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Grill $\mathcal{V}$ -Space

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Abstract

This scientific study aims to introduce a new type of topological space, called "Grill $\mathcal{V}$-space", with the aim of contributing to scientific knowledge in this field. New generalizations were developed for both the local function and the Kuratowski closure function in order to generalize their concepts. Subsequently, these generalizations were used to define a new topology based on the concepts associated with the grill and the $\mathcal{V}$ -space. The set of $\mathcal{V}$ -open in $\mathcal{V}$ -space is defined as the sum of the set of $\mathcal{V}$ -open forms in $\mathcal{V}$ -space when the grill consists of the subsets of P(X) except the empty set. Many of the properties of this new space were demonstrated, and a number of illustrative examples were given.

Keywords: grill, $\mathcal{V}$-open set, $\mathcal{V}$ -closed set, $\mathcal{V}$ -interior, grill $\mathcal{V}$ -space.

1. Introduction

In 1947, [1] introduced the notion of grill. After that, each of the researchers, [2], [3] studied many concepts for different types of spaces using this term. In 2007, [4] introduced the notion of grill topological space. [5-7] used the studied concept to introduce new generalized sets and to study these generalizations and their properties in detail. In 2022, [8] introduced the concept of $\mathcal{V}$-open set for the first time and named the ordered pair of the universal set and the family of all $\mathcal{V}$-open sets a $\mathcal{V}$-space and proved that this space represents a topological space under certain conditions. In this work, the concept of the grill is extending from the grill $\mathcal{V}$-space A new kind of topological space has been obtained. Grill topological spaces are considered one of the most important areas in mathematics, and there are many researchers who have studied generalizations of topological properties in this context, since the aim is to understand how these properties evolve and change when dealing with generalized topological spaces, see [9-24].
Perhaps soft topological spaces are one of the modern spaces that many researchers are interested in studying. Therefore, the topological properties in soft topological spaces have been studied using the grill concept, see [25- 30].

2. Preliminaries

Definition 2.1[8]

Let X represent a set that is not empty, and let $\{\tau_k\}_{k\epsilon J}$, $k \geq 2$ be any topologies on K and let the family $\mathcal{V}O_X = \{\mathcal{N} \subseteq K: \mathcal{N} = \emptyset \; or \; \exists \; \mathcal{T} \in \bigcap_{k\epsilon J} \tau_k \; \ni \; \emptyset \neq \mathcal{T} \subseteq \mathcal{N}\}$ satisfying the following axioms:

1. $K, \emptyset \in \mathcal{V}O_X$.
2. $\bigcup_{i\in I} \mathcal{N}_i \in \mathcal{V}O_X \ \forall \{\mathcal{N}_i\}_{i\in I} \in \mathcal{V}O_X$.
3. $\bigcap_{i=1}^{n} \mathcal{N}_i \in \mathcal{V}O_X \ \forall \{\mathcal{N}_i\}_{i=1}^{n} \in \mathcal{V}O_X$.

Then $(X, \mathcal{V}O_X)$ is called to be $\boldsymbol{\mathcal{V}}$-space and the elements of $\mathcal{V}O_X$ are called $\mathcal{V}-$ open sets and the complement $\mathcal{V}-$ open set is $\mathcal{V}-$ closed set. We denote the set of all $\mathcal{V}-$ closed of X by $\mathcal{V}C_X$.

Definition 2.2 [8]

For any $\boldsymbol{\mathcal{V}}$-space $(K, \mathcal{V}O_X)$ and let Ɱ $\subseteq X$. Then

1. The $\boldsymbol{\mathcal{V}}$-closure of Ɱ is symbolized by $cl_{\mathcal{V}}$(Ɱ) and is equal

$cl_{\mathcal{V}}$(Ɱ) $= \bigcap\{\mathcal{F} \subseteq K: \mathcal{F}\ is\ \mathcal{V}-$ closed and Ɱ $\subseteq \mathcal{F}\}$.

2. The $\boldsymbol{\mathcal{V}}$-interior of Ɱ is symbolized by $int_{\mathcal{V}}$(Ɱ) and is equal

$int_{\mathcal{V}}$(Ɱ) $= \bigcup\{\mathcal{N} \subseteq K: \mathcal{N}\ is\ \mathcal{V}-$ open and $\mathcal{N} \subseteq$ Ɱ$\}$.

Theorem 2.3

For any $\boldsymbol{\mathcal{V}}$-space $(K, \mathcal{V}O_X)$, let Ɱ $\subseteq K, k \in K$ .So $k \in cl_{\mathcal{V}}$(Ɱ) if and only if $\mathcal{N} \cap$ Ɱ $\neq \emptyset$ for all $\mathcal{V}-$ open set $\mathcal{N}$; $k \in \mathcal{N}$.

Proof:

The "if" part

Let $k \in$ cl$_{\boldsymbol{\mathcal{V}}}$(Ɱ) $= \bigcap\{\mathcal{F} \subseteq K: \mathcal{F}\ is\ \mathcal{V}-$ closed and Ɱ $\subseteq \mathcal{F}\}$ and let be there exists a $\mathcal{V}-$ open set $\mathcal{N}$ containing x such that $\mathcal{N} \cap$ Ɱ $= \emptyset$, it follows that Ɱ $\subseteq \mathcal{N}^c$ which is $\mathcal{V}-$ closed set with $x \notin \mathcal{N}^c$, so $x \notin \bigcap\{\mathcal{F} \subseteq k: \mathcal{F}\ is\ \mathcal{V}-$ closed and Ɱ $\subseteq \mathcal{F}\}$ which is a contradiction.

The "only if" part

Assume that every $\mathcal{V}-$ open set $\mathcal{N}$ containing x intersects Ɱ and suppose that $k \notin$ cl$_{\boldsymbol{\mathcal{V}}}$(Ɱ) $= \bigcap\{\mathcal{F} \subseteq K: \mathcal{F}\ is\ \mathcal{V}-$ closed and Ɱ $\subseteq \mathcal{F}\}$, then there exists a $\mathcal{V}-$ closed set $\mathcal{F}$ with Ɱ $\subseteq \mathcal{F}$ and $k \notin \mathcal{F}$, so $k \in \mathcal{F}^c$ that is $\mathcal{V}-$ open set , but $\mathcal{F}^c \cap$ Ɱ $= \emptyset$, which is contradiction.

Definition 2.4 [9]

A collection Q of subsets of a nonempty subset in a topological space (K, τ) is referred to as a grill on K if it fulfils the following conditions:

1. $\emptyset \notin \boldsymbol{Q}$.
2. Ɱ $\in \boldsymbol{Q}$ & Ɱ $\subseteq$ B therefore B $\in \boldsymbol{Q}$**.**
3. Ɱ $\notin \boldsymbol{Q}$ & B $\notin \boldsymbol{Q}$ therefore Ɱ $\cup$ B $\notin \boldsymbol{Q}$.

A topological space (K, τ) with a grill $\boldsymbol{Q}$ on K is named a grill topological space and is symbolize by $(X, \tau, \boldsymbol{Q})$.

Definition 2.5 [4]

Let Q denote a grill on a topological space (K, τ). Consider the map $\theta : \mathcal{P}(K) \to \mathcal{P}(K)$ such that

θ(Ɱ) $= \{x \in K: S \cap$ Ɱ $\in \boldsymbol{Q}\ \forall S \in \tau, x \epsilon S\}$ for each Ɱ $\subseteq K$. Therefore, the function ω : $\mathcal{P}(K) \to \mathcal{P}(K)$ where ω(Ɱ) = Ɱ$\cup\theta$(Ɱ) is a Kuratowski's closure operator and the topological origin, which is finer than τ and defined as $\tau_{\boldsymbol{Q}} = \{$Ɱ $\subseteq$ X$: \Psi(K -$ Ɱ$) = K -$ Ɱ$\}$.

3. Closure Operator in Grill $\boldsymbol{\mathcal{V}}$-spaces

Definition 3.1

A nonempty family $\mathfrak{S}$ of nonempty sets of a $\boldsymbol{\mathcal{V}}$-space $(K, \mathcal{V}O_X)$ is named a grill on K, if it satisfies the following conditions:

1. $\emptyset \notin \mathfrak{S}$.
2. Ɱ $\in \mathfrak{S}$ ^ Ɱ $\subseteq$ B therefore P $\in \mathfrak{S}$.

3. $\mathcal{M} \notin \mathfrak{S} \wedge P \notin \mathfrak{S}$ therefore $\mathcal{M} \cup P \notin \mathfrak{S}$.

Any $\boldsymbol{\mathcal{V}}$-space $(K, \mathcal{V}O_X)$ with a grill $\mathfrak{S}$ on K is named a grill $\boldsymbol{\mathcal{V}}$-space and is symbolize by $(K, \mathcal{V}O_X, \mathfrak{S})$.

Definition 3.2

Let $\mathfrak{S}$ be a grill on $\boldsymbol{\mathcal{V}}$-space $(K, \mathcal{V}O_X)$. The map $\Upsilon_{\mathfrak{S}} : \mathcal{P}(K) \to \mathcal{P}(K)$ such that $\Upsilon_{\mathfrak{S}}(\mathcal{M}) = \{x \in X : S \cap \mathcal{M} \in \mathfrak{S}\ \forall S \in \mathcal{V}O_X, x \epsilon S\}$ for each $\mathcal{M} \subseteq K$, is named the local map commitment to a grill $\mathfrak{S}$ with the topology $\mathcal{V}O_X$.

Theorem 3.3

Suppose that $(K, \mathcal{V}O_X)$ be a $\boldsymbol{\mathcal{V}}$-space. so, the following are satisfying:

1. For any grill $\mathfrak{S}$ on K , then $A \subseteq B$ implies $\Upsilon_{\mathfrak{S}}(A) \subseteq \Upsilon_{\mathfrak{S}}(B)$.
2. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two grilles on X and $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, implies that $\Upsilon_{\mathfrak{S}_1}(A) \subseteq \Upsilon_{\mathfrak{S}_2}(A)$ for each $A \subseteq K$.
3. whenever $\mathfrak{S}$ a grill from K, if $A \notin \mathfrak{S}$, implies that $\Upsilon_{\mathfrak{S}}(A) = \emptyset$.

Proof:

1. Let $x \in \Upsilon_{\mathfrak{S}}(A)$, so $\forall S \in \mathcal{V}O_X, x \epsilon S$, we have $S \cap A \in \mathfrak{S}$. But $S \cap A \subseteq S \cap B$ since $A \subseteq B$, it follows from Definition 3.1 that $S \cap B \in \mathfrak{S}\ \forall S \in \mathcal{V}O_X, x \epsilon S$, then $x \in \Upsilon_{\mathfrak{S}}(B)$. Hence $\Upsilon_{\mathfrak{S}}(A) \subseteq \Upsilon_{\mathfrak{S}}(B)$.
2. Let $x \in \Upsilon_{\mathfrak{S}_1}(A)$, so $\forall S \in \mathcal{V}O_X, x \epsilon S$, we have $S \cap A \in \mathfrak{S}_1$. But $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, so $S \cap A \in \mathfrak{S}_2\ \forall S \in \mathcal{V}O_X, x \epsilon S$, it follows from Definition 3.2 that $x \in \Upsilon_{\mathfrak{S}_2}(A)$. Hence $\Upsilon_{\mathfrak{S}_1}(A) \subseteq \Upsilon_{\mathfrak{S}_2}(A)$.
3. Suppose that $A \notin \mathfrak{S}$, and $\Upsilon_{\mathfrak{S}}(A) \neq \emptyset$, then there exists $x \in \Upsilon_{\mathfrak{S}}(A)$, it follows from Definition 3.2 that $S \cap A \in \mathfrak{S}\ \forall S \in \mathcal{V}O_X, x \epsilon S$. But $S \cap A \subseteq A$, it follows from Definition 3.1(2) that $A \in \mathfrak{S}$, which is a contradiction.

Theorem 3.4

For a grill $\boldsymbol{\mathcal{V}}$-space $(X, \mathcal{V}O_X, \mathfrak{S})$. And for all $A, B \subseteq X$, the following are true:

1. $\Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B) = \Upsilon_{\mathfrak{S}}(A \cup B)$.
2. $\Upsilon_{\mathfrak{S}}(A) \subseteq \mathrm{cl}_{\mathcal{V}}(A)$.
3. $\Upsilon_{\mathfrak{S}}(\Upsilon_{\mathfrak{S}}(A)) \subseteq \Upsilon_{\mathfrak{S}}(A)$

Proof:

1. Let $x \in \Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B)$, then $x \in \Upsilon_{\mathfrak{S}}(A)$ or $x \in \Upsilon_{\mathfrak{S}}(B)$. If $x \in \Upsilon_{\mathfrak{S}}(A)$, so $\forall S \in \mathcal{V}O_X, x \epsilon S$, we have $S \cap A \in \mathfrak{S}$. Since $A \subseteq A \cup B$, so $S \cap A \subseteq S \cap (A \cup B)$. From Definition 3.1(2) we get $S \cap (A \cup B) \in \mathfrak{S}\ \ \forall S \in \mathcal{V}O_X, x \epsilon S$. Hence $x \in \Upsilon_{\mathfrak{S}}(A \cup B)$ Similarly, we can prove that $x \in \Upsilon_{\mathfrak{S}}(A \cup B)$ whenever $x \in \Upsilon_{\mathfrak{S}}(B)$, and so $\Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B) \subseteq \Upsilon_{\mathfrak{S}}(A \cup B)$……(1). Now let $x \in \Upsilon_{\mathfrak{S}}(A \cup B)$, so $\forall S \in \mathcal{V}O_X, x \epsilon S$, we have $S \cap (A \cup B) \in \mathfrak{S}$. Distributing the intercept to the union, we get $\forall S \in \mathcal{V}O_X, x \epsilon S$, $(S \cap A) \cup (S \cap B) \in \mathfrak{S}$, it follows from Definition 3.1(3) $S \cap A \in \mathfrak{S}$ or $S \cap B \in \mathfrak{S}$, if $S \cap A \in \mathfrak{S}$, then $x \in \Upsilon_{\mathfrak{S}}(A)$, implies $x \in \Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B)$. If $S \cap B \in \mathfrak{S}$, then $x \in \Upsilon_{\mathfrak{S}}(B)$, implies $x \in \Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B)$. Thus $\Upsilon_{\mathfrak{S}}(A \cup B) \subseteq \Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B)$……(2). From (1) and (2) , we prove $\Upsilon_{\mathfrak{S}}(A) \cup \Upsilon_{\mathfrak{S}}(B) = \Upsilon_{\mathfrak{S}}(A \cup B)$.
2. If $x \notin \mathrm{cl}_{\mathcal{V}}(A)$, it follows from Theorem 2.3 that there exists a $\mathcal{V}-$open set S containing x such that $S \cap A = \emptyset$, it follows from Definition 3.1(1) that $S \cap A \notin \mathfrak{S}$ implies $x \notin \Upsilon_{\mathfrak{S}}(A)$. Thus $\Upsilon_{\mathfrak{S}}(A) \subseteq \mathrm{cl}_{\mathcal{V}}(A)$.
3. Let $x \in \Upsilon_{\mathfrak{S}}(\Upsilon_{\mathfrak{S}}(A))$, so $\forall S \in \mathcal{V}O_X, x \epsilon S$, we have $S \cap \Upsilon_{\mathfrak{S}}(A) \in \mathfrak{S}$, implies $S \cap \Upsilon_{\mathfrak{S}}(A) \neq \emptyset$, let S^* be $\mathcal{V}-$open set containing x and; $y \in S^* \cap \Upsilon_{\mathfrak{S}}(A)$, therefore $y \in S^*$ and $y \in \Upsilon_{\mathfrak{S}}(A)$, it follows

that $S^{*}\cap A \in \mathfrak{S}$. For each $S \in \mathcal{V}O_X, x\in S$ we can find an element $y \in S\cap \Upsilon_{\mathfrak{S}}(A)$ which implies that $S\cap A \in \mathfrak{S}$. Hence $x \in \Upsilon_{\mathfrak{S}}(A)$, and so $\Upsilon_{\mathfrak{S}}(\Upsilon_{\mathfrak{S}}(A)) \subseteq \Upsilon_{\mathfrak{S}}(A)$.

Remark 3.5

The convers does not hold in general in (2) of Theorem 3.4.

For example

Let $X = \{ħ_1, ħ_2, ħ_3, ħ_4\}$, and let $\{\tau_i\}_{i=1}^{3}$ be a family of topologies defined on X as follows:

$\tau_1 = \mathbb{P}(X), \tau_2 = \{X, \emptyset, \{Ƒ_1\}, \{Ƒ_1, Ƒ_2\}, \{Ƒ_3\}, \{Ƒ_1, Ƒ_3\}, \{Ƒ_1, Ƒ_2, Ƒ_3\}\}, \tau_3 = \{X, \emptyset, \{Ƒ_1\}, \{Ƒ_1, Ƒ_2\}, \{Ƒ_3, Ƒ_4\}, \{Ƒ_1, Ƒ_3, Ƒ_4\}\}$. Then, $\bigcap_{i=1}^{3}\tau_i = \{X, \emptyset, \{Ƒ_1\}, \{Ƒ_1, Ƒ_2\}\}$, and so

$\mathcal{V}O_X = \{X, \emptyset, \{Ƒ_1\}, \{Ƒ_1, Ƒ_2\}, \{Ƒ_1, Ƒ_4\}, \{Ƒ_1, \}, \{Ƒ_1, Ƒ_2, Ƒ_3\}, \{Ƒ_1, Ƒ_3, Ƒ_4\}, \{Ƒ_1, Ƒ_2, Ƒ_4\}\}$.

$\mathcal{V}C_X = \{\emptyset, X, \{Ƒ_2, Ƒ_3, Ƒ_4\}, \{Ƒ_3, Ƒ_4\}, \{Ƒ_2, Ƒ_3\}, \{Ƒ_2, Ƒ_4\}, \{Ƒ_4\}, \{Ƒ_2\}, \{Ƒ_3\}\}$

Let $\mathfrak{S} = \{\{Ƒ_1\}, \{Ƒ_1, Ƒ_2\}, \{Ƒ_1, Ƒ_3\}, \{Ƒ_1, Ƒ_4\}, \{Ƒ_1, Ƒ_2, Ƒ_3\}, \{Ƒ_1, Ƒ_2, Ƒ_4\}, \{Ƒ_1, Ƒ_3, Ƒ_4\}, X\}$ be a grill on X.

Let $A = \{Ƒ_2, Ƒ_3\}$

$Ƒ_{\mathfrak{S}}(A) = \{x \in X: S\cap A \in \mathfrak{S}\ \forall S \in \mathcal{V}O_X, x\in S\} = \emptyset$

$cl_{\mathcal{V}}(A) = A$ since A is $\mathcal{V}-$ closed.

Remark 3.6

The convers does not always hold in (3) of Theorem 3.4.

For example

Let $X = \{ŧ_1, ŧ_2, ŧ_3, ŧ_4\}$, and let $\{\tau_i\}_{i=1}^{3}$ be a family of topologies defined on X as follows:

$\tau_1 = \mathbb{P}(X), \tau_2 = \{X, \emptyset, \{ŧ_1\}, \{ŧ_1, ŧ_2\}, \{ŧ_3\}, \{ŧ_1, ŧ_3\}, \{ŧ_1, ŧ_2, ŧ_3\}\}, \tau_3 = \{X, \emptyset, \{ŧ_1, ŧ_2\}, \{ŧ_3, ŧ_4\}\}$.

Then, $\bigcap_{i=1}^{3}\tau_i = \{X, \emptyset, \{ŧ_1, ŧ_2\}\}$, and so

$\mathcal{V}O_X = \{X, \emptyset, \{ŧ_1, ŧ_2\}, \{ŧ_1, ŧ_2, ŧ_3\}, \{ŧ_1, ŧ_2, ŧ_4\}\}$

$\mathfrak{S} = \{\{ŧ_1, ŧ_2, ŧ_3\}, \{ŧ_1, ŧ_2, ŧ_4\}, \{ŧ_1, ŧ_3, ŧ_4\}, X\}$

Let $A = \{ŧ_1, ŧ_2, ŧ_3\}$,

$ŧ_{\mathfrak{S}}(A) = \{x \in X: S\cap A \in \mathfrak{S}\ \forall S \in \mathcal{V}O_X, x\in S\} = \{c, d\}$

$ŧ_{\mathfrak{S}}(ŧ_{\mathfrak{S}}(A)) = \emptyset$

Definition 3.7

Let $\mathfrak{S}$ be a grill on $\boldsymbol{\mathcal{V}}$-space $(X, \mathcal{V}O_X)$. The map $\mathcal{V}_{\mathfrak{S}} - cl: \mathbb{P}(X) \to \mathbb{P}(X)$ where $\mathcal{V}_{\mathfrak{S}} - cl(A) = A\cup L_{\mathfrak{S}}(A)$ is a Kuratowski's closure operator and hence induces a topology on X defined as $\mathcal{V}_{\mathfrak{S}} = \{G \subseteq X: \mathcal{V}_{\mathfrak{S}} - cl(X - G) = X - G\}$.

For example

Let $X = \{ŧ_1, ŧ_2, ŧ_3\}$, and let $\{\tau_i\}_{i=1}^{3}$ be a family of topologies defined on X as follows:

$\tau_1 = \{X, \emptyset, \{ŧ_1\}, \{ŧ_1, ŧ_2\}\}, \tau_2 = \{X, \emptyset, \{ŧ_1\}, \{ŧ_1, ŧ_3\}\}, \tau_3 = \{X, \emptyset, \{ŧ_1\}, \{ŧ_2, ŧ_3\}\}$,

then $\bigcap_{i=1}^{3}\tau_i = \{X, \emptyset, \{ŧ_1\}\}$

$\mathcal{V}O_X = \{X, \emptyset, \{ŧ_1\}, \{ŧ_1, ŧ_2\}, \{ŧ_1, ŧ_3\}\}$, let $\mathfrak{S} = \{X, \{ŧ_2\}, \{ŧ_1, ŧ_2\}, \{ŧ_2, ŧ_3\}\}$ be a grill on X.

$\mathcal{V}_{\mathfrak{S}} = \{X, \emptyset, \{ŧ_1\}, \{ŧ_2\}, \{ŧ_3\}, \{ŧ_1, ŧ_2\}, \{ŧ_1, ŧ_3\}, \{ŧ_2, ŧ_3\}\}$.

Theorem 3.8

Suppose that $(\mathfrak{k}, \mathcal{V}O_{\mathfrak{k}})$ be a $\boldsymbol{\mathcal{V}}$-space:

1. If $\mathfrak{S}$ is any grill on $\mathfrak{k}$ and $A \notin \mathfrak{S}$ then A is $\mathcal{V}_{\mathfrak{S}}-$ closed set in $(\mathfrak{k}, \mathcal{V}_{\mathfrak{S}})$.
2. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two grilles on $\mathfrak{k}$ with $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, then $\mathcal{V}_{\mathfrak{S}_2} \subseteq \mathcal{V}_{\mathfrak{S}_1}$.
3. For any grill $\mathfrak{S}$ on $\mathfrak{k}$ and any subset A of k, $\Upsilon_{\mathfrak{S}}(A)$ is $\mathcal{V}_{\mathfrak{S}}-$ closed.
4. If A is $\mathcal{V}_{\mathfrak{S}}-$ closed, then $\Upsilon_{\mathfrak{S}}(\mathrm{A}) \subseteq \mathrm{A}$.

Proof:

1. Since $A \notin \mathfrak{S}$, so $\Upsilon_{\mathfrak{S}}(A) = \emptyset$, it follows that $\mathcal{V}_{\mathfrak{S}} - cl(A) = A \cup \emptyset = A$, it means $\mathfrak{f} - A \in \mathcal{V}_{\mathfrak{S}}$. Hence A is $\mathcal{V}_{\mathfrak{S}}-$ closed set in $(\mathfrak{k}, \mathcal{V}_{\mathfrak{S}})$.
2. Let $G \in \mathcal{V}_{\mathfrak{S}_2}$, then $\mathcal{V}_{\mathfrak{S}_2} - cl(\mathfrak{f} - G) = \mathfrak{f} - G$, and so $\mathfrak{f} - G \cup \Upsilon_{\mathfrak{S}_2}(\mathfrak{f} - G) = \mathfrak{f} - G$, it follows that $\Upsilon_{\mathfrak{S}_2}(\mathfrak{f} - G) \subseteq \mathfrak{f} - G$, but $\Upsilon_{\mathfrak{S}_1}(\mathfrak{f} - G) \subseteq \Upsilon_{\mathfrak{S}_2}(\mathfrak{f} - G)$, so $\Upsilon_{\mathfrak{S}_1}(\mathfrak{f} - G) \subseteq \mathfrak{f} - G$, implies that $\mathfrak{f} - G \cup \Upsilon_{\mathfrak{S}_1}(\mathfrak{f} - G) = \mathfrak{f} - G$, therefor $G \in \mathcal{V}_{\mathfrak{S}_1}$. Thus $\mathcal{V}_{\mathfrak{S}_2} \subseteq \mathcal{V}_{\mathfrak{S}_1}$.
3. Since $\Upsilon_{\mathfrak{S}}(\Upsilon_{\mathfrak{S}}(\mathrm{A})) \subseteq \Upsilon_{\mathfrak{S}}(\mathrm{A})$, so $\mathcal{V}_{\mathfrak{S}} - cl(\Upsilon_{\mathfrak{S}}(\mathrm{A})) = \Upsilon_{\mathfrak{S}}(\mathrm{A}) \cup \Upsilon_{\mathfrak{S}}(\Upsilon_{\mathfrak{S}}(\mathrm{A})) = \Upsilon_{\mathfrak{S}}(\mathrm{A})$. Hence $\Upsilon_{\mathfrak{S}}(A)$ is $\mathcal{V}_{\mathfrak{S}}-$ closed.
4. Let A is $\mathcal{V}_{\mathfrak{S}}-$ closed set in $(\mathfrak{k}, \mathcal{V}_{\mathfrak{S}})$, then $\mathcal{V}_{\mathfrak{S}} - cl(A) = A \cup \Upsilon_{\mathfrak{S}}(\mathrm{A}) = A$. Thus $\Upsilon_{\mathfrak{S}}(\mathrm{A}) \subseteq A$.

Theorem 3.9

For a grill $\boldsymbol{\mathcal{V}}$-space $(X, \mathcal{V}O_X, \mathfrak{S})$. Then the collection $\mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_K}) = \{\mathcal{N} - A : \mathcal{N} \in \mathcal{V}O_X$ and $A \notin \mathfrak{S}\}$ is a basis for $\mathcal{V}_{\mathfrak{S}}$.

Proof:

Let $\mathrm{Ɱ} \in \mathcal{V}_{\mathfrak{S}}$ and $x \in \mathrm{Ɱ}$, then $x \notin \mathrm{K} - \mathrm{Ɱ}$, but $\mathrm{K} - \mathrm{Ɱ}$ is $\mathcal{V}_{\mathfrak{S}}-$ closed set**,** so $\Upsilon_{\mathfrak{S}}(\mathrm{K} - \mathrm{Ɱ}) \subseteq \mathrm{K} - \mathrm{Ɱ}$, it follows that $x \notin \Upsilon_{\mathfrak{S}}(\mathrm{K} - \mathrm{Ɱ})$. From Definition 3.2, there exists a $\mathcal{V}-$ open set $\mathcal{N}$ containing x such that $\mathcal{N} \cap (\mathrm{K} - \mathrm{Ɱ}) \notin \mathfrak{S}$. Let $A = \mathcal{N} \cap (\mathrm{K} - \mathrm{Ɱ})$, then $x \in \mathcal{N} - A \subseteq \mathrm{Ɱ}$ such that $\mathcal{N} \in \mathcal{V}O_X$ and $A \notin \mathfrak{S}$. Thus Ɱ is the union of subsets in $\mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_K})$. Easily $\mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_K})$ is closed when the intersection are finite that is if $\mathcal{N}_1 - A_1$ and $\mathcal{N}_2 - A_2$ are in $\mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_K})$, then $(\mathcal{N}_1 - A_1) \cap (\mathcal{N}_2 - A_2) = (\mathcal{N}_1 \cap \mathcal{N}_2) - (A_1 \cup A_2)$, where $\mathcal{N}_1 \cap \mathcal{N}_2 \in \mathcal{V}O_X$ and $A_1 \cup A_2 \notin \mathfrak{S}$. Hence $\mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_K})$ is a basis for $\mathcal{V}_{\mathfrak{S}}$.

Theorem 3.10.

From any $(X, \mathcal{V}O_X, \mathfrak{S})$ grill $\boldsymbol{\mathcal{V}}$-space. Then $\mathcal{V}O_X \subseteq \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X}) \subseteq \mathcal{V}_{\mathfrak{S}}$. And $\mathfrak{S} = \mathbb{P}(X) - \{\emptyset\}$, therefore $\mathcal{V}O_X = \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X}) = \mathcal{V}_{\mathfrak{S}}$.

Proof:

Let $\mathcal{N} \in \mathcal{V}O_X$, implies $\mathcal{N} = \mathcal{N} - \emptyset$ where $\emptyset \notin \mathfrak{S}$, so $\mathcal{N} \in \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X})$. Thus $\mathcal{V}O_X \subseteq \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X})$. Now let $G \in \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X})$, then there exists a $\mathcal{V}-$ open set $\mathcal{N}$ and $A \notin \mathfrak{S}$ such that $G = \mathcal{N} - A$, therefor $\mathcal{V}_{\mathfrak{S}} - cl(G^c) = \mathcal{V}_{\mathfrak{S}} - cl((\mathcal{N} - A)^c) = (\mathcal{N} - A)^c \cup \Upsilon_{\mathfrak{S}}((\mathcal{N} - A)^c) = (\mathcal{N}^c \cup A) \cup \Upsilon_{\mathfrak{S}}(\mathcal{N}^c \cup A)$, now by Theorem 3.4(1) that $\mathcal{V}_{\mathfrak{S}} - cl(G^c) = (\mathcal{N}^c \cup A) \cup \Upsilon_{\mathfrak{S}}(\mathcal{N}^c) \cup \Upsilon_{\mathfrak{S}}(A)$. But $A \notin \mathfrak{S}$, so $\Upsilon_{\mathfrak{S}}(A) = \emptyset$. Since $\mathcal{N}^c$ is $\mathcal{V}_{\mathfrak{S}}-$ closed, then $\Upsilon_{\mathfrak{S}}(\mathcal{N}^c) \subseteq \mathcal{N}^c$. So we get $\mathcal{V}_{\mathfrak{S}} - cl(G^c) = (\mathcal{N}^c \cup A) = (\mathcal{N} - A)^c = G^c$, which implies that $G \in \mathcal{V}_{\mathfrak{S}}$. Hence $\mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X}) \subseteq \mathcal{V}_{\mathfrak{S}}$.

If $\mathfrak{S} = \mathbb{P}(X) - \{\emptyset\}$, then we have to show that $\mathcal{V}O_X \supseteq \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X}) \supseteq \mathcal{V}_{\mathfrak{S}}$. Let $G \in \mathcal{V}_{\mathfrak{S}}$, therefor G^c is $\mathcal{V}_{\mathfrak{S}}-$ closed, then $\Upsilon_{\mathfrak{S}}(G^c) \subseteq G^c$ and so $G \subseteq X - \Upsilon_{\mathfrak{S}}(G^c)$, that means for each $x \in G$ there exists $S \in \mathcal{V}O_X$ such that $S \cap G^c \notin \mathfrak{S}$, which implies that $S \cap G^c = \emptyset$, then $S \subseteq G$, it follows that $G \in \mathcal{V}O_X$ and so $\mathcal{V}O_X \supseteq \mathcal{V}_{\mathfrak{S}}$.

Hence $\mathcal{V}O_X = \mathcal{V}_{\mathfrak{S}}$. Now let $\mathcal{K} \in \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X})$, then $\mathcal{K} = \mathcal{N} - A \ni \mathcal{N} \in \mathcal{V}O_X$ and $A \notin \mathfrak{S}$, but $A = \emptyset$, so $\mathcal{K} = \mathcal{N}$, then $\mathcal{K} \in \mathcal{V}O_X$ and so $\mathcal{V}O_X \supseteq \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X})$. Thus $\mathcal{V}O_X = \mathcal{B}(\mathfrak{S}, \mathcal{V}\mathrm{O_X})$.

Corollary 3.11

For a grill $\boldsymbol{\mathcal{V}}$-space $(X, \mathcal{VO}_X, \mathfrak{S})$. If $\mathcal{N} \in \mathcal{VO}_X$, then $\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(Ɱ) = \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ)$ for each $Ɱ \subseteq X$.

Proof:

Let $\mathcal{N} \in \mathcal{VO}_X$, we know that $\mathcal{N} \cap Ɱ \subseteq Ɱ$, it follows from Theorem 3.3(1) that $\Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ) \subseteq \Upsilon_{\mathfrak{S}}(Ɱ)$, then $\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ) \subseteq \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(Ɱ)$. On the other hand, let $x \in \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(Ɱ)$, then $x \in \mathcal{N} \wedge x \in \Upsilon_{\mathfrak{S}}(Ɱ)$. for each $S \in \mathcal{VO}_X \ni x \in S$, we have $x \in \mathcal{N} \cap S \in \mathcal{VO}_X$, but $x \in \Upsilon_{\mathfrak{S}}(Ɱ)$, then $(\mathcal{N} \cap S) \cap Ɱ \in \mathfrak{S}$, that is, $(\mathcal{N} \cap Ɱ) \cap S \in \mathfrak{S}$, therefor $x \in \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ)$ and so $x \in \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ)$, which implies that $\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(Ɱ) \subseteq \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ)$. Thus $\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(Ɱ) = \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ)$.

Corollary 3.12

From *any* $(X, \mathcal{VO}_X, \mathfrak{S})$ grill $\boldsymbol{\mathcal{V}}$-space. If $\mathcal{VO}_X - \{\emptyset\} \subseteq \mathfrak{S}$, implies $\mathcal{N} \subseteq \Upsilon_{\mathfrak{S}}(\mathcal{N})$ for each $\mathcal{N} \in \mathcal{VO}_X$.

Proof:

Let $\mathcal{N} \in \mathcal{VO}_X$, if $\mathcal{N} = \emptyset$, then $\Upsilon_{\mathfrak{S}}(\mathcal{N}) = \emptyset$. If $\mathcal{VO}_X - \{\emptyset\} \subseteq \mathfrak{S}$, for each $\mathcal{N} \in \mathcal{VO}_X$, we have from Corollary 3.11, $\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(X) = \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap X)$, but $\Upsilon_{\mathfrak{S}}(X) = X$, it follows that $\mathcal{N} \cap X = \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N})$ and so $\mathcal{N} = \mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N})$, that means $\mathcal{N} \subseteq \Upsilon_{\mathfrak{S}}(\mathcal{N})$.

Corollary 3.13

For a grill $\boldsymbol{\mathcal{V}}$-space $(X, \mathcal{VO}_X, \mathfrak{S})$. If $\mathcal{N} \in \mathcal{VO}_X$ and $Ɱ \subseteq X$, then $\mathcal{N} \cap \mathcal{V}_{\mathfrak{S}} - cl(Ɱ) \subseteq \mathcal{V}_{\mathfrak{S}} - cl(\mathcal{N} \cap Ɱ)$.

Proof:

$\mathcal{N} \cap \mathcal{V}_{\mathfrak{S}} - cl(Ɱ) = \mathcal{N} \cap \left(Ɱ \cup \Upsilon_{\mathfrak{S}}(Ɱ)\right)$ (By Definition 3.6)

$= (\mathcal{N} \cap Ɱ) \cup \left(\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(Ɱ)\right)$ (Distribution of the intersection on the union)

$= (\mathcal{N} \cap Ɱ) \cup \left(\mathcal{N} \cap \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ)\right)$ (By Corollary 3.11)

$\subseteq (\mathcal{N} \cap Ɱ) \cup \Upsilon_{\mathfrak{S}}(\mathcal{N} \cap Ɱ) = \mathcal{V}_{\mathfrak{S}} - cl(\mathcal{N} \cap Ɱ)$ (By Definition 3.6).

Definition 3.14

For any $(X, \mathcal{VO}_x, \mathfrak{S})$, and $Ɱ \subseteq X$;

i. $Ɱ$ is named $\mathcal{V}_{\mathfrak{S}}$*preopen* set if $Ɱ \subseteq \mathcal{V}_{\mathfrak{S}}\text{int}\,\mathcal{V}_{\mathfrak{S}}\text{cl}(Ɱ)$.

ii. $Ɱ$ is named $\mathcal{V}_{\mathfrak{S}}$*semiopen* set if $Ɱ \subseteq \mathcal{V}_{\mathfrak{S}}\text{cl}\,\mathcal{V}_{\mathfrak{S}}\text{int}(Ɱ)$.

iii. $Ɱ$ is named $\mathcal{V}_{\mathfrak{S}}\alpha$*open* set if $Ɱ \subseteq \mathcal{V}_{\mathfrak{S}}\text{int}\,\mathcal{V}_{\mathfrak{S}}\text{cl}\,\mathcal{V}_{\mathfrak{S}}\text{int}\,(Ɱ)$.

iv.

When the $(\mathcal{V}_{\mathfrak{S}}\text{int}(Ɱ), \mathcal{V}_{\mathfrak{S}}\text{cl}(Ɱ))$ are the (interior, exterior) of $Ɱ$ for $(X, \mathcal{VO}_X, \mathfrak{S})$, resp.,

The *following* diagram show the relationships among the above notions:

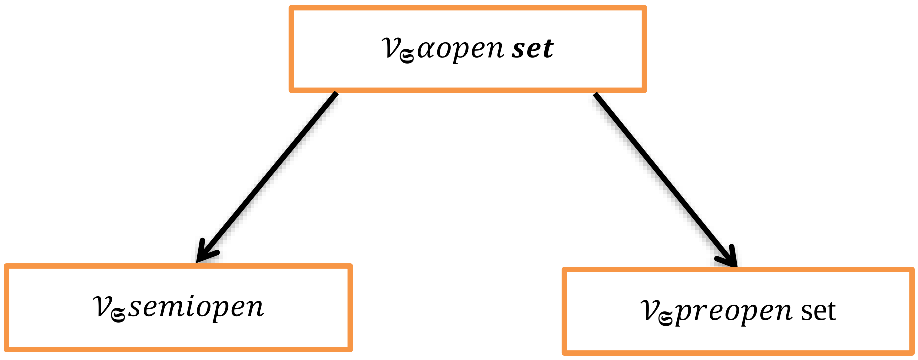

Diagram -1-

4. Conclusions

The main objective of this research is to define a new type of topological spaces and to understand and analyze the geometric and topological properties of these spaces in such a way that mathematicians and researchers in the field of topological engineering can carry out deeper and more effective studies in this field, as well as to classify spaces and understand, classify, organize and define different types of topological spaces. The differences between them. In this way, mathematicians can understand how the different properties and relationships between spaces are interwoven. In addition to the applications of space studied in mathematics and other sciences, it also contributes to the development of related theories and technologies by finding applications in other areas of mathematics and science, which may have implications for physics, computer science and data science, for example.

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Conflict of Interest

"Conflict of Interest: The authors declare that they have no conflicts of interest."

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