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A Comparison of some non-parametric kernel estimators using simulations

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Abstract

In this research, we conducted a comprehensive investigation of various kernel functions employed in the estimation of nonparametric regression functions. In particular, we investigated the Nadaraya-Watson method and local polynomial regression techniques involving linear, quadratic and cubic forms. These methods were evaluated using five different kernel functions: Gaussian, Epanechnikov, uniform, triangular and quartic, collectively referred to as GEUTQ.
The main objective of the work was to determine the best estimators for non-parametric kernel functions. To achieve this, we performed a rigorous comparison with simulation methods, different regression models and different sample sizes. The evaluation of the performance of the estimators was based on the mean absolute percentage error (AMAPE) assuming a standard normal distribution with a mean of zero and a variance of one.
Our simulation results and plots clearly show that the quadratic estimator (LP2) using the kernel function (G, E) consistently has the lowest (AMAPE) across all sample sizes and two models. Similarly, the local linear estimator (LP1) in the functions (U, T, Q) has the lowest (AMAPE) for all sample sizes and two models. As for the optimal functions, function is identified as the most effective kernel function among the options considered, leading to the lowest average values.
Furthermore, it is observed that as the sample size increases, the average values for the following methods decrease: Nadaraya, the linear method and the quadratic method. Conversely, the LP3 estimators, especially the linear cubic regression, turn out to be the least favorable and have relatively high values compared to the other estimators. These results provide valuable insights into the performance of different estimators and kernel functions in nonparametric regression models and thus contribute to future research and decision-making processes.

Keywords: nonparametric regression, Nadaraya-Watson estimator, kernel functions, Quadratic, local polynomial estimator.

1. Introduction

In 1988, [1] introduced and studied Kernel Smoothing for estimating partial linear models, [2] conducted a study titled "A Comparison of Some Semiparametric Estimators for Consumption Function Regression". In 2013, [3] presented a paper titled "Performance of Nonparametric Regression Estimation with Diverse Covariates Pak." In 2014, [4] introduced a paper comparing semiparametric estimators models using different smoothing methods. Additionally, [5] contributed to the literature with a paper titled "Hazard Rate Estimation Using Varying Kernel Function for Censored Data Type I.". In 2020, [6] employed wavelet and kernel smoothers to estimate stock return rates. Additionally, in 2020, [7] introduced a comparison between various nonparametric methods, utilizing the methodology of quantile regression models.

In 2021, [8] presented a paper titled 'Estimation of Nonparametric Autoregressive Curves Using the Smoothing Spline Method.' Furthermore, in the same year, [9] made their own introduction.

(Linear Regression Model to Study the Effects of Weather Variables) In the past, numerous researchers have incorporated the Nadaraya-Watson technique into their studies. Some notable examples include [10-12], all of whom conducted research involving this technique.

Regression analysis aims to uncover the relationship between the dependent variable and the explanatory variables. It involves incorporating these variables into a mathematical model to predict future values of the dependent variable. Regression models can be categorized into three types:

The first model is a parametric regression model, which requires knowledge of the distribution and has assumptions and limitations [13]. The second model is when data analysis is challenging in this model, we use the non-parametric regression models, which offer greater flexibility for analyzing variable relationships, do not require knowledge of the distribution, and do not necessitate restrictions or special conditions as in parametric models. This flexibility makes non-parametric models attractive to many researchers. However, they are susceptible to the problem of the curse of dimensionality, which arises when the number of variables increases. The last model combines both parametric and non-parametric elements [14,13].

The objective of this study is to compare non-parametric estimators, including Nadaraya-Watson and local linear, quadratic, and cubic estimators, using multiple kernel functions (GEUTQ). We employed Monte Carlo simulation to identify the best estimator based on mean absolute percentage errors (MAPE). Simulation results have demonstrated that the LP1 estimator for the quartic (Q) function consistently outperforms other estimators across various sample sizes and models.

Section 2 provides the theoretical foundation, encompassing methods for estimating non-parametric regression models, kernel functions, the Local Polynomial Regression Estimator, and bandwidth. Section 3 We studied five of the kernel functions. Section 4 presents the results of our simulation study, conducted using MATLAB 19. In Section 5, we present the results and engage in discussion. Section 6 summarizes the key findings of this study, and Section 7 offers recommendations for future research.

2. Methods for Estimating Non-parametric Regression Models

Non-parametric regression, initially proposed by researcher Jacob Wolfowib in 1942, focuses on estimating the curve of the regression function rather than estimating specific parameters. This is achieved through the application of non-parametric regression functions, ultimately leading to the identification of the most suitable method for non-parametric regression models [8].

$$Yi = m\left(Xi\right) + \varepsilon_i \tag{1}$$

where $i = 1, 2, \dots, n$

Y_i is the variable of interest, $m(X_i)$ is an unknown function to be determined using sample data

e_i is an error term assumed to be N (0, σ^2) under the model.[16]

The methodology of estimating non-parametric regression models is widely employed to determine the average treatment effect when implementing regression discontinuity designs.

Instead of simply calculating averages, this method helps reduce bias, and any remaining bias can be treated as normal [17,18]. For a more in-depth exploration of non-parametric regression, refer to the works of Some´[19-32], among others.

kernel function (Nadaraya-Watson- estimator NW)

In 1988, Robinson constructed an estimate of the non-parametric component using the least squares method in conjunction with the Nadaraya-Watson estimator. Subsequently, in 1997, Hamilton, Genentech, and Truong applied the Local Linear Smoother, which is based on the Speckman method [33].

Partial Linear Model Definition:

The partial linear model encompasses two types of Nadaraya-Watson estimators.

The first type employs Nadaraya and Watson with a fixed smoothing parameter. This estimator is among the most widely used nonparametric regression estimators. It is also known for utilizing a fixed smoothing parameter. An essential characteristic of this estimator is its ability to provide a continuous estimate of the regression when a continuous kernel function is used. The mathematical representation of this estimator is as follows [22].

$$\hat{m}^F(x) = \frac{\sum_{i=1}^{n} K\left(\frac{X-x}{h}\right) Y_i}{\sum_{i=1}^{n} K\left(\frac{X-x}{h}\right)}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{K_{h(x_i)}(X_i - x)}{n^{-1}K_{h(x_i)}(X_i - x)}\right] Y_i$$

$$= \frac{1}{2}\sum_{i=1}^{n} W_h(X_i)\, Y_i$$

The second is Nadaraya and Watson with variable smoothing parameters'

This estimator relies on the utilization of variable smoothing parameters as opposed to fixed smoothing parameters. It earns its name "variable" due to its adaptability, wherein the smoothing parameter varies at each point. This variability is achieved by employing large windows in areas with low data density and small windows in areas with high data density. The formula for this estimator is given as follows [3]:

$$\hat{m}^v(x) = \frac{\sum_{i=1}^{n} K_{h(x_i)}(X_i - x) Y_i}{\sum_{i=1}^{n} K_{h(x_i)}(X_i - x)}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{K_{h(x_i)}(X_i - x)}{n^{-1}K_{h(x_i)}(X_i - x)}\right] Y_i$$

This modification ensures that the estimator adjusts its smoothing parameter based on the local data density, contributing to its adaptability in various contexts.

The Local Polynomial Regression Estimator:

Consider a random sample $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ of bivariate data taken from a finite population. From the model in (1), to estimate the unknown function $m(Xi)$, the procedure below follows [20]. For more on Local Polynomial Regression Estimator [34-39]:

$m(X) = E[Y / X = x]$

The approximation for this can be achieved using Taylor's series, as outlined in [46]

$$m(x) \approx m(x_o) + m'(x_o) + \frac{m''(x_o)}{2!}(x - x_0)^2 + \ldots + \frac{m^{(p)}(x_o)}{p!}(x - x_0)^p \quad (2)$$

When utilizing the local polynomial X_0, and minimizing the Weighted Least Square Regression, we arrive at the following result. For more comprehensive information [34].

for further details:

$$\min_B \sum_{k=1}^{n} \left\{ y_i - B_0 - B_1(X_K - x_0) - \ldots - B_p(X_K - x_0)^p \right\}^2 K_h(X_K - x_0) \quad (3)$$

Where: $B = (B_0, B_1, .., ..B_P)^T$

K=kernel function

h: Represents a bandwidth

Then we obtain the estimator of Weighted Least Square (WLS) in form; [50]

$$\hat{B}(x) = (X' WX)^{-1} X' WY \quad (4)$$

Where

$$X = \begin{Bmatrix} 1 & (X_1 - x_0) & \cdots & (X_1 - x_0)^p \\ 1 & (X_2 - x_0) & \ldots & (X_2 - x_0)^p \\ \vdots & \vdots & \ldots & \vdots \\ 1 & (X_n - x_0) & \cdots & (X_n - x_0)^p \end{Bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$W = \begin{Bmatrix} K_h(X_0 - x_1) & 0 & \cdots & 0 \\ 0 & K_h(X_0 - x_2) & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & K_h(X_0 - x_n) \end{Bmatrix} = diag(K_h(X - x))$$

So, the local polynomial regression estimator for the regression function (smoothing) m will be [39]:

$\hat{m}_{p,h}(x) = \hat{B}_0(X)$

And $m(x) \approx B_0(X)$

- When the degree of the polynomial (p=0), we obtain the local constant estimator (Nadaraya-Watson).
- When the degree of the polynomial (p=1), we obtain the local linear estimator.
- When the degree of the polynomial (p=2), we obtain the local quadratic estimator.
- When the degree of the polynomial (p=3), we obtain the local cubic estimator.
- When the degree of the polynomial is *p*, we obtain the local polynomial regression.

In order to emphasize the importance of linked kernels in multiple regression, various types have been used in numerous studies. These include discrete univariate kernels, continuous univariate kernels and bivariate kernels. Further information on the different kernel types can be found in the work of [5,31,40,41].

3. kernel functions

1-Epanechnikov kernel

The interconnected kernel defines the kernel (Epan) Epanechniko [42]

$$K(x) = \frac{3}{4}\left(1 - x^2\right) \quad , \quad I_{\{|X| \le 1\}} \tag{5}$$

2- Gaussian kernel

the Gaussian kernel for multiple regression is,

$$k(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \quad , \quad I_{\{|x| \le \infty\}} \tag{6}$$

3-Uniform kernel

$$K(x) = \frac{1}{2} \quad , \quad I_{\{|x| \le 1\}} \tag{7}$$

4- Triangular kernel,

$$K(x) = \left(1 - |x|\right) \quad , \quad I_{\{|x| \le 1\}} \tag{8}$$

5-Quartic kernel,

$$K(x) = \frac{15}{16}\left(1 - x^2\right)^2 \quad , \quad I_{\{|X| \le 1\}} \tag{9}$$

Bandwidth has been a subject of study by numerous researchers in various studies, including[43,44].

In the context of multivariate analysis, the bandwidth is denoted by the symbol H, while in univariate analysis it is denoted by the symbol h. It is often referred to as 'bandwidth' or 'window size' A small bandwidth is usually associated with areas of dense data, while a large bandwidth is preferred in regions of sparse data. The value of bandwidth can be determined using the cross-validation (CV) method [21,44]. The bandwidth matrix selection was originally proposed by [15]. Further results can be found in [45, 46]. For our study, we employed the leave-one-out cross-validation (CV) method proposed by Scott and Terrell in1987 [47].

The basic idea of this method is to choose a value that reduces the value of CV(h) and delete all controls xi and yi from the data set and prepare the prediction for the remaining n-1 controls according to the following formula:

$$lcv(h) = \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{m}_{(-1)}(x_i) - y_i\right)^2 \tag{10}$$

$\widehat{m}_{(-1)}(.)$ Denotes the smoothing estimator when the odd data point xi,yi is deleted from the dataset, and only the remaining data n-1 is used to calculate this estimate, This calculation depends on leaving out of one for regression estimates $\widehat{m}_{(-1)}(.)$ which can be illustrated by the following equation:

$$\widehat{m}_{(-1)}(x_i) = \frac{\widehat{m}_{(-1)}(x_i) - L_i(X_i)Y_i}{1 - L_i(X_i)}$$

Substituting the above equation into equation (10) becomes a criterion (cv)as follows:

$$lcv(h) = \frac{1}{2}\sum_{i=1}^{n}\frac{\left(Y - \widehat{m}_{(-1)}(x_i)\right)^2}{\left(1 - L_i(X_i)\right)^2}$$

Naturally, we choose the bandwidth that minimizes cv(h), which is known to minimize the Kullback-Leibler distance between $\widehat{m}(x)$ and $m(x)$

4. Simulation study

The primary objective of this research was to conduct several simulation experiments to select the best estimators. We considered two regression functions taken from previously published research, namely:

M(x)=x-3 [31] linear smoothing function

M(x)=x+2exp(-16x^2) nonlinear smoothing function

The simulation was performed on the following two models:

Y= M(x-3) + e

y=exp+2exp(-16x^2) +e

The illustrative variable *x* was generated using a uniform distribution based on the Box-Muller method, while the random error followed a standard normal distribution with a mean of zero and a variance of one. For our simulation experiments, we employed five different sample sizes: n=10,30,50,70,100*n*=10,30,50,70,100 for each model. The experiments were repeated R=500 times to ensure accurate results.

In the estimation of nonparametric regression functions, we utilized the following kernel functions: Gaussian (G), Epanechnikov (E), uniform (U), Triangular (T), and Quartic (Q). To compare the models, we employed the Mean Absolute Percentage Error (MAPE) [48].

This rigorous approach allowed us to make meaningful comparisons between the different models and assess their performance effectively."

$$AMAPE = \frac{1}{R}\sum_{i=1}^{R} MAPE$$

where

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{y-\hat{y}}{y}\right|$$

n= sample size, y=real values

$\hat{y}$=estimator values, R=is the replication

and used (cv) to find the smoothing bandwidth for the nonparametric estimators

5. Results and Discussion

In our research, we utilized MATLAB 19 for the thorough analysis of results obtained from each model.

First model

We utilized a simple linear model for the smoothing NW Nadaraya-Watson- Estimator, LP1 Local linear Estimator, LP2 Local quadratic Estimator, and LP3 Local cubic Estimator and the results are in the table below:

Table 1. MAPE for nonparametric estimators' size (n=10,30,50,70,100) for the first model

Methods kernel function		NW	LP1	LP2	LP3	Best
G	10	0.642346	1.003498	0.581124	0.665476	LP2(G)
E		0.634757	0.558149	0.574258	0.665494	LP1(E)
U		0.613096	0.442113	0.554662	0.667669	LP1(U)
T		0.621812	0.341636	0.562547	0.668542	LP1(T)
Q		0.619036	0.318724	0.560036	0.669462	LP1(Q)
Best		NW(U)	Lp1(Q)	LP2(U)	LP3(G)	LP1(Q)

G	30	0.545729	0.93469	0.519102	0.818688	LP2(G)
E		0.548701	0.556902	0.521929	0.823759	LP2(E)
U		0.555327	0.442176	0.528232	0.81483	LP1(U)
T		0.552742	0.350689	0.525774	0.817275	LP1(T)
Q		0.55185	0.327352	0.524925	0.8121	LP1(Q)
Best		NW(G)	LP1(Q)	LP2(G)	LP3(Q)	LP1(Q)
G	50	0.532882	0.920043	0.515409	0.875729	LP2(G)
E		0.534376	0.555453	0.516854	0.874997	LP2(E)
U		0.535663	0.442501	0.518098	0.876267	LP1(U)
T		0.539932	0.348913	0.522227	0.872257	LP1(T)
Q		0.534709	0.326183	0.517176	0.8794	LP1(Q)
Best		Nw(G)	LP1(Q)	LP2(G)	LP3(T)	LP1(Q)
G	70	0.526177	0.913865	0.513184	0.903238	LP2(G)
E		0.527437	0.554494	0.514414	0.903668	LP2(E)
U		0.525973	0.444323	0.512985	0.904481	LP1(U)
T		0.526167	0.353313	0.513174	0.905332	LP1(T)
Q		0.529208	0.326448	0.51614	0.904708	LP1(Q)
Best		NW(U)	LP1(Q)	LP2(U)	LP3(E)	LP1(Q)
G	100	0.522963	0.908445	0.512607	0.923784	LP2(G)
E		0.524939	0.552334	0.514544	0.923153	LP2(E)
U		0.522031	0.443607	0.511694	0.922888	LP1(U)
T		0.522288	0.353527	0.511945	0.92167	LP1(T)
Q		0.521718	0.328334	0.511387	0.924304	LP1(Q)
Best		NW(Q)	LP1(Q)	LP2(Q)	LP3(T)	LP1(Q)

The results of the first model showed that in the table (1) we note:
1- Each function will be compared with the methods at size 10
For function G the LP2 method is better. For functions (E, U, T, Q) the LP1 method is better.
Each function will be compared with the methods at size 30,50,70,100
For function (G, E) The LP2 method is better. For function (U, T,Q) the LP1 method is better
2- We find that the best for all methods, for all functions, and for all sizes is: LP1(Q)
3- MAPE values decrease when the sample size increases for most estimators.
4- To compare the best functions for each estimator, for the estimator Nadaraya Watson, the lowest value of MAPE was when N=100 at the function Quartic (Q)
5- for the Local linear LP1 estimator, the lowest value of MAPE for all sizes was the Quartic (Q)
6-For the Local quadratic LP2 estimator, the lowest value of MAPE when N=100 is the function Quartic (Q)
7-For the LP3 estimator, the lowest value of MAPE was when N = 10, which is the function Gaussian (G)
8-The worst estimators were LP3 for all functions and all size
Figures 1-5 illustrate what was mentioned above

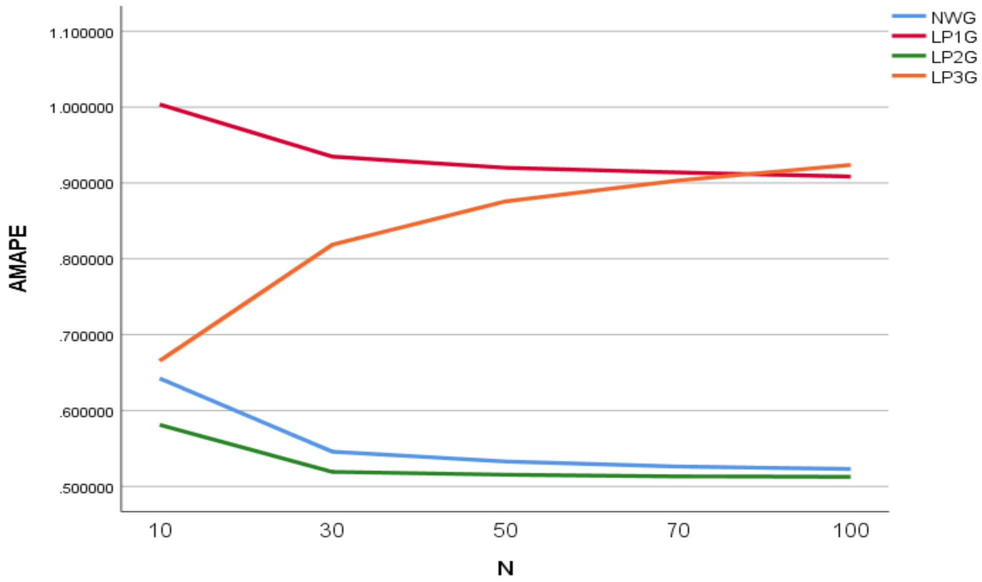

Figure 1. AMAPE for the parametric methods at the function G for the first model

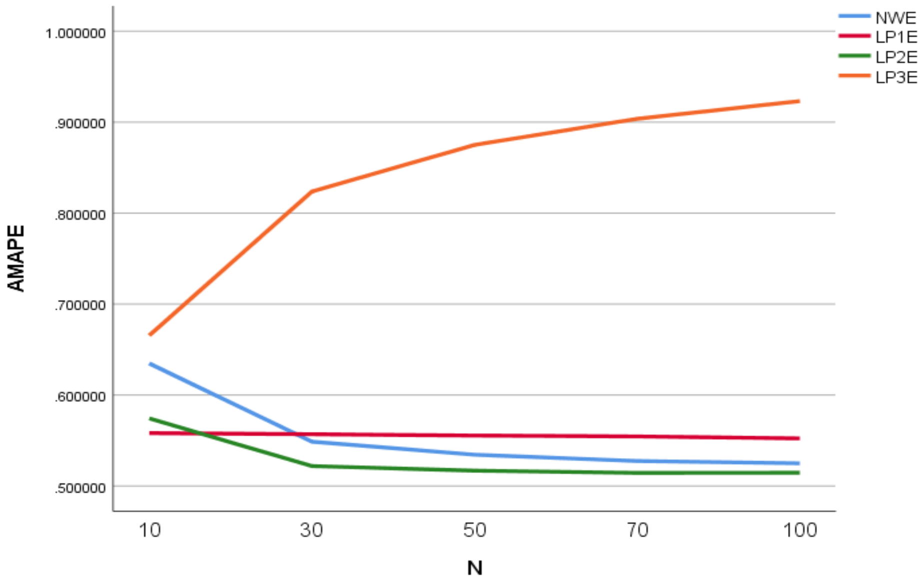

Figure 2. AMAPE for the parametric methods at the function E for the first model

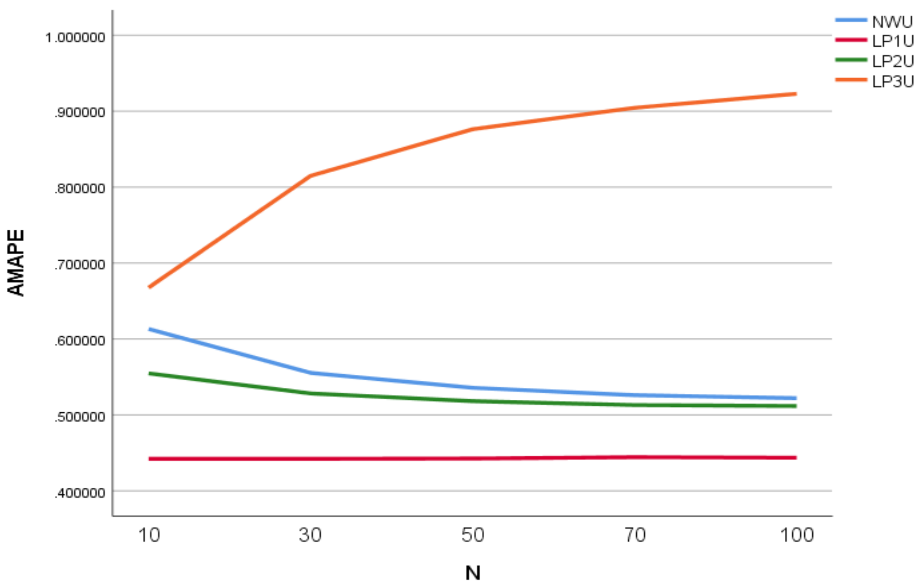

Figure 3. AMAPE for the parametric methods at the function U for the first model

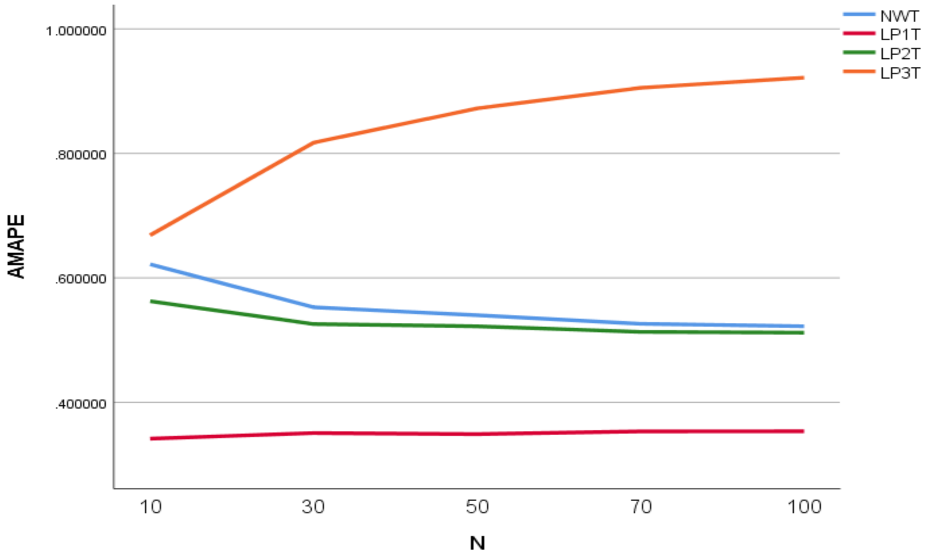

Figure 4. AMAPE for the parametric methods at the function T for the first model

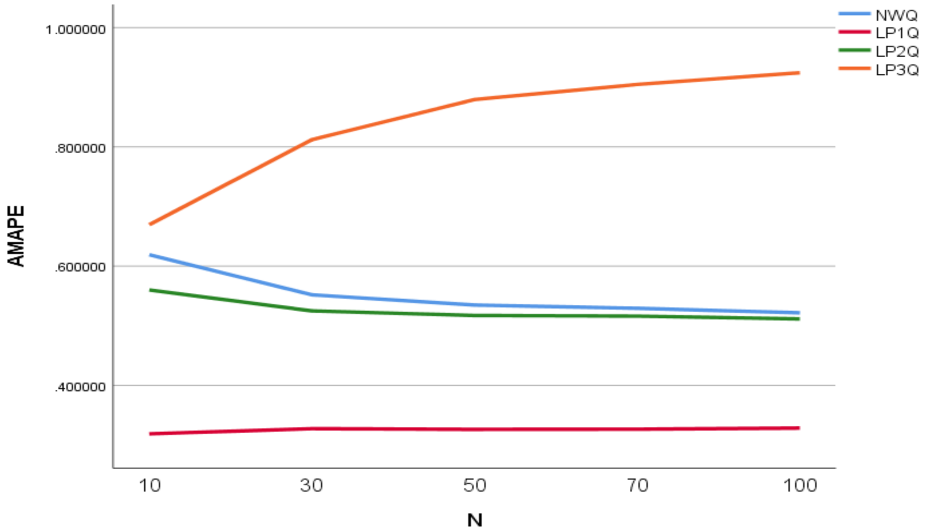

Figure 5. AMAPE for the parametric methods at the function Q for the first model

Table 2. MAPE for nonparametric estimators' size (n=10,30,50,70,100) for the second model

kernel function	Methods	NW	LP1	LP2	LP3	Best
G	10	0.624293	0.995722	0.564791	0.667828	LP2(G)
E		0.6297	0.555418	0.569683	0.674076	LP1(E)
U		0.616122	0.445208	0.557399	0.662098	LP1(U)
T		0.624639	0.341685	0.565105	0.664842	LP1(T)
Q		0.613763	0.324112	0.555265	0.656937	LP1(Q)
Best		NW(Q)	LP1(Q)	LP2(Q)	LP3(Q)	LP1(Q)
G	30	0.55542	0.93609	0.528321	0.816388	LP2(G)
E		0.553955	0.55382	0.526928	0.823293	LP2(E)
U		0.552651	0.442828	0.525687	0.818546	LP1(U)
T		0.552655	0.350887	0.52569	0.817904	LP1(T)
Q		0.554171	0.323977	0.527132	0.817301	LP1(Q)
Best		NW(U)	LP1(Q)	LP2(U)	LP3(G)	LP1(Q)
G	50	0.532262	0.921187	0.514809	0.873312	LP2(G)

E		0.539639	0.552673	0.521944	0.874695	LP2(E)
U		0.536898	0.442926	0.519294	0.873117	LP1(U)
T		0.539634	0.350005	0.521939	0.872166	LP1(T)
Q		0.531799	0.32971	0.514362	0.872262	LP1(Q)
Best		NW(Q)	LP1(Q)	LP2(Q)	LP3(T)	LP1(Q)
G	70	0.527445	0.912154	0.514421	0.906164	LP2(G)
E		0.526648	0.555864	0.513643	0.900793	LP2(E)
U		0.528657	0.4429	0.515603	0.903672	LP1(U)
T		0.525111	0.353748	0.512144	0.906952	LP1(T)
Q		0.527096	0.327854	0.51408	0.904502	LP1(Q)
Best		NW(T)	LP1(Q)	LP2(T)	LP3(E)	LP1(Q)
G	100	0.520777	0.909646	0.510464	0.922083	LP2(G)
E		0.524435	0.55284	0.514049	0.922157	LP2(E)
U		0.526603	0.441297	0.516174	0.921811	LP1(U)
T		0.519077	0.355318	0.508798	0.924151	LP1(T)
Q		0.521945	0.328704	0.511609	0.921277	LP1(Q)
Best		NW(T)	LP1(Q)	LP2(T)	LP3(Q)	LP1(Q)

The results of the second model showed that in the table (2) we note:

1. Each function will be compared with the methods:
   At size 10 For function G the LP2 method is better. For function (E, U, T, Q) the LP1 method is better.
   At size 30,50,70,100
   For function (G, E) The LP2 method is better. For functions (U, T, Q) the LP1 method is better
2. decrease in MAPE values when increasing sample size for most estimators
3. To compare the best functions for each estimator, for the estimator Nadaraya Watson, the lowest value of MAPE was when N=100 at the function the Triangular (T)
4. FOr the Local linear LP1 estimator, the lowest value of MAPE for all sizes was the quartic (Q)
5. For the Local quadratic LP2 estimator, the lowest value of MAPE when N=100 is the Triangular (T)
6. For the LP3 estimator, the lowest value of MAPE was when N = 10, which is the quartic (Q)
7. The worst estimators were LP3 for all functions and all size

Figures 6-10 illustrate what was mentioned above

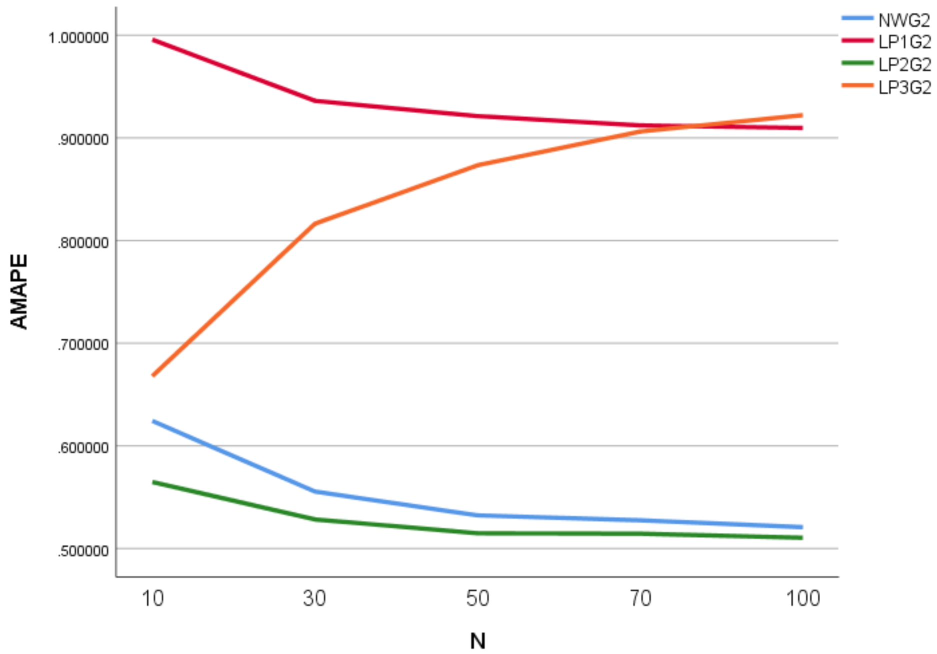

Figure 6. AMAPE for the parametric methods at the function G for the second model

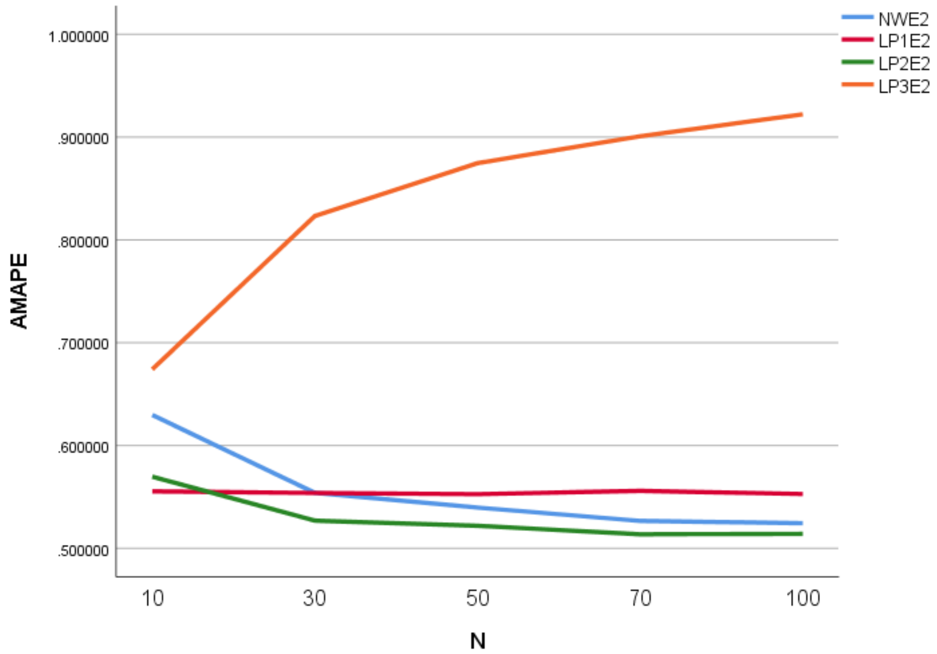

Figure 7. AMAPE for the parametric methods at the function E for the second model

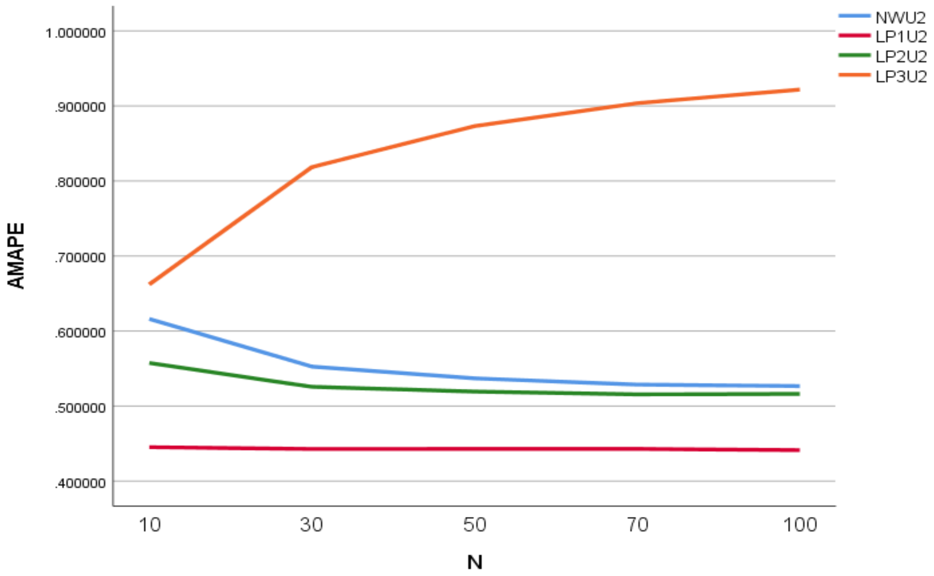

Figure 8. AMAPE for the parametric methods at the function U for the second model

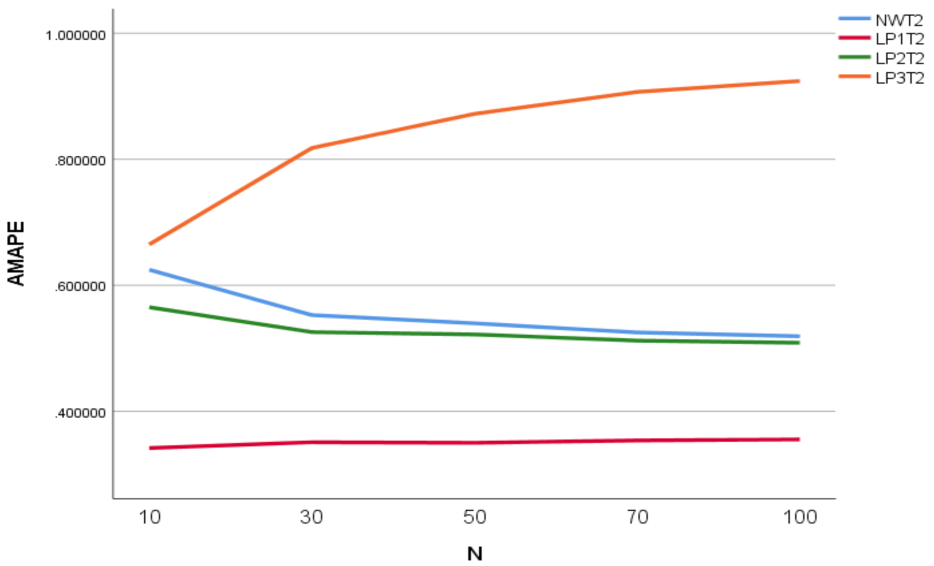

Figure 9. AMAPE for the parametric methods at the function T for the second model

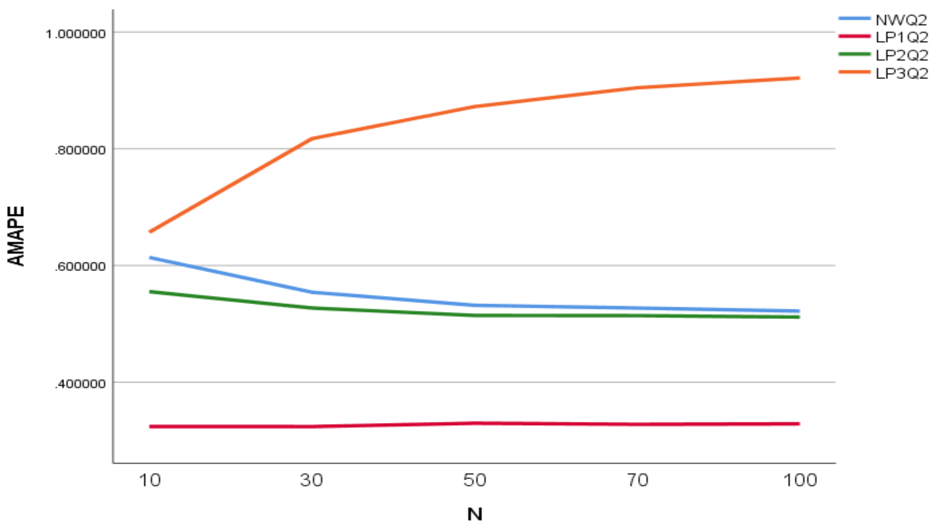

Figure 10. AMAPE for the parametric methods at the function Q for the second model

6. Conclusions

Based on the results of the simulation experiments, the following conclusions can be drawn:

1. The LP1 estimator is preferred when using the quartic function across all sample sizes and for two models. This is closely followed by the LP2 estimator, which performs well when quartic and triangular kernels are used.
2. The quartic function proves to be the best kernel function among the options considered.
3. In contrast, the LP3 estimators (linear cubic regression estimators), which have relatively large values compared to the other estimators, are the worst.

These results provide valuable insights into the performance of different estimators and kernel functions in nonparametric regression models and will help in future research and decision-making processes.

7. Recommendations

1. We recommend considering the use of cubic smoothing estimators, in addition to kernel and local polynomial regression functions, for future research.
2. It is advisable to investigate methods for detecting outliers and assessing linear correlations among variables when conducting future studies.
3. Exploring alternative functions beyond those employed in this research is encouraged for a more comprehensive analysis.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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