Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2025, 38(2)

Fixed Point Results and Application of Cyclic Contractive Maps in b-Metric Spaces

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Received:10 November 2023 Accepted:16 January 2024 Published: 20 April 2025
doi.org/10.30526/38.2.3828

Abstract

One of the generalizations of the usual metric function is the b-metric, which provides researchers with a broader field for deriving numerous results and applications related to fixed point theory. The aim of this paper is to develop three new fixed point principles in the complete b-metric space (£,ρ) when ρ is a continuous function in two variables. Here there are three directions to prove the existence and uniqueness of fixed points. First, we derive a result in terms of Branciari's theorem by combining integral contractive conditions with the notion of a cyclic map. Second, we apply the notion of cyclic representation to maps satisfying general weak conditions, including a changing distance function, to simulate the content of Boyd and Wong's theorem. Using this result, an application to the existence and uniqueness of the solution of an integral equation is given. Finally, an implicit relation with a changing distance function is used to construct a cyclic contractive map. Some examples are also presented to analyze and illustrate the main results.

Keywords: Alternating distance functions, Complete b-metric spaces, Contractive conditions, Cyclic representation, Fixed points.

1. Introduction

In 2003, Cyclic contraction was first introduced by Kirk et al. (1), who introduced results dealing with mappings of the type $f: £_i \rightarrow £_{i+1}$, $i = 1, 2, \cdots, p+1$, with $£_i = £_{i+1}$, where the contractive hypotheses are restricted to pairs $(a, b) \in £_i \times £_{i+1}$.

Extensions of the Banach's theorem and an extension of the Caristi theorem were proved. In addition, results related to non-expansive mappings in a Banach space were included. Several authors have contributed to research on fixed points in various cases for cyclic contractions (2-9). Backhtin (10) presented a definition of b-metric by replacing the triangle inequality as the following

Definition1.1: Let £ be a nonempty set and $q \geq 1$. A function $\rho : £ \times £ \rightarrow R^+$ is said to be a b-metric on £ if (10):

1. $\rho(a, b) = 0$ if and only if $a = b$;

2. $\rho(a,b) = \rho(b,a)$ for all $a,b \in £$;
3. $\rho(a,b) \leq q(\rho(a,c) + \rho(c,b))$ for all $a,b,c \in £$.

The pair $(£,\rho)$ is called a b-metric space.

Definition 1.2: Let $(£,\rho)$ be a b-metric space, $a \in £$ and (a_n) be a sequence in £. Then

1. (a_n) converges to a if and only if $\lim_{n\to\infty} \rho(a_n, a) = 0$. We denote this by $\lim_{n\to\infty} a_n = a$ or $a_n \to a$(as $n \to \infty$).
2. (a_n) is Cauchy if and only if $\lim_{n,m\to\infty} \rho(a_n, a_m) = 0$.
3. $(£,\rho)$ is complete if and only if every Cauchy sequence in £ is convergent.

Remark 1.3: In a b-metric space $(£,\rho)$, the following assertions hold(11,12):

1. A convergent sequence has a unique limit.
2. Each convergent sequence is Cauchy.
3. In general, a b-metric is not continuous.

As in the usual metric space **(**1**)**, we reform the following definition:

Definition1.4: Let $\{£_i\}_{i=1}^{n}$be a nonempty subsets of a b-metric space $(£,\rho), £ = \bigcup_{i=1}^{n} £_i$ and $f: £ \to £$ such that

1) $f(£_1) \subset £_2, \dots, f(£_{n-1}) \subset £_n, f(£_n) \subset £_1$ for $1 \leq i \leq n$;
2) $\exists k \in (0,1)$ such that $\rho(fa, fb) \leq k\, \rho(a,b)\ \forall a \in £_i, b \in £_{i+1}$ for $1 \leq i \leq n$

Then f is the cyclic contraction map and £ is cyclic representation w.r.t., f.

In the field of fixed point theory for cyclicity, see (13-20)

Example1.5: Let $£ = [-1,1], \rho(a,b) = |a-b|^2$ is b-metric with $s = 2$ and $£_1 = [-1,0], £_2 = [0,1], £_3 = [-1,0], £_4 = [0,1], £_5 = [-1,0], £_6 = [0,1]$. So, $£ = \bigcup_{i=1}^{6} £_i$. Define $f: \bigcup_{i=1}^{6} £_i \to \bigcup_{i=1}^{6} £_i$ such that $fa = -\frac{a}{2+a}$, $\forall a \in \bigcup_{i=1}^{6} £_i$. Here £ is cyclic representation w.r.t.,f. $(f(£_1) \subset £_2, f(£_2) \subset £_3, f(£_3) \subset £_4$, and $f(£_4) \subset £_5, f(£_5) \subset £_6, f(£_6) \subset £_1)$ and $f$is cyclic contraction with constant $0 < k = \frac{1}{2}$, where $a \in £_i\ b \in £_{i+1}$.

Definition 1.6: A point a is called a fixed point of a map $f: £ \to £$ if $f(a) = a$ (21).

An example, $a = 0$ is a fixed point of f in the previous example.

This work includes four main fixed point theorems based on different cyclic contractive conditions. Here, $(£,\rho)$ denote to complete b-metric space where ρ is continuous.

2. Materials and Methods

In this section, there are three axes; the first one is

Fixed point for cyclic contractive maps with integral condition

Not that, A Lebesgue-integrable function $¥: [0,1) \to [0,1)$ is called summable if $\int ¥(r)dr < \infty$ (22).

Theorem 2.1: Let $(£,\rho)$ be b-metric space, $k \in (0,1)$, and let $f: £ \to £$ be a map such that for each $a, b \in £$

$$\int_0^{\rho(fa,\ fb)} ¥(r)\, dr \leq k \int_0^{\rho(a,\ b)} ¥(r) dr, \forall a \in £_i,\ b \in £_{i+1} \tag{1}$$

where, ¥ is summable on each compact subset of $[0,\infty)$, nonnegative and for any $\varepsilon > 0, \int_0^{\varepsilon} ¥(r)\, dr > 0$. Then $\exists!\, c \in \bigcap_{i=1}^{n} £_i$, moreover, $\forall a \in £$, $\lim_{m\to\infty} f^m(a) = c$.

Proof: Let $a_0 \in £ = \bigcup_{i=1}^{n} £_i$ and consider $a_{m+1} = fa_m$ for each $m \in N \bigcup\{0\}$, so for any $m \in N \bigcup\{0\}$, $\exists i_m \in \{1,2,\dots,n\}$ such that $a_m \in £_{i_m}$ and $a_{m+1} \in £_{i_m+1}$.

If $a_{m_0} = a_{m_0+1}$ for some m_0 then, since $a_{m_0+1} = fa_{m_0} = a_{m_0}$ this mean a_{m_0} is a fixed point of f. Thus, assume that $a_m \neq a_{m+1}$ for all $m \in N \cup \{0\}$. So,

$$\int_0^{\rho(fa_m,\ fa_{m+1})} ¥(r)\, dr \leq k \int_0^{\rho(a_m, a_{m+1})} ¥(r)\, dr. \qquad (2)$$

By repeating the inequality (2) m times, it follows directly

$$\int_0^{\rho(fa_m,\ fa_{m+1})} ¥(r)\, dr \leq k \int_0^{\rho(a_m, a_{m+1})} ¥(r)\, dr = k^m \int_0^{\rho(a_0,\ fa_0)} ¥(r)\, dr.$$

As consequence, since $k \in (0,1)$, obtaining a monotone decreasing sequence $\left(\int_0^{\rho(fa_m,\ fa_{m+1})} ¥(r)\, dr\right) = \left(\int_0^{\rho(a_{m+1}, a_{m+2})} ¥(r)\, dr\right)$ which has lower bound is 0. We have $\rho(fa_m, fa_{m+1}) \to 0$ as$m \to \infty$. By properties of real sequence, $\left(\int_0^{\rho(fa_m,\ fa_{m+1})} ¥(r)\, dr\right)$ convergences $\varepsilon \geq 0$ such that $\lim_{m\to\infty} \int_0^{\rho(a_{m+1},a_{m+2})} ¥(r)\, dr = \varepsilon$. Suppose that $\varepsilon > 0$, it is enough to assume $\lim_{m\to\infty} sup\, \rho(a_{m+1}, a_{m+2}) = \varepsilon > 0$. Then there exists a $u_\varepsilon \epsilon N$ and a sequence $(fa_{m_u})_{u \geq u_\varepsilon}$ such that $\rho(fa_{m_u}, fa_{m_u+1}) \to \varepsilon > 0$ as $u \to \infty$ and $\rho(fa_{m_u}, fa_{m_u+1}) \geq \frac{\varepsilon}{2}$.

For each $u \geq u_\varepsilon$, by

$$\frac{\varepsilon}{2} \leq \lim_{m\to\infty} \rho\left(fa_{m_u}, fa_{m_u+1}\right) = \lim_{m\to\infty} \sup \rho\left(fa_{m_u}, fa_{m_u+1}\right) = \varepsilon > 0,$$

this is true only if $\varepsilon = 0$. The next step is proving that for each $a_0 \in £$, (a_m) is a Cauchy sequence. For $j > n$ define $£_j = £_i$ if $j = i \bmod n$.

Claim I**:** for all $\varepsilon > 0$ there exist $m \in N$ such that for all $j, i \geq m$, $j - i \equiv 1 (\bmod\, n)$ then $\rho(a_j, a_i) < \varepsilon$.

Suppose that there exists $\varepsilon > 0$ such that for each $m \in N$, one can find $j > i > m$ with $j - i \equiv 1 (\bmod\, n)$ satisfying $\rho(a_j, a_i) \geq \varepsilon$. Clearly, $\rho(a_m, a_{m+1}) < \varepsilon$.

Now, take $m \geq 2 (\bmod\, n)$. Then, corresponding to $i \geq m$ use can choose j in such a way that it is the smallest integer with $j > i$ satisfying $j - i \equiv 1 (\bmod\, n)$ and $\rho(a_j, a_i) \geq \varepsilon$. Therefore, $\rho(a_{j-n}, a_i) \leq \varepsilon$. By triangular inequality

$$\varepsilon \leq \rho\left(a_j, a_i\right) \leq q\left(\rho\left(a_j, a_{i-n}\right) + \sum_{k=1}^{n} \rho(a_{i-k}, a_{i-k+1})\right)$$
$$\leq q\left(\sum_{k=1}^{n} \rho(a_{i-k}, a_{i-k+1})\right) + q\,\varepsilon \to q\,\varepsilon \text{ as } n \to \infty.$$

Again, by triangular inequality

$$\varepsilon \leq \rho\left(a_j, a_i\right) \leq q\,\rho\left(a_j, a_{j+1}\right) + q\rho\left(a_{j+1}, a_{i+1}\right) + q\rho(a_{i+1}, a_i) \to q\,\varepsilon \quad \text{as } j, i \to \infty,$$

we get

$$\int_0^{\rho(fa_{j+1}, fa_{i+1})} ¥(r)\, dr \leq k \int_0^{\rho\left(a_{j+1}, a_{i+1}\right)} ¥(r)\, dr. \qquad (3)$$

Letting $j, i \to \infty$ implies to $\int_0^{q\varepsilon} ¥(r)\, dr \leq k \int_0^{q\varepsilon} ¥(r)\, dr$, which is a contradiction. Therefore, the (Claim I) is proved.

Now, to prove (a_m) is Cauchy sequence in $(£, \rho)$. Fix $\varepsilon > 0$. By the claim, $\exists m_0$ such that if $j, i \geq m_0$ with $j - i \equiv 1 (\bmod\, n)$ $\rho(a_j, a_i) \leq \frac{\varepsilon}{n}$.

Since $\lim_{m\to\infty} \rho(a_m, a_{m+1}) = 0$, $\exists m_1 \in N$ such that $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{n}$, $\forall m \geq m_1$

Suppose $c, v \geq max\{m_0, m_1\}$ and $c > v$. Then there exists $h \in \{1,2, \dots\dots, n\}$ such that $v - c \equiv h (\bmod\, n)$. Therefore, $v - c + r \equiv 1 (\bmod\, n)$ for $r = n - h + 1$. So, getting

$$\rho(a_c, a_v) \leq q(\rho(a_c, a_{v+r}) + \rho(a_{v+r}, a_v)).$$
$$\leq q\,\rho(a_c, a_{v+r}) + q^2\,\rho(a_{v+r}, a_{v+r-1}) + q^3 \rho(a_{v+r-1}, a_{v+r-2}) + \cdots + q^r \rho(a_{v+1}, a_v)$$

By $\rho(a_j, a_i) \leq \frac{\varepsilon}{n}$ and $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{n}$ and from the last inequality,

$$\rho(a_c, a_v) \leq q\frac{\varepsilon}{n} + q^2\frac{\varepsilon}{n} + q^3\frac{\varepsilon}{n} + \cdots + q^r\frac{\varepsilon}{n} \leq q\frac{\varepsilon}{n}\left(\frac{1}{1-q}\right) \to 0, \text{ as } n \to \infty$$

This proves that (a_m) is the Cauchy sequence. The completeness of $(£, \rho)$ implies to exist $c \in$

£ such that $\lim_{m\to\infty} a_m = c$.

To prove c is a fixed point for f. Since $£ = \bigcup_{i=1}^{n} £_i$ is a cyclic representation of £ w.r.t., f, the sequence (a_m) has infinite terms in each $£_{i_m}$for $i_m \in \{1,2,\ldots,n\}$. Closeness of $£_{i_m}$ for $i_m \in \{1,2,\ldots,n\}$ implies to $c \in \bigcap_{i=1}^{n} £_i$. Suppose that $c \in £_i$ and $fc \in £_{i+1}$. Since $(£,\rho)$ is complete, there exists appoint $c \in £ = \bigcup_{i=1}^{n} £_i$ such that $c = \lim_{m\to\infty} fa_m$

$0 < \rho(c, fc) \le q\,\rho(c, a_{m+1}) + q\,\rho(fa_m, fc) \to 0$ as $m \to \infty$.

Indeed both $\rho(c, fa_{m_r})$ and $\rho(fa_{m_r}, fc)$ converge to 0 as $m \to \infty$, for the first one it is obvious, while for the second one we have

$\int_0^{\rho(fa_m,fc)} ¥(r)\,dr \le k \int_0^{\rho(a_m,c)} ¥(r)\,dr \to 0$ as $m \to \infty$.

Now, if $\rho(fa_m, fc)$ does not converge to 0 as $m \to \infty$, then there exists a subsequence $(a_{m_r})_{r\in N}$ of (a_m) with $a_{m_r} \in £_{i-1}$ such that $\rho\big(fa_{m_r}, fc\big) \ge \varepsilon$ for a certain $\varepsilon > 0$, we have the following contradiction $0 < \int_0^{\varepsilon} ¥(r)\,dr \le \int_0^{\rho(fa_{m_r},fc)} ¥(r)\,dr \to 0$ as $r \to \infty$. This means that $\rho(c, fc) \le 0$ thus, $\rho(c, fc) = 0$, c is the fixed point of f.

For the uniqueness of fixed point c. Assume a fixed point w of f differs from c, i.e., $fw = w$. The cyclic character of f and c, $w \in £ = \bigcup_{i=1}^{n} £_i$ are fixed points of f implying that $c, w \in \bigcap_{i=1}^{n} £_i$. By (1), we obtain

$\int_0^{\rho(c,w)} ¥(r)\,dr = \int_0^{\rho(fc,fw)} ¥(r)\,dr \le k \int_0^{\rho(c,w)} ¥(r)\,dr$,

which is contradiction, consequently, $c = w$ for each $a \in £$, $\lim_{m\to\infty} f^m a = c$.

Example 2.2 : Let $£ = [-1, 1]$ and $\rho(a,b) = |a-b|^2$ is b-metric with $q = 2$. Suppose $£_1 = [-1, 0]$, $£_2 = [0, 1]$ and $£ = \bigcup_{i=1}^{2} £_i$. Define $f\colon \bigcup_{i=1}^{2} £_i \to \bigcup_{i=1}^{2} £_i$ such that $(a) = \frac{-a}{2}\ \forall a$. So, $f(£_1) \subset £_2$, $f(£_2) \subset £_1$ and f is contraction of integral type with constant $k = \frac{1}{2} \in (0,1)$ and $¥(r) = \frac{r}{3}$, for $a \in £_1, b \in £_2$

$\int_0^{\rho(fa,fb)} \frac{r}{3}\,dr = \int_0^{\frac{1}{2}|b-a|^2} \frac{r}{3}\,dr \le \frac{1}{2}\int_0^{\rho(a,b)} \frac{r}{3}\,dr.$

Hence f satisfies the hypothesis of (Theorem 2.1), which has a unique fixed point at 0.

Fixed point for general cyclic$(\emptyset - \psi)$ weak contractive maps

Recall the following two definitions:

Definition 2.3: Let ψ is function where $\psi\colon [0,\infty) \to [0,\infty)$ is called altering distance function if satisfies(23).

1. ψ is monotone increasing and lower semi-continuous;
2. $\psi(r) = 0$ if and only if $r = 0$.

As in usual metric spaces **(**4**)**, below, we reform many concepts in a b-metric space

Definition 2.4: Let $(£,\rho)$ be a b-metric space, n a positive integer $£_1$, $£_2$, …, $£_n$ nonempty closed subsets of £ and $£ = \bigcup_{i=1}^{n} £_i$. An operator $f : £ \to £$ is said to be a cyclic weak $(\emptyset - \psi)$-contraction if

1) $£ = \bigcup_{i=1}^{n} £_i$ is a cyclic representation of £ w.r.t., f.
2) $\emptyset(\rho(fa, fb)) \le \emptyset(\rho(a,b)) - \psi(\rho(a,b))$, for any $a \in £_i$, $b \in £_{i+1}$, $i = 1, 2, \ldots, n$, where $£_{n+1} = £_1$ and $\emptyset, \psi\colon [0,\infty) \to [0,\infty)$ is a non-decreasing and continuous function satisfying $\emptyset(r) > 0, \psi(r) > 0$ for $r \in (0,\infty)$ and $\emptyset(0) = 0, \psi(0) = 0$.

Theorem 2.5: Let f be a self-map of $(£,\rho)$ satisfies

$$\emptyset(\rho(fa, fb)) \le \emptyset\big(M(a,b)\big) - \psi\big(N(a,b)\big),\ \forall a \in £_i, b \in £_{i+1} \qquad (4)$$

where

$M(a,b) = t\max\{\rho(a,b), \rho(a,fa), \rho(b,fb), \rho(a,fb), \rho(b,fa)\}$,
and
$N(a,b) = t\min\{\rho(a,b), \rho(a,fa), \rho(b,fb), \rho(a,fb), \rho(b,fa)\}$,
where $t \in (0,1)$, and $\psi, \emptyset: [0,\infty) \to [0,\infty)$ are altering distance functions. Then $\exists\, c \in \bigcap_{i=1}^{n} £_i$, c is a unique fixed point of f.

Proof: Let $a_0 \in \bigcup_{i=1}^{n} £_i$ and consider $a_{m+1} = fa_m$ for each $m \in N \cup \{0\}$, so for any $m \in N \cup \{0\}$, $\exists i_m \in \{1, 2, \dots, n\}$ such that $a_m \in £_{i_m}$ and $a_{m+1} \in £_{i_m+1}$.

If $a_{m_0} = a_{m_0+1}$ for some m_0 then, since $a_{m_0+1} = f^{m_0} a_0 = a_{m_0}$ this means a_{m_0} is fixed point of f. Thus, assume that $a_m \neq a_{m+1}$ for all $m \in N \cup \{0\}$. getting

$$\emptyset(\rho(a_m, a_{m+1})) \leq \emptyset(M(a_{m-1}, a_m)) - \psi(N(a_{m-1}, a_m)), \quad (5)$$

where

$$M(a_{m-1}, a_m) = t\max\{\rho(a_{m-1}, a_m), \rho(a_m, a_{m+1}), \rho(a_{m-1}, a_{m+1}), \rho(a_m, a_m)\},$$

and

$$N(a_{m-1}, a_m) = t\min\{\rho(a_{m-1}, a_m), \rho(a_m, a_{m+1}), \rho(a_{m-1}, a_{m+1}), 0\} = 0.$$

By triangular inequality, so

$$M(a_{m-1}, a_m) = t\max\{\rho(a_{m-1}, a_m), \rho(a_m, a_{m+1}), q\,\rho(a_{m-1}, a_m) + q\,\rho(a_m, a_{m+1}), 0\},$$

if $\rho(a_{m-1}, a_m) < \rho(a_m, a_{m+1})$, then $M(a_{m-1}, a_m) \leq t\,2\,q\,\rho(a_m, a_{m+1})$. And by (5), we obtain $\emptyset(\rho(a_m, a_{m+1})) \leq \emptyset(t\,2\,q\,\rho(a_m, a_{m+1}))$. Since $\emptyset$ is altering distance function, subsequently $\rho(a_m, a_{m+1}) \leq t\,2\,q\,\rho(a_m, a_{m+1})$, this not true for all $t \in (0,1)$, as result

$$M(a_{m-1}, a_m) = t\,2\,q\,\rho(a_{m-1}, a_m) \text{ and } N(a_{m-1}, a_m) = 0. \quad (6)$$

Now put (6) in (5)

$$\emptyset(\rho(a_m, a_{m+1})) \leq \emptyset(t\,2\,q\,\rho(a_{m-1}, a_m)). \quad (7)$$

Since $\emptyset$ is altering distance function, as a result $\rho(a_m, a_{m+1}) \leq t\,2\,q\,\rho(a_{m-1}, a_m)$. Thus for all $m \in N \cup \{0\}$ we have a monotone decreasing sequence $(\rho(a_m, a_{m+1})) = (\rho(fa_{m-1}, fa_m))$. By properties of real sequence, $(\rho(a_m, a_{m+1}))$ there exist $\varepsilon \geq 0$ such that $\lim_{m\to\infty} \rho(a_m, a_{m+1}) = \varepsilon$. On letting $m \to \infty$ in (7), obtaining $\emptyset(\varepsilon) \leq \emptyset(t\,2\,q\,\varepsilon)$. Assume that $\varepsilon \neq 0$. So since $\emptyset$ is altering distance function, getting $\varepsilon \leq t\,2\,q\,\varepsilon < \varepsilon$, which a contradiction. Hence $\lim_{m\to\infty} \rho(a_m, a_{m+1}) = 0$.

For $j > n$ define $£_j = £_i$ if $j = i \bmod n$.

Claim I: for all $\varepsilon > 0$ there exist $m \in N$ such that for all $j, i \geq m$, $j - i \equiv 1 (\bmod\, n)$ then $\rho(a_j, a_i) < \varepsilon$.

Suppose that there exists $\varepsilon > 0$ such that for each $m \in N$ we can find $j > i > m$ with $j - i \equiv 1 (\bmod\, n)$ satisfying $\rho(a_i, a_j) \geq \varepsilon$.

Now, take $m \geq 2(\bmod\, n)$. Then, corresponding to $i \geq m$ use can choose j in such a way that it is the smallest integer with $j > i$ satisfying $j - i \equiv 1(\bmod\, n)$ and $\rho(a_j, a_i) \geq \varepsilon$. Therefore, $\rho(a_{j-n}, a_i) \leq \varepsilon$. By triangular inequality

$$\varepsilon \leq \rho(a_j, a_i) \leq q\,\rho(a_j, a_{j-n}) + q\,\sum_{k=1}^{n} \rho(a_{i-k}, a_{i-k+1}) \leq q\,\sum_{k=1}^{n} \rho(a_{i-k}, a_{i-k+1}) + q\varepsilon.$$

Taking $n \to \infty$ and since $\lim_{m\to\infty} \rho(a_m, a_{m+1}) = 0$, we obtain $\lim_{i,j\to\infty} \rho(a_j, a_i) = q\varepsilon$.

Again, by triangular inequality

$$\varepsilon \leq \rho(a_j, a_i) \leq 2q\rho(a_{j+1}, a_j) + q\rho(a_j, a_i) + 2q\rho(a_{i+1}, a_i).$$

Letting $j, i \to \infty$ and since $\lim_{m\to\infty} \rho(a_m, a_{m+1}) = 0$, so $\lim_{i,j\to\infty} \rho(a_{j+1}, a_{i+1}) = q\varepsilon$.

Since a_i, a_j belong to different sets $£_i$ and $£_{i+1}$, and using (2.4)

$$\emptyset\left(\rho\left(fa_j, fa_i\right)\right) \leq \emptyset\left(M\left(a_j, a_i\right)\right) - \psi\left(N\left(a_j, a_i\right)\right), \quad (8)$$

where

$$M\left(a_j, a_i\right) = t \max\{\rho\left(a_j, a_i\right), \rho\left(a_j, a_{j+1}\right), \rho\left(a_i, a_{i+1}\right), \rho\left(a_j, a_{i+1}\right), \rho\left(a_i, a_{j+1}\right)\}$$

$$N\left(a_j, a_i\right) = t \min\{\rho\left(a_j, a_i\right), \rho\left(a_j, a_{j+1}\right), \rho\left(a_i, a_{i+1}\right), \rho\left(a_j, a_{i+1}\right), \rho\left(a_i, a_{j+1}\right)\}.$$

Since $\lim_{i,j\to\infty} \rho\left(a_j, a_i\right) = q\varepsilon$, $\lim_{i,j\to\infty} \rho\left(a_{j+1}, a_{i+1}\right) = q\varepsilon$, and by the triangle inequality

$$\rho\left(a_{i+1}, a_{j+1}\right) \leq q\rho(a_{i+1}, a_i) + q\rho\left(a_i, a_j\right) + q\rho\left(a_j, a_{j+1}\right) \to q\varepsilon \text{ as } i, j \to \infty.$$

Again by the triangle inequality

$$\rho(a_j, a_{i+1}) \leq q\rho(a_j, a_{j+1}) + q\rho(a_{j+1}, a_{i+1}) \to q\varepsilon \text{ as } i, j \to \infty.$$

Again by the triangle inequality

$$\rho(a_i, a_{j+1}) \leq q\rho(a_i, a_{i+1}) + q\rho(a_{i+1}, a_{j+1}) \to q\varepsilon \text{ as } i, j \to \infty.$$

Letting $i, j \to \infty$, we have $M\left(a_j, a_i\right) = t \max\{q\varepsilon, 0, 0, q\varepsilon, q\varepsilon\} \to tq\varepsilon$ and $N(a_j, a_i) \to 0$ and by the inequality (8), we have $\emptyset(\varepsilon) \leq \emptyset(tq\varepsilon) - \psi(0)$.

Since $\emptyset$ is altering distance function and $t \in (0,1)$, then $\varepsilon \leq tq\varepsilon$ which a contradiction. Therefore, the claim (I) is held.

Now, we will prove (a_m) is a Cauchy sequence in $(£, \rho)$. Fix $\varepsilon > 0$. By the claim, $\exists m_0$ such that if $j, i \geq m_0$ with $j - i \equiv 1(\text{mod}\, n)$ such that $\rho(a_j, a_i) \leq \frac{\varepsilon}{2}$.

Since $\lim_{m\to\infty} \rho(a_m, a_{m+1}) = 0$, also $\exists m_1 \in N$ such that $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{2n}$, $\forall m \geq m_1$.

Suppose $c, v \geq \max\{m_0, m_1\}$ and $c > v$. Then there exists $h \in \{1, 2, \ldots, n\}$ such that $v - c \equiv h(\text{mod}\, n)$. Therefore, $v - c + r \equiv 1(\text{mod}\, n)$ for $r = n - h + 1$. So, we have

$$\rho(a_c, a_v) \leq q\, \rho(a_c, a_{v+r}) + q^2\, \rho(a_{v+r}, a_{v+r-1}) + q^3 \rho(a_{v+r-1}, a_{v+r-2}) + \\ + q^r \rho(a_{v+1}, a_v) \quad (9)$$

By $\rho(a_j, a_i) \leq \frac{\varepsilon}{2}$ and $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{2n}$ and from (9), $\rho(a_c, a_v) \leq q\, \frac{\varepsilon}{n}(\frac{1}{1-q}) \to 0$, as $n \to \infty$

This proves that (a_m) is a Cauchy sequence. The completeness of $(£, \rho)$ implies to exists $c \in £$ such that $\lim_{m\to\infty} a_m = c$.

Now, to prove c is a fixed point for f. Since $£ = \bigcup_{i=1}^{n} £_i$ is a cyclic representation of £ w.r.t., f, the sequence (a_m) has infinite terms in each $£_{i_m}$ for $i_m \in \{1, 2, \ldots, n\}$. Closeness of $£_{i_m}$ for $i_m \in \{1, 2, \ldots, n\}$ implies to $c \in \bigcap_{i=1}^{n} £_i$. Suppose that $c \in £_i$ and $fc \in £_{i+1}$ and take a subsequence $(a_{m_r})_{r\in N}$ of (a_m) with $a_{m_r} \in £_{i-1}$ and take $a = c, b = a_{m_r}$ in (4)

$$\emptyset\left(\rho\left(fc, fa_{m_r}\right)\right) \leq \emptyset\left(M\left(a_{m_r}, c\right)\right) - \psi\left(N\left(a_{m_r}, c\right)\right) \quad (10)$$

where

$$M\left(a_{m_r}, c\right) = t \max\{\rho\left(a_{m_r}, c\right), \rho\left(a_{m_r}, fa_{m_r}\right), \rho(c, fc), \rho\left(a_{m_r}, fc\right), \rho\left(c, fa_{m_r}\right)\},$$

$$N\left(a_{m_r}, c\right) = t \min\{\rho\left(a_{m_r}, c\right), \rho\left(a_{m_r}, fa_{m_r}\right), \rho(c, fc), \rho\left(a_{m_r}, fc\right), \rho\left(c, fa_{m_r}\right)\}.$$

Taking $r \to \infty$, hence

$M\left(c, a_{m_r}\right) = t\, \rho(c, fc)$. And $\left(c, a_{m_r}\right) = 0$, using (10) subsequently

$$\emptyset\left(\rho(fc, c)\right) \leq \emptyset\left(t\, \rho(c, fc)\right) - \psi(0) \leq \emptyset\left(t\, \rho(c, fc)\right).$$

Since $\emptyset$ is altering distance function, then $\rho(fc, c) < t\, \rho(c, fc)$, which a contradiction because $t \in (0,1)$, hence $fc = c$. Thus c is a fixed point of f.

For the uniqueness, suppose that there are two distinct points c, w with c and w fixed points of f. The cyclic character of f and the fact that $c, w \in £ = \bigcup_{i=1}^{n} £_i$ are fixed points of f imply that $c, w \in \bigcap_{i=1}^{n} £_i$. Using (4), we can obtain

$\emptyset(\rho(fc, fw)) \leq \emptyset(M(c,w)) - \psi(N(c,w))$,

Where، $M(c,w) = t\,\rho(c,w)$ and $N(c,w)\} = 0$. Hence $\emptyset(\rho(c,w)) = \emptyset(\rho(fc,fw)) \leq \emptyset(t\,\rho(c,w))$. And since $\emptyset$ is altering distance function, we obtain $\rho(c,w) \leq q\,\rho(c,w)$ which a is contradiction since it is not true for all $t \in (0,1)$, then $c = w$. Hence f has a unique fixed point in £ and $c \in \bigcap_{i=1}^{n} £_i$.

Example 2.6: Let $£ = \{6, 7, 8, 9, 10\}$ with $\rho: £ \times £ \to [0, \infty)$ defined by

$$\rho(a,b) = \begin{cases} 0, if\ a = b \\ 6, if\ a \neq b, a, b \in \{6,7,8,9\} \\ 17, if\ a, b \in \{9,10\} \text{ and } a \neq b \\ 40, if a \in \{6,7,8\} \text{ and } b = 10 \text{ (or } b \in \{6,7,8\} \text{ and } a = 10) \end{cases}.$$

Since all Cauchy sequences in £ are constant. Therefore, are convergent. Then $(£, \rho)$ is complete b-metric space with $q = 2$. And $£_1 = \{6,8,10\}$ and $£_2 = \{6,7,9\}$, $£ = \bigcup_{i=1}^{2} £_i$. Define $f: \bigcup_{i=1}^{2} £_i \to \bigcup_{i=1}^{2} £_i$ such that $f(a) = 6$ and $if\ a \in \{6,7,8,9\}$ and $f(10) = 8$. So, $(£_1) \subset £_2, f(£_2) \subset £_1$, for $a \in £_1, b \in £_2$, and take $t = \frac{1}{4}$. Let $\emptyset, \psi: [0, \infty) \to [0, \infty)$ such that $\emptyset(r) = \frac{r}{4}$ and $\psi(r) = \frac{r}{2}$. Then $\emptyset$ and ψ are altering distance functions. It is easy to check condition (4) holds with fixed point $a = 6$.

An application for solving integral equations. Consider the integral equation (24,25)

$$w(t) = \int_0^J Q(t,r)T(r, w(r))\,dr \quad \text{for all } t \in [0, J], \tag{11}$$

where $J > 0$, $T: [0,J] \times R \to R$ and $Q: [0,J] \times [0,J] \to [0, \infty)$ are continuous functions.

In this section, we look for a nonnegative solution to (11) in $£ = C([0,J], R)$ by (Theorem 2.5). Let $£ = C[0,J]$ be the set of real valued continuous functions on $[0,J]$, where $[0,J]$ is a closed and bounded interval in R. For $p > 1$, define $\rho: [0,J] \times [0,J] \to R$ by

$\rho(w,v) = \max_{t \in [0,J]} |w(t) - v(t)|^p$, for all $w, v \in £$.

Therefore, $(£\ ,\rho)$is a complete b-metric space with $q = 2^{p-1}$. Let $\alpha, \beta \in £$ and $\alpha_0, \beta_0 \in R$ such that

$$\alpha_0 \leq \alpha(t) \leq \beta(t) \leq \beta_0, \quad \forall\, t \in [0,J]. \tag{12}$$

Suppose that for all $t \in [0,J]$, we have

$$\alpha(t) \leq \int_0^J Q(t,r)T(r, \beta(r))\,dr, \tag{13}$$

and

$$\beta(t) \geq \int_0^J Q(t,r)T(r, \alpha(r))\,dr. \tag{14}$$

We suppose that $\forall\, r \in [0,J], T(r, .)$be a decreasing function, that

$$a, b \in R, \quad a \geq b \text{ then } T(r,a) \leq T(r,b). \tag{15}$$

Assume that $k > 0$ is such that

$$k(\max_{t \in [0,J]} \int_0^J Q(t,r)\,dr) < 1. \tag{16}$$

Define a map $f: £ \to £$ by $fw(t) = \int_0^J Q(t,r)T(r, w(r))\,dr$, for all $t \in [0,J]$.

Suppose that $\forall r \in [0,J]$ and $a, b \in £$ with $(a(r) \leq \alpha_0$ and $b(r) \leq \beta_0)$ or vice versa,

$$0 \leq [T(q, a(r)) - T(r, b(r))] \leq k \max\{|a(r) - b(r)|^p, 0, |b(r) - fb(r)|^p,$$

$$|a(r) - fb(r)|^p, |b(r) - a(r)|^p\})^{\frac{1}{p}} \tag{17}$$

Theorem 2.7: Under the assumptions (12)-(17), the integral equation (11) has a solution in the set $\{w \in C([0,J]): \alpha \leq w \leq \beta\}$.

Proof : We omit the details because the proof steps are classic with some minor differences due to the specificity of the b-metric space.

Fixed points by implicit conditions

The following list of implicit functions under various conditions (26). Let Ω be the set of all real continuous functions $M: R_+^6 \to R$, satisfying the following conditions:

M_1) is non-increasing in variables r_2, r_3, r_4, r_5, r_6

M_2) there exists a right continuous function $A: [0, \infty) \to [0, \infty), A(0) = 0, A(r) < r$, for $r > 0$ such that for $c \geq 0$ $M(c, u, c, u, 0, c + u) \leq 0$ or $M(c, u, 0,0, u, u) \leq 0$, implies $c \leq A(u)$.

M_3) $M(c, 0, c, 0,0, c) > 0$ and $M(c, c, 0,0, c, c) > 0$ $\forall$ $c > 0$.

Also, let Ψ ≔ the set of functions $\psi: [0, \infty) \to [0, \infty)$ such that:

i) ψ is monotone increasing and continuous;

ii)$\psi(r) = 0$ if and only if $r = 0$;

iii) ψ is subadditive, i.e., $\forall r_1, r_2 \in [0, +\infty), \psi(r_1 + r_2) = \psi(r_1) + \psi(r_2)$.

Lemma 2.8: Let $A: [0, \infty) \to [0, \infty)$ be a right continuous function such that $A(r) < r$, for $r > 0$. (27) . Then $\lim_{m\to\infty} A^m(r) = 0$, where A^m≔m times repeated composition of A.

Theorem 2.9: If $M \in \Omega$ exists and £ = $\bigcup_{i=1}^{n}$ £$_i$ is a cyclic representation of £ w.r.t., f: £ → £. If for any $(a, b) \in$ £$_i \times$ £$_{i+1}$, $i = 1,2, \dots, n$

$M(\psi(\rho(fa, fb)), \psi(\rho(a, b)), \psi(\rho(a, fa)), \psi(\rho(b, fb)), \psi(\rho(a, fb)), \psi(\rho(b, fa))) \leq 0$

(18), and $\psi \in \Psi$. ∃! $c \in \bigcap_{i=1}^{n}$ £$_i$, c is a unique fixed point. Moreover, $\lim_{m\to\infty} f^m(a) = c$, for any $a \in$ £.

Proof: Let $a_0 \in \bigcup_{i=1}^{n}$ £$_i$ and a_m define by $a_{m+1} = fa_m$. So for $m \geq 0$, $\exists i_m \in \{1,2, \dots n\}$ such that $a_{m-1} \in$ £$_{i_m}$ and $a_m \in$ £$_{i_m+1}$. If $a_{m_0} = a_{m_0-1}$for some m_0, $a_{m_0} = fa_{m_0-1} = a_{m_0-1}$ then $a_{m_0}$is fixed point of f. Thus, suppose that $a_m \neq a_{m-1}$, for all $m \in N \cup \{0\}$. By using (2.18) therefore

$M(\psi(\rho(a_{m+1}, a_m)), \psi(\rho(a_m, a_{m-1})), \psi(\rho(a_m, a_{m+1})), \psi(\rho(a_{m-1}, a_m)),$
$\psi(\rho(a_m, a_m)), \psi(\rho(a_{m-1}, a_{m+1}))) \leq 0.$

And since $\psi(r) = 0$ if and only if $r = 0$, also by using triangle inequality and since ψ is subadditive, therefore

$M(\psi(\rho(a_{m+1}, a_m)), \psi(\rho(a_m, a_{m-1})), \psi(\rho(a_m, a_{m+1})), \psi(\rho(a_{m-1}, a_m)),0,$
$\psi(q\rho(a_{m-1}, a_m)) + \psi(q\rho(a_m, a_{m+1}))) \leq 0.$

And from M_2, there exists a right continuous function A: $[0, \infty) \to [0, \infty)$, A$(0) = 0$, A$(r) < r$, for $r > 0$, such that for all $m \in N \cup \{0\}$

$\psi(\rho(a_{m+1}, a_m) \leq A(\psi(\rho(a_m, a_{m-1}))).$

If we use this procedure, we get

$\psi(\rho(a_{m+1}, a_m) \leq A(\psi(\rho(a_m, a_{m-1}))) \leq \cdots \leq$ A^m $(\psi(\rho(a_1, a_0)).$ (19)

And by (Lemma 8) and continuity of ψ, subsequently

$\lim_{m\to\infty} \psi(\rho(a_{m+1}, a_m)) = 0 = \psi(\lim_{m\to\infty} \rho(a_{m+1}, a_m))$. Since $\psi(r) = 0$ if and only if $r = 0$, therefore

$\lim_{m\to\infty} \rho(a_{m+1}, a_m) = 0.$ (20)

To prove for each $a_0 \in$ £, (a_m) is a Cauchy sequence. Assume it is false. Then we can find a $\varepsilon > 0$ and $\{p_r\}, \{d_r\}, d_r > p_r \geq r$ where $\{p_r\}, \{d_r\}$ two subsequences of integers with

$\psi(\rho(a_{p_r}, a_{d_r})) \geq \varepsilon$ for $n \in \{1,2, \dots\}$. (21)

We also assume

$$\psi(\rho(a_{p_r}, a_{d_r-1})) < \varepsilon. \tag{22}$$

By selecting d_r to be the least number surpassing p_r for which inequality (21) holds, now by (19) and (21), (22), and since ψ is subadditive, getting

$$\varepsilon \leq \psi(\rho(a_{p_r}, a_{d_r-1})) \leq \psi(q\rho(a_{p_r}, a_{d_r-1})) + \psi(q\rho(a_{d_r-1}, a_{p_r}))$$
$$\leq q\varepsilon + A^{d_r-1}\psi(q\rho(a_0, a_1)). \tag{23}$$

And so

$$\lim_{r\to\infty} \psi(\rho(a_{d_r-1}, a_{p_r})) = q\varepsilon. \tag{24}$$

On the other hand, $\forall r$, $\exists i_r \in \{1,2,\dots,n\}$ such that $d_r - p_r + i_r \equiv 1 (\text{mod } n)$. Then $a_{p_r-i_r}$(for r large enough, $p_r > i_r$) and a_{d_r}belong to different sets $£_i$ and $£_{i+1}$ for $i \in \{1,2,\dots,n\}$. By the triangle inequality, also ψ is subadditive, obtaining

$$\psi\left(\rho(a_{d_r}, a_{p_r})\right) \leq \psi(q\rho(a_{d_r}, a_{d_r-i_r}) + q\rho(a_{d_r-i_r}, a_{p_r}))$$
$$\leq \psi(q\rho(a_{d_r}, a_{d_r-i_r})) + \psi(q\rho(a_{d_r-i_r}, a_{p_r})).$$

Now taking $r \to \infty$, by (19) and from (24), as a results

$$\psi\left(\rho(a_{p_r}, a_{d_r})\right) \leq A^{d_r-i_r}\psi(q\rho(a_0, a_1)) + q\varepsilon.$$

And so,

$$\lim_{r\to\infty} \psi(\rho(a_{d_r}, a_{p_r})) = q\varepsilon. \tag{25}$$

By using (20), so

$$\operatorname*{Lim}_{r\to\infty} \psi(\rho(a_{d_r+1}, a_{d_r})) = 0, \operatorname*{Lim}_{r\to\infty} \psi(\rho(a_{p_r-i_r+1}, a_{p_r-i_r})) = 0. \tag{26}$$

And by using the triangle inequality, then

$$\psi\left(\rho(a_{p_r-i_r}, a_{d_r})\right) \leq \psi(q\rho(a_{p_r-i_r}, a_{p_r})) + \psi(q\rho(a_{p_r}, a_{d_r})).$$

Letting $r \to \infty$ in the last inequality and using (2.19) and (2.25), consequently

$$\psi\left(\rho(a_{p_r-i_r}, a_{d_r})\right) \leq A^{p_r-i_r}\psi(q\rho(a_0, a_1)) + q\varepsilon$$
$$\lim_{r\to\infty} \rho\left(\rho(a_{p_r-i_r}, a_{d_r})\right) = q\varepsilon. \tag{27}$$

Again, by using the triangle inequality, obtaining

$$\psi\left(\rho(a_{p_r-i_r}, a_{q_r+1})\right) \leq \psi(q\rho(a_{p_r-i_r}, a_{d_r})) + \psi(q\rho(a_{d_r}, a_{d_r+1})).$$

Letting $r \to \infty$ in the last inequality and using (26) and (27), therefore

$$\lim_{r\to\infty} \psi(\rho(a_{p_r-i_r}, a_{d_r+1})) = q\varepsilon. \tag{28}$$

And in the same way, as a results

$$\psi(\rho(a_{d_r}, a_{p_r-i_r+1})) \leq \psi(q\rho(a_{d_r}, a_{p_r-i_r})) + \psi(q\rho(a_{p_r-i_r}, a_{p_r-i_r+1})).$$

Letting $r \to \infty$ and using (27) and (26), getting

$$\lim_{r\to\infty} \psi(\rho(a_{d_r}, a_{p_r-i_r+1})) = q\varepsilon. \tag{29}$$

Again, by using the triangle inequality, therefore

$$\psi\left(\rho(a_{p_r-i_r+1}, a_{d_r+1})\right) \leq \psi(q\rho(a_{p_r-i_r+1}, a_{d_r})) + \psi(q\rho(a_{d_r}, a_{d_r+1})).$$

Letting $r \to \infty$ in the last inequality and using (2.26) and (2.29), having

$$\operatorname*{Lim}_{r\to\infty} \psi(\rho(a_{p_r-i_r+1}, a_{d_r+1})) = q\varepsilon. \tag{30}$$

Using (2.18) for $a = a_{p_r-i_r}$ and $b = a_{d_r}$, subsequently

$$M(\psi(\rho(a_{p_r-i_r+1}, a_{d_r+1})), \psi(\rho(a_{p_r-i_r}, a_{d_r})), \psi(\rho(a_{p_r-i_r}, a_{p_r-i_r+1})),$$

$\psi(\rho(a_{d_r}, a_{d_r+1})), \psi(\rho(a_{d_r+1}, a_{p_r-i_r})), \psi(\rho(a_{d_r}, a_{p_r-i_r+1}))) \leq 0.$

Letting $r \to \infty$, and using (30), (27), (26), (28), (29), then, by continuity of M and $\psi(r) = 0$ if and only if $r = 0$, $M(q\varepsilon, q\varepsilon, 0,0, q\varepsilon, q\varepsilon) \leq 0$, a contradiction with M_3. Thus, (a_m) is Cauchy sequence in $(£, \rho)$. Now to prove that c is fixed point of f. In fact $fa_m \to c$ and since $£ = \bigcup_{i=1}^{n} £_i$ is cyclic representation of £ w.r.t., f, the sequence (a_m) has infinite terms in each $£_{i_m}$for $i_m \in \{1,2, \dots, n\}$. Considering that $£_{i_m}$ is closed for $i_m \in \{1,2, \dots, n\}$ we have $c \in \bigcap_{i=1}^{n} £_i$.

Suppose that $c \in £_i$ and $fc \in £_{i+1}$, and take a subsequence $(a_{m_r})_{r\in N}$of (a_m) with $a_{m_r} \in £_{i-1}$, using (2.18), take $a = c$ and $b = a_{m_r}$, as result

$M(\psi(\rho(fc, fa_{m_r})), \psi(\rho(c, a_{m_r})), \psi(\rho(c, fc)), \psi(\rho(a_{m_r}, fa_{m_r})),$

$\psi(\rho(c, fa_{m_r})), \psi(\rho(a_{m_r}, fc))) \leq 0.$

Taking $r \to \infty$, hence

$M(\psi(\rho(fc, c)), \psi(\rho(c, c)), \psi(\rho(c, fc)), \psi(\rho(c, c)), \psi(\rho(c, c)), \psi(\rho(c, fc))) \leq 0,$

and $\psi(r) = 0$ if and only if $r = 0$, then $M(\psi(\rho(fc, c)), 0, \psi(\rho(c, fc)), 0,0, \psi(\rho(c, fc))) \leq 0$, which is contradiction to M_3. Thus $\psi(\rho(fc, c)) = 0$ then $\rho(fc, c) = 0$, then $fc = c$. We obtain c is a fixed point of f. Suppose that there are two distinct points c, w with c and w are two fixed points of f. The cyclic nature of f, also, the fact $c, w \in £ = \bigcup_{i=1}^{n} £_i$ are fixed points of f imply that $c, w \in \bigcap_{i=1}^{n} £_i$ and by (18), obtaining

$M(\psi(\rho(fc, fw)), \psi(\rho(c, w)), \psi(\rho(c, fc)), \psi(\rho(w, fw)), \psi(\rho(c, fw)), \psi(\rho(w, fc))) \leq 0.$

Since c, w are fixed points of f, $\psi(r) = 0$ if and only if $r = 0$, we get

$M(\psi(\rho(c, w)), \psi(\rho(c, w)), 0,0, \psi(\rho(c, fw)), \psi(\rho(w, c))) \leq 0,$

which is a contradiction to M_3, then $\psi(\rho(c, w)) = 0$, hence $\rho(c, w) = 0$, that is, $c = w$. Hence f has a unique fixed point in £ and $c \in \bigcap_{i=1}^{n} £_i$.

Remark 2.10: It is worth noting, it is worth noting that we can obtain good results by including the concept of cyclicity in cases of **(**28,29**)**.

3. Discussion

This work is classified within the field depending on the classification 2010 MSC: 47H09, 47H10. Our study of this topic is the first in Iraq (to the best of our knowledge) in the field of fixed points for cyclic maps, and it is taken from a master's thesis by researcher Abbas Karim Nahi. The paper included new results in the field of integral contractions in the b-metric spaces of the cyclic type, as well as new generalizations of the results of other researchers in the case of the b-metric space. In the future, we would like to study the results in (30) in the case of cyclic maps.

4. Conclusions

In this paper, new theorems were established to find fixed points. First, by merging integral contractive conditions with the concept of cyclic map, and second, by applying the concept of cyclic representation with respect to maps satisfying general weak conditions, including a changing distance function. And third, by merging the changing distance function with the sub additive to find theorems in the field of fixed points through the concept of cyclic representation for maps satisfying an implicit relation including a changing distance function.

Acknowledgments

I would like to thank the previous researchers whose results have played an important role in obtaining our current results. Especially Professors W. A. Kirk, P.S. Srinivasan and P. Veeramani.

Conflict of Interest

The authors declare no conflict of interest.

Funding

This research received no external funding.

References

1. Kirk WA, Srinivasan PS, Veeramani P. Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory. 2003;4(1):79-89.
2. Asem V, Singh YM, Khan MS, Sessa S. On (α, p)-Cyclic Contractions and Related Fixed Point Theorems. Symmetry. 2023;15(10):1826. http://doi.org/10.3390/sym15101826
3. Konar P, Bhandari SK, Chandok S, Mukheimer A. Multivalued weak cyclic δ-contraction mappings. Journal of Inequalities and Applications. 2020;2020(1):1-11. http://doi.org/10.1186/s13660-020-02373-1
4. Karapinar E, Sadarangani K. Fixed point theory for cyclic (ϕ-ψ)-contractions. Fixed Point Theory and Applications. 2011;2011(1):1-8. http://doi.org/10.1186/1687-1812-2011-1
5. Hussain N, Parvaneh V, Roshan JR, Kadelburg Z. Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications. Fixed Point Theory and Applications. 2013;2013(1):1-18. http://doi.org/10.1186/1687-1812-2013-1
6. Hiranmoy G, Karapınar E, Kanta Dey L. Best proximity point results for contractive and cyclic contractive type mappings. Numerical Functional Analysis and Optimization. 2021;42(7):849-864. http://doi.org/10.1080/01630563.2021.1929578
7. Chaipunya P, Cho YJ, Sintunavarat W, Kumam P. Fixed point and common fixed point theorems for cyclic quasi-contractions in metric and ultrametric spaces. Advances in Pure Mathematics. 2012;2(6):401-407. http://doi.org/10.4236/apm.2012.26059
8. Abou Bakr, Sahar MA. Cyclic G-Ω-Weak Contraction-Weak Nonexpansive Mappings and Some Fixed Point Theorems in Metric Spaces. Abstract and Applied Analysis. 2021;2021:1-9.
9. Karapınar E, Romaguera S, Taş K. Fixed points for cyclic orbital generalized contractions on complete metric spaces. Open Mathematics. 2013;11(3):552-560. http://doi.org/10.2478/s11533-013-0245-5
10. Bakhtin I. The contraction mapping principle in quasimetric spaces. Functional Analysis. 1989;30:26-37.
11. Al-Bundi SS. Iterated function system in metric spaces. Bol Soc Paran Mat. 2022;40:1-10.
12. Abed SS. Fixed point principles in general b-metric Spaces and b-Menger probabilistic spaces. Journal of AL-Qadisiyah for Computer Science and Mathematics. 2018;10(2):42.
13. Karapınar E. Fixed point theory for cyclic weak ϕ-contraction. Applied Mathematics Letters. 2011;24(6):822-825. http://doi.org/10.1016/j.aml.2011.01.021
14. Hammad HA, De la Sen M. Solution of nonlinear integral equation via fixed point of cyclic α L ψ-rational contraction mappings in metric-like spaces. Bulletin of the Brazilian Mathematical Society. 2020;51(1):81-105. http://doi.org/10.1007/s00574-019-00162-1
15. Jabbar HA, Kadhim SN, Abed SS. Best approximation in b-modular spaces. Baghdad Science Journal. 2023. http://doi.org/10.21123/bsj.2023.8230

16. Qawaqneh H, Noorani M, Shatanawi W, Alsamir H. Common fixed point theorems for generalized geraghty (α, ψ, ϕ)-quasi contraction type mapping in partially ordered metric-like spaces. Axioms. 2018;7(4):74. http://doi.org/10.3390/axioms7040074
17. Mustafa Z, Mustafa Z, Khan SU, Jaradat MMM, Arshad M, Jaradat HM. Fixed point results of F-rational cyclic contractive mappings on 0-complete partial metric spaces. Forum. 2018;40:394-409.
18. Păcurar M, Rus IA. Fixed point theory of cyclic operators. Journal of Fixed Point Theory and Applications. 2022;24(4):79.
19. Theivaraman R, Srinivasan P, Radenovic S, Park C. New approximate fixed point results for various cyclic contraction operators on E-metric spaces. Journal of the Korean Society for Industrial and Applied Mathematics. 2023;27(3):160-179.
20. Kumari PS, Panthi D. Cyclic contractions and fixed point theorems on various generating spaces. Fixed Point Theory and Applications. 2015;2015(1):1-17. http://doi.org/10.1186/s13663-015-0365-7
21. Hussein Luaibi H, Abed SS. Fixed point theorems in general metric space with an application. Baghdad Science Journal. 2021;18(1 Suppl.):0812.
22. Branciari A. A fixed point theorem for mappings satisfying a general contractive condition of integral type. International Journal of Mathematics and Mathematical Sciences. 2002;29:531-536. http://doi.org/10.1155/S0161171202007524
23. Khan MS, Swaleh M, Sessa S. Fixed point theorems by altering distances between the points. Bulletin of the Australian Mathematical Society. 1984;30(1):1-9. http://doi.org/10.1017/S0004972700001659
24. Nieto JJ, Rodríguez-López R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order. 2005;22(3):223-239. http://doi.org/10.1007/s11083-005-9018-5
25. Rabaiah A, Tallafha A, Shatanawi W. Common fixed point results for mappings under nonlinear contraction of cyclic form in b-metric spaces. Nonlinear Functional Analysis and Applications. 2021;26(2):289-301.
26. Nashine HK, Kadelburg Z, Kumam P. Implicit-relation-type cyclic contractive mappings and applications to integral equations. Abstract and Applied Analysis. 2012;2012:1-15.
27. Matkowski J. Fixed point theorems for mappings with a contractive iterate at a point. Proceedings of the American Mathematical Society. 1977;62(2):344-348. http://doi.org/10.2307/2041931
28. Abed SS, Abed AN. Convergence and Stability of Iterative Scheme for a Monotone Total Asymptotically Non-expansive Mapping. Iraqi Journal of Science. 2022;63(1):241-250.
29. Abed SS, Taresh NS. On stability of iterative sequences with error. Mathematics. 2019;7(8):765. http://doi.org/10.3390/math7080765
30. Monje ZAM, Ahmed BA. A study of stability of first-order delay differential equations using fixed point theorem Banach. Iraqi Journal of Science. 2019;60(12):2719-2724