)\right\}},\\ \mathcal{U}_2 &= 2Re\{C_1(0)\},\\ \mathcal{T}_2 &= \frac{-Im\{C_1(0)\}+\mathcal{M}_2\ Im\left\{\frac{d\mu}{d\tau}(\tau_0)\right\}}{\omega_0\tau_0}. \end{aligned}\right\} \qquad (39)$$

and $\mathcal{G}_{11}$, $\mathcal{G}_{20}$, $\mathcal{G}_{02}$ and $\mathcal{G}_{21}$ are given in the proof

Proof. Let $\mathfrak{U}_1(t) = S(t) - S^*$, $\mathfrak{U}_2(t) = I(t) - I^*$, $\mathfrak{U}_3(t) = Z(t) - Z^*$, and $\tau = \tau_0 + \mathcal{S}$, here τ_0 is define by equation (34) and $\mathcal{S} \in R$. It is possible to convert system (2) into a functional differential equation in $C = C([-1,0], R^3)$ then

$$\mathfrak{U}'(t) = L_{\mathcal{S}}(\mathfrak{U}_t) + \mathcal{F}(\mathcal{S}, \mathfrak{U}_t), \qquad (40)$$

Where
$\mathfrak{U}(t) = (\mathfrak{U}_1(t), \mathfrak{U}_2(t), \mathfrak{U}_3(t))^T \in C = C([-1,0], R^3)$ and $L_{\mathcal{S}}: C \to R^3$, $\mathcal{F}: R \times C \to R^3$ are given by:

$$L_{\mathcal{S}}(\Gamma) = (\mathcal{S} + \tau_0)[\mathfrak{D}_1\Gamma(0) + \mathfrak{D}_2\Gamma(-1)] \tag{41}$$

The nonlinear is

$$\mathcal{F}(\mathcal{S}, \Gamma) = (\mathcal{S} + \tau_0)\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\end{pmatrix}$$

where

$$\mathfrak{D}_1 = \begin{bmatrix}\mathcal{F}_{10}^{(1)} & \mathcal{F}_{01}^{(1)} & 0\\ \mathcal{F}_{1000}^{(2)} & \mathcal{F}_{0100}^{(2)} & 0\\ 0 & 0 & \mathcal{F}_{100}^{(3)}\end{bmatrix} = \begin{bmatrix} r - R_1 & -R_2 S^* & 0\\ CI^* & R_5 & 0\\ 0 & 0 & -R_6\end{bmatrix},$$

$$\mathfrak{D}_2 = \begin{bmatrix}0 & 0 & 0\\ \mathcal{F}_{0010}^{(2)} & 0 & \mathcal{F}_{0001}^{(2)}\\ \mathcal{F}_{010}^{(3)} & 0 & \mathcal{F}_{001}^{(3)}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ -\alpha R_3 & 0 & -\alpha R_4\\ hR_3 & 0 & hR_4\end{bmatrix},$$

with R_1, R_2, R_3 and R_4 are define in the JF_3, while

$$\mathcal{H}_1 = \sum_{i+j\geq 2} \frac{1}{i!j!}\mathcal{F}_{ij}^{(1)}\Gamma_1^i(0)\Gamma_2^j(0),$$

$$\mathcal{H}_2 = \sum_{i+j+m+n\geq 2} \frac{1}{i!\,j!\,m!\,n!}\mathcal{F}_{ijmn}^{(2)}\Gamma_1^i(0)\Gamma_2^j(0)\tilde{\Gamma}_1^m(-1)\tilde{\Gamma}_2^n(-1),$$

$$\mathcal{H}_3 = \sum_{k+m+n\geq 2} \frac{1}{k!\,m!\,n!}\mathcal{F}_{kmn}^{(3)}\Gamma_3^k(0)\tilde{\Gamma}_1^m(-1)\tilde{\Gamma}_2^n(-1),$$

where, $\Gamma(\upsilon_0) = \big(\Gamma_1(\upsilon_0), \Gamma_2(\upsilon_0), \Gamma_3(\upsilon_0)\big) \in C, -1 \leq \upsilon_0 \leq 0$, and

$$\mathcal{F}_{ij}^{(1)}\Gamma_1^i(0)\Gamma_2^j(0), = \frac{\partial^{i+j}\mathcal{F}^{(1)}}{\partial\Gamma_1^i\Gamma_2^j}\bigg|_{(\Gamma_1,\Gamma_2)=(0,0)},$$

$$\mathcal{F}_{ijmn}^{(2)}\Gamma_1^i(0)\Gamma_2^j(0)\tilde{\Gamma}_1^m(-1)\tilde{\Gamma}_2^n(-1) = \frac{\partial^{i+j+m+n}\mathcal{F}^{(2)}}{\partial\Gamma_1^i\Gamma_2^j\tilde{\Gamma}_1^m\tilde{\Gamma}_2^n}\bigg|_{(\Gamma_1,\Gamma_2,\tilde{\Gamma}_1,\tilde{\Gamma}_2)=(0,0,-1,-1)},$$

$$\mathcal{F}_{kmn}^{(3)}\Gamma_3^k(0)\tilde{\Gamma}_1^m(-1)\tilde{\Gamma}_2^n(-1) = \frac{\partial^{k+m+n}\mathcal{F}^{(3)}}{\partial\Gamma_3^k\tilde{\Gamma}_1^m\tilde{\Gamma}_2^n}\bigg|_{(\Gamma_3,\tilde{\Gamma}_1,\tilde{\Gamma}_2)=(0,-1,-1)}.$$

Based on the Riesz representation theorem, a 3×3 matrix function $\mathcal{M}_0\,(\upsilon_0, \mathcal{S})$ exists for $-1 \leq \upsilon_0 \leq 0$ such that.

$$L_{\mathcal{S}}(\Gamma) = \int_{-1}^{0} d\,\mathcal{M}_0(\upsilon_0, \mathcal{S})\Gamma(\upsilon_0) \;\; for \;\; \Gamma \epsilon C. \tag{42}$$

In actuality, it can be chosen.

$$\mathcal{M}_0(\upsilon_0, \mathcal{S}) = (\tau_0 + \mathcal{S})\,(\mathfrak{D}_1\sigma(\upsilon_0) - \mathfrak{D}_2\sigma(\upsilon_0 + 1))\,, \tag{43}$$

here, σ is called the Dirac delta function and

$$\sigma(\upsilon_0) = \begin{cases}1 & \upsilon_0 = 0\\ 0 & \upsilon_0 \neq 0\end{cases}.$$

For $\Gamma \in C([-1,0], R^3)$, define

$$\mathcal{A}(\mathcal{S})\Gamma(\upsilon_0) = \begin{cases}\dfrac{d\Gamma(\upsilon_0)}{d\upsilon_0}, & -1 \leq \upsilon_0 < 0\,,\\ \int_{-1}^{0} d\,\eta_0(\mathfrak{H}_0, \sigma_0)\varphi_0(\mathfrak{H}_0), & \upsilon_0 = 0\,,\end{cases} \tag{44}$$

and

$$\mathcal{R}(\mathcal{S})\Gamma(\upsilon_0) = \begin{cases} 0, & -1 \le \upsilon_0 < 0, \\ \mathcal{F}(\mathcal{S}, \Gamma), & \upsilon_0 = 0. \end{cases} \tag{45}$$

Hence, the system (39) is equivalent

$$\mathfrak{U}'(t) = \mathcal{A}(\mathcal{S})\mathfrak{U}_t + \mathcal{R}(\mathcal{S})\mathfrak{U}_t \ . \tag{46}$$

Where, $\mathfrak{U}_t = \mathfrak{U}(\mathrm{t} + \upsilon_0), -1 \le \upsilon_0 \le 0$.

For $\Psi_0 \in C^1([-1,0], R^3)$, the adjoint operator $\mathcal{A}^*$of $\mathcal{A}(0)$ is

$$\mathcal{A}^* \Psi_0(\mathfrak{H}_0) = \begin{cases} -\frac{d\Psi_0(\mathfrak{H}_0)}{d\mathfrak{H}_0}, & 0 < \mathfrak{H}_0 \le 1 , \\ \int_{-1}^{0} d\, \Gamma^T(\mathcal{E}_0, 0)\Psi_0(-\mathcal{E}_0), & \mathfrak{H}_0 = 0 . \end{cases} \tag{47}$$

For $\Gamma \in C([-1,0], R^3)$, and $\Psi_0 \in (C^1[-1,0], (R^3)^*)$. we define the bilinear inner product

$$\langle \Psi_0(\mathfrak{H}_0), \Gamma(\upsilon_0) \rangle = \overline{\Psi_0}(0)\Gamma(0) - \int_{\upsilon_0=-1}^{0} \int_{\varsigma_0=0}^{\upsilon_0} \overline{\Psi_0}^T(\varsigma_0 - \upsilon_0)\, d\mathcal{M}_0(\upsilon_0)\Gamma(\varsigma_0) d\varsigma_0 \ , \tag{48}$$

Given $\mathcal{M}_0(\upsilon_0) = \mathcal{M}_0(\upsilon_0, 0.$, it follows that $\mathcal{A} = \mathcal{A}$ (0) and $\mathcal{A}^*$ are adjoint operators. Referring to the previous theorem 2, we can deduce that $\pm i\, \omega_0$ are eigenvalues of $\mathcal{A}$ (0) and $\mathcal{A}^*$, respectively. By performing a straightforward calculation, it becomes evident that.

$$p(\upsilon_0) = (1, p_1, p_2)^T\, e^{iw_0\tau_0\upsilon_0}$$

$$p^*(\mathfrak{H}_0) = D_0(1, p_1^*, p_2^*)^T\, e^{-iw_0\tau_0\mathfrak{H}_0}$$

Here

$$p_1 = \frac{iw_0 - \mathcal{F}_{10}^{(1)}}{\mathcal{F}_{01}^{(1)}}, \quad p_2 = -\frac{\mathcal{F}_{1000}^{(2)} + \mathcal{F}_{0010}^{(2)} e^{-iw_0\tau_0} + (\mathcal{F}_{0100}^{(2)} - iw_0)p_1}{f_{0001}^{(2)}\, e^{iw_0\tau_0}},$$

$$p_1^* = -\frac{\mathcal{F}_{10}^{(1)}}{\mathcal{F}_{0100}^{(2)} + +iw_0}, p_1^* = -\frac{\mathcal{F}_{10}^{(2)} + iw_0 + (\mathcal{F}_{1000}^{(2)} + \mathcal{F}_{0010}^{(2)}\, e^{-i\tau_0 w_0})p_1^*}{\mathcal{F}_{010}^{(2)}\, e^{-iw_0\tau_0}}.$$

From equation (48), we can get

$$\begin{aligned} \langle p^*(\mathfrak{H}_0), p(\upsilon_0) \rangle = \overline{D_0} \Big[1 + \bar{p}_1^* p_1 + \bar{p}_2^* p_2 + \bar{p}_1^* \tau_0 e^{-iw_0\tau_0} \left(\mathcal{F}_{00010}^{(2)} + \mathcal{F}_{00001}^{(2)} p_2 \right) \\ + \bar{p}_2^* \tau_0 e^{-iw_0\tau_0} \left(\mathcal{F}_{010}^{(3)} + \mathcal{F}_{001}^{(3)} p_2 \right) \Big]. \end{aligned} \tag{49}$$

Let, $D_0 = \Big[1 + p_1^* \bar{p}_1 + p_2^* \bar{p}_2 + \tau_0\ e^{-iw_0\tau_0} \Big[p_1^* \left(\mathcal{F}_{00010}^{(2)} + \mathcal{F}_{00001}^{(2)} \bar{p}_2 \right) + p_2^* \Big(\mathcal{F}_{010}^{(3)} + \mathcal{F}_{001}^{(3)} \bar{p}_2 \Big) \Big] \Big]^{-1}$, where, $\overline{D_0}$ represent the conjugate complex number of D_0, such that $\langle p^*, p \rangle = 1$ and

$\langle p^*, \bar{p} \rangle = 0$.

Next, using Hassard et al.'s algorthems (26), we can obtain the Hopf bifurcation's properties:

$$\left.\begin{aligned} \mathcal{G}_{20} &= 2\tau_0 \overline{D_0}(\hbar_1 + \hbar_5 \bar{p}_1^* + \hbar_9 \bar{p}_2^*) \\ \mathcal{G}_{11} &= \tau_0 \overline{D_0}(\hbar_2 + \hbar_6 \bar{p}_1^* + \hbar_{10} \bar{p}_2^*) \\ \mathcal{G}_{02} &= 2\tau_0 \overline{D_0}(\hbar_3 + \hbar_7 \bar{p}_1^* + \hbar_{11} \bar{p}_2^*) \\ \mathcal{G}_{21} &= 2\tau_0 \overline{D_0}(\hbar_4 + \hbar_8 \bar{p}_1^* + \hbar_{12} \bar{p}_2^*) \end{aligned}\right\} \tag{50}$$

Where

$\hbar_1 = \mathcal{F}_{11}^{(1)}\ P_1 + \mathcal{F}_{20}^{(1)}$,

$\hbar_2 = \mathcal{F}_{11}^{(1)}\ (P_1 + \bar{P}_1) + 2\mathcal{F}_{20}^{(1)}$,

$\hbar_3 = \mathcal{F}_{11}^{(1)}\ \bar{P}_2 + \mathcal{F}_{20}^{(1)}$,

$$\begin{aligned} \hbar_4 = \mathcal{F}_{11}^{(1)} \Big(P_1\, w_{11}^{(1)}(0) + \tfrac{1}{2}\ \bar{P}_1\ w_{20}^{(1)}(0) + \tfrac{1}{2}\ w_{20}^{(2)}(0) + w_{11}^{(2)}(0) \Big) \\ + \mathcal{F}_{02}^{(1)} \Big(w_{20}^{(1)}(0) + 2\, w_{11}^{(1)}(0) \Big) \end{aligned},$$

$\hbar_5 = \mathcal{F}_{1100}^{(2)}\ P_1 + \mathcal{F}_{0020}^{(2)}\ P_1^{(2)} e^{-2iw_0\tau_0} + \mathcal{F}_{0011}^{(2)}\ P_1 P_2\ e^{-2iw_0\tau_0}$,

$\hbar_6 = \mathcal{F}_{1100}^{(2)}(P_1 + \bar{P}_1) + 2\mathcal{F}_{0020}^{(2)} P_1\bar{P}_1 + 2 P_1 P_2 \mathcal{F}_{0011}^{(2)}$,

$\hbar_7 = \mathcal{F}_{1100}^{(2)} \bar{P}_1 + f_{0020}^{(2)} \bar{P}_1^2 e^{2iw_0\tau_0} + \bar{P}_1\bar{P}_2 e^{2iw_0\tau_0}$,

$$\hbar_8 = \mathcal{F}_{1100}^{(2)}\left(\frac{1}{2}\bar{P}_1 w_{20}^{(1)}(0) + P_1 w_{11}^{(1)}(0) + \frac{1}{2}w_{20}^{(2)}(0) + w_{11}^{(2)}(0)\right)$$
$$+\mathcal{F}_{0020}^{(2)}\left(\bar{P}_1 w_{20}^{(2)}(-1)e^{iw_0\tau_0} + 2P_1 w_{11}^{(2)}(-1)e^{-iw_0\tau_0}\right)$$
$$+\mathcal{F}_{0011}^{(2)}\left(\frac{1}{2}\overline{(P_1} + \bar{P}_2) w_{20}^{(2)}(-1) e^{iw_0\tau_0} + (P_1 + P_2) w_{11}^{(3)}(-1) e^{-iw_0\tau_0}\right)$$

$\hbar_9 = \mathcal{F}_{020}^{(3)} P_1^2 e^{-2iw_0\tau_0} + \mathcal{F}_{011}^{(3)} P_1 P_2 e^{-2iw_0\tau_0}$,

$\hbar_{10} = 2\mathcal{F}_{020}^{(3)} P_1\bar{P}_1 + 2\mathcal{F}_{011}^{(3)} P_1 P_2$,

$\hbar_{11} = \mathcal{F}_{020}^{(3)} \bar{P}_1^2 e^{2iw_0\tau_0} + \mathcal{F}_{011}^{(3)} \bar{P}_1\bar{P}_2 e^{2iw_0\tau_0}$,

$$\hbar_{12} = \mathcal{F}_{020}^{(3)}\left(\bar{P}_1 w_{20}^{(1)}(-1) e^{iw_0\tau_0} + 2P_1 w_{11}^{(1)}(-1)e^{-iw_0\tau_0}\right)$$
$$+\mathcal{F}_{011}^{(3)}\left(\frac{1}{2}(\bar{p}_1 + \bar{P}_2) w_{20}^{(2)}(-1) e^{iw_0\tau_0} + (P_1 + P_2) w_{11}^{(3)}(-1) e^{-iw_0\tau_0}\right)$$

with

$$w_{20}(v_0) = \frac{i\mathcal{G}_{20}}{w_0\tau_0}P(0)e^{iw_0\tau_0 v_0} + \frac{i\bar{\mathcal{G}}_{02}}{3w_0\tau_0}\bar{P}(0)e^{-iw_0\tau_0 v_0} + \mathcal{L}_1 e^{2iw_0\tau v_0}. \quad (51)$$

$$w_{11}(v_0) = -\frac{i\mathcal{G}_{11}}{w_0\tau_0} P(0) e^{iw_0\tau_0 v_0} + \frac{i\bar{\mathcal{G}}_{11}}{w_0\tau_0}\bar{P}(0) e^{-iw_0\tau_0 v_0} + \mathcal{L}_2. \quad (52)$$

Her $\mathcal{L}_1 = \left(\mathcal{L}_1^{(1)}, \mathcal{L}_1^{(2)}, \mathcal{L}_1^{(3)}\right)^T$ and $\mathcal{L}_2 = \left(\mathcal{L}_2^{(1)}, \mathcal{L}_2^{(2)}, \mathcal{L}_2^{(3)}\right)^T$ can be calculated by the following equations:

$$\jmath_1^* \mathcal{L}_1 = 2\tau_0 \jmath_1. \quad (53)$$

$$\jmath_2^* \mathcal{L}_2 = -\tau_0 \jmath_2 . \quad (54)$$

Where,

$$\jmath_1^* = \left(2iw_0\tau_0 I - \int_{-1}^{0} d\mathcal{M}_0(v_0) e^{2iw_0\tau_0 v_0}\right),$$

$$\jmath_2^* = \left(\int_{-1}^{0} d\mathcal{M}_0(v_0)\right),$$

$\jmath_1 = (\hbar_1 \ \hbar_5 \ \hbar_9)^T$,

$\jmath_2 = (\hbar_2 \ \hbar_6 \ \hbar_{10})^T$.

Accordingly, it is determined that:

$$\mathcal{L}_1 = 2\jmath_1 \begin{pmatrix} 2iw_0 - \mathcal{F}_{10}^{(1)} & -\mathcal{F}_{01}^{(1)} & 0 \\ -\mathcal{F}_{10000}^{(2)} - \mathcal{F}_{0010}^{(2)} e^{2iw_0\tau_0 v_0} & 2iw_0 - \mathcal{F}_{0100}^{(2)} & \mathcal{F}_{0001}^{(2)} e^{2iw_0\tau_0 v_0} \\ -\mathcal{F}_{010}^{(3)} e^{2iw_0\tau_0 v_0} & 0 & 2iw_0 - \mathcal{F}_{100}^{(3)} - \mathcal{F}_{001}^{(3)} e^{2iw_0\tau_0 v_0} \end{pmatrix}^{-1}.$$

$$\mathcal{L}_1 = -\jmath_2 \begin{pmatrix} -\mathcal{F}_{10}^{(1)} & -\mathcal{F}_{01}^{(1)} & 0 \\ -\mathcal{F}_{1000}^{(2)} - \mathcal{F}_{0010}^{(2)} & -\mathcal{F}_{0100}^{(2)} & -\mathcal{F}_{0001}^{(2)} \\ -\mathcal{F}_{010}^{(3)} & 0 & -\mathcal{F}_{100}^{(3)} - \mathcal{F}_{001}^{(3)} \end{pmatrix}^{-1}.$$

Thus, equations (51)– (54) can be used to calculate $w_{20}(\upsilon_0)$ and $w_{11}(\upsilon_0)$. Following that, the proof can be completed by determining the expressions given in equation (39) based on those in equation (50).

3. Numerical Simulation

In this section, we present essential findings through numerical representation, utilizing a set of biologically reasonable hypothetical values as provided below. The goal is to validate the theoretically generated outcomes and gain insights into the parameters' impact on the system dynamics (2).

$$r = 0.009 \,, k = 40 \,, c = 0.0002 \,, E_1 = 0.001, \alpha = 0.001, \gamma = 1.2 \,, \quad \lambda = 0.02 \\ E_2 = 0.005; \theta = 0.001, h = 0.03, E_3 = 0.02, \tau = 15.0, n = 5000 \qquad (55)$$

It is noted that for all $\tau \geq 0$ and the data provided the solution of system (2) has globally asymptotically stable F_1 as shown in **Figure 1.**

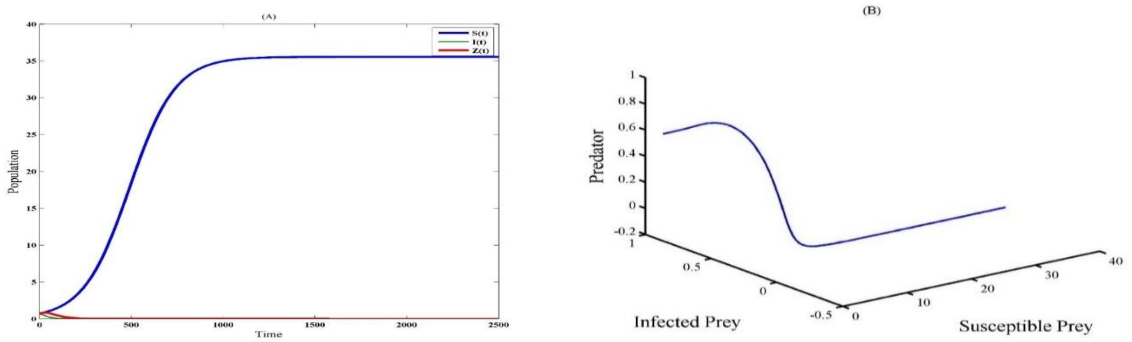

Figure 1. The system's (2) trajectory based on the data provided in equation (55).

(A) The system's (2) time series gradually converge to F_1. (B) 3D phase diagram representing the globally asymptotic stability of point F_1.

It is observed that the same data from equation (55), with $c = 0.02$ and $h = 0.001$ system (2) exhibits global asymptotic stability forF_2 as depicted in **Figure 2**.

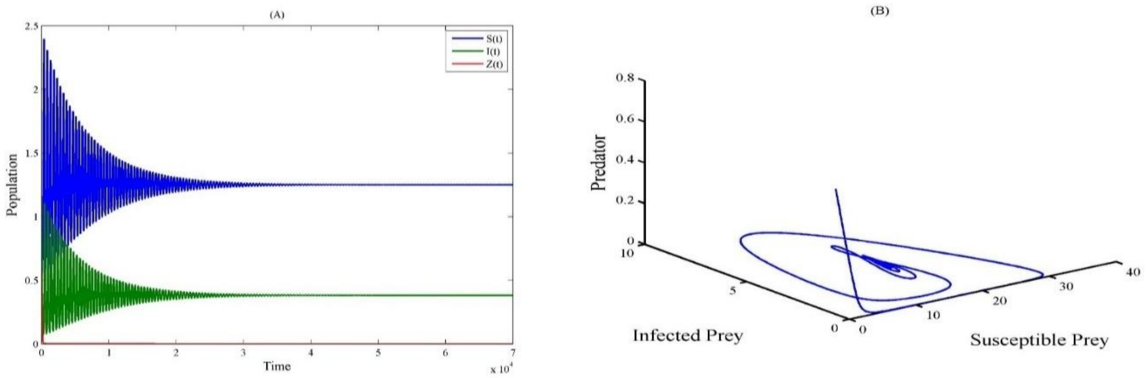

Figure 2. The system's (2) trajectory based on the data provided in equation (55) with $c = 0.02$ and $h = 0.001$.(A) The system's (2) time series gradually converge to F_2. (B) 3D phase diagram representing the globally asymptotic stability of point F_2.

Here, we discuss the impact of time delays on the behavior of the system (2) near the F_3 point It is noticed that the conditions of theorem 2 are met for the parameters in the data of equation (55) with $= 0.02$. it is observed that when $\tau = 15$ the F_3 point is asymptotic stable as shown in Fig. (3) while for $\tau = \tau_0 = 45$ a Hopf bifurcation occurs at F_3 as shown in **Figure 4.** On other hand for $\tau = 60 > \tau_0 = 45$ increasing period as τ increases as shown in **Figure 5.**

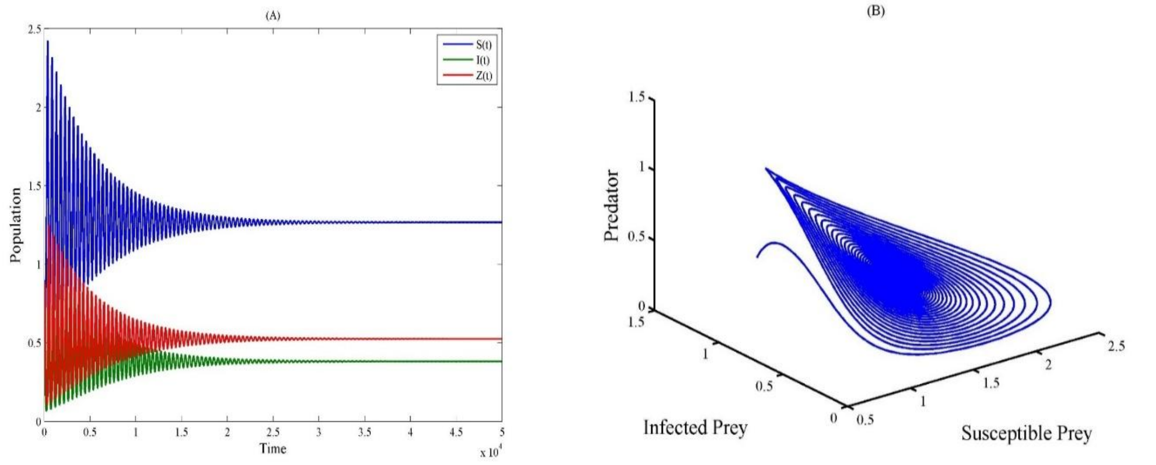

Figure 3. The system's (2) trajectory based on the data provided in equation (55) with $c = 0.02$ and $\tau = 15$ (A)The system's (2) time series gradually converge to F_3. (B) 3D phase diagram representing the globally asymptotic stability of point.

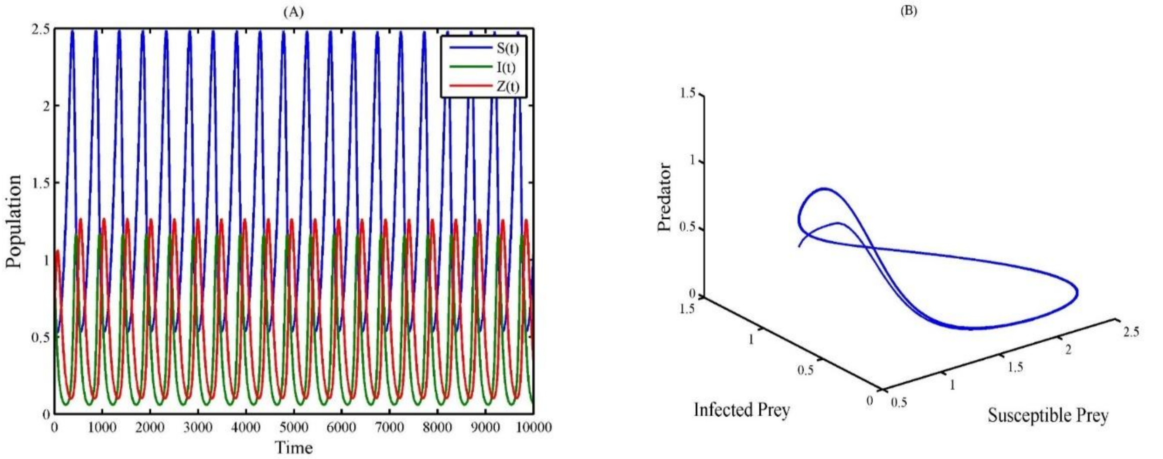

Figure 4. The system's (2) trajectory based on the data provided in equation(55) with $c = 0.02$ and $\tau = \tau_0 = 45$.(A) a periodic solution's existence near F_3 near F_3 (B) 3D periodic solution.

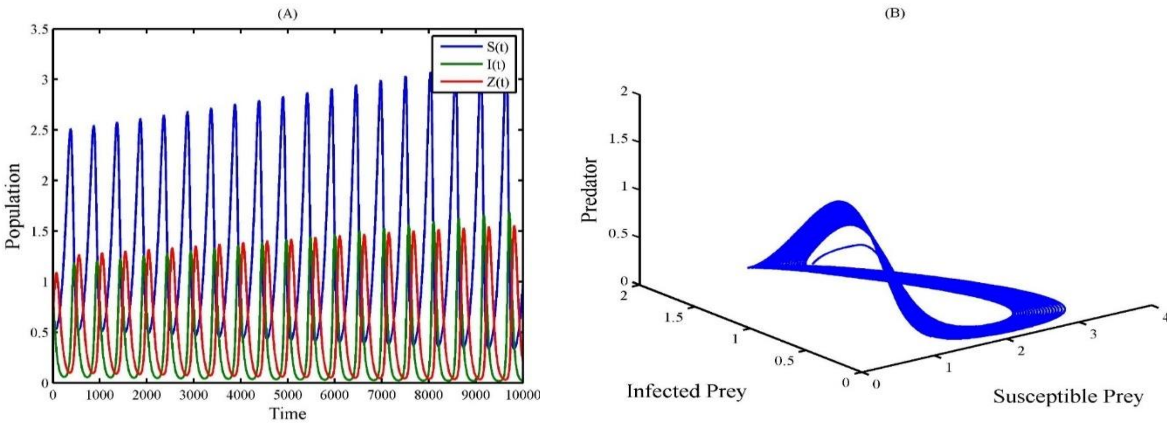

Figure 5. The system's (2) trajectory based on the data provided in equation(55) with $c = 0.02$ and $\tau = 60 > \tau_0 = 45$.(A) a periodic solution's existence near F_3 (B) 3D periodic solution.

4. Conclusions

In this study, a delayed predator-prey model with harvests is proposed. Our aim is to understand how the stability of the model is affected by the latent time of predation and the gestation period of the predators. All properties of the solution, such as positivity and boundedness, were investigated. It was found that the system (2) can have four equilibrium points. The stability analysis has shown that the discrete time delay has no effect on the stability of the axial equilibrium point and the planer equilibrium point, so it is still locally asymptotically stable for all $\tau \geq 0$. It has been demonstrated that the coexistence equilibrium point is characterized by asymptotic stability when the delay does not approach a critical value of ,τ-0.. However, a Hopf bifurcation occurs when =,τ-0. , making it an unstable point, and the solution approaches asymptotically to periodic dynamics for τ>,τ-0.. Additionally, the center manifold theorem was applied to investigate the stability and direction of bifurcating periodic dynamics. Finally, the Matlab program is employed for a numerical investigation of the system's global dynamics. For the given data in equation (55), ,-1. is globally asymptotically stable. When the same data from equation (55) is used with =0.02 and h=0.001 , it is observed that ,F-2 . is globally asymptotically stable. For the data in equation (55) with =0.02 , it is noted that ,F-3. is asymptotically stable when τ=15. However, when ,=τ-0.=45, a Hopf bifurcation occurs.

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Conflict of Interest

The authors declare that there are no competing interests regarding the publication of this paper.

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