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New lifetime Shanker Weibull distribution: Structure and Properties

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Abstract

This paper introduces a new model with two parameters titled "Shanker Weibull distribution". This distribution is obtained by merging the Shanker distribution with a scale of one parameter and the Weibull distribution with a shape parameter. First, we present the mathematical structure of the new distribution, which depends on the survival function for each of the Shanker and Weibull distributions. The statistical functions of this new distribution are presented, such as the cumulative, probability density, hazard and survival functions. In addition, we discuss the behavior of the probability density function and the hazard function by mxamining their shape. We also present the statistical properties of this distribution, which include mode, median and moments around the origin. As a result of studying the moments around the origin, we obtain the variance and the first expected value (mean), the moment generating function, the skewness, the kurtosis, the characteristic function, the factorial generating function, the quantile function and the mean time to failure.

Keywords: Shanker distribution, exponential distribution, survival function, moments arond the origin, moment generating function.

1. Introduction

Although the process of mixing the distributions to create a new distribution is not new, it has gone through several stages of development depending on how to choose which distributions to mix (1). All of this is done using the data available to the researcher about the properties and nature of the work of each distribution (2). Many researchers have combined two distributions. For example, in 2015, Rama (3-4) proposed modelling lifetime data with a new one-parameter lifetime distribution, the "Shanker distribution", by combining the 'gamma(2,β) distribution" with the "exponential distribution" where are the (p.d.f.) and the (c.d.f.) of this distribution:

$$f_{SH}(z,\beta)=\frac{\beta^2}{\beta^2+1}(\beta+z)e^{-\beta z};\ z>0,\beta>0 \tag{1}$$

$$F_{SH}(z)=1-\frac{(\beta^2+1)+\beta z}{\beta^2+1}e^{-\beta z} \tag{2}$$

and survival (reliability) function defined as:

$$S_{SH}(z)=\frac{(\beta^2+1)+\beta z}{\beta^2+1}e^{-\beta z} \tag{3}$$

Gauss (5), introduced a three-parameter model in 2014, which is a mixture of exponential and Weibull distribution. In addition, the three-parameter lifetime model was first introduced in 2015 by Faton and Ibrahim (6). This is a mixture of Weibull and Rayleigh distribution to obtain a new distribution. The structure was based on PDF integration and the bounds of the integration ranged from zero to the survival function. Suleman (7) presented a new distribution in 2016, which is a mixture of Weibull and Rayleigh distribution. Mohammed, Maysaa and Hussein (8) presented a new mixture distribution of Weibull, Rayleigh and exponential distribution in 2018. In addition, Mundher et al. (9) presented an extension of the Burr type X distribution. Also, Maysaa and Iden (10) estimated the parameters of the new mixture distribution using classical methods. In addition, Ali and Iden (11) estimated the density function using a hybrid Breslow and a semi-symmetric wavelet. Also, Hameed et al. (12) presented the estimation of ($Y<X$) in the case of the inverse Kumaraswamy distribution. In addition, Maysaa and Ali (13) presented the inverse exponential Rayleigh distribution (IERD). There are many applications of continuous distributions, such as Wang (14), who presents a new model with a bathtub-shaped failure rate using the Burr XII distribution. Moreover, Rama (15) introduced a new distribution that depends on Gama (3,θ). Moreover, Kilany (16) introduced a weighted Lomax distribution in 2016. In addition, Oguntunde et al. (17) introduced the inverted weighted exponential distribution. Okagbue, et al. (18) also compared some of the estimation methods. Jebur, et al. 2021 (19) also presented the efficient shrinkage estimation method for the generalized inverse Rayleigh distribution. Maysaa and Ali (20) estimated the parameters of the new inverse exponential Rayleigh distribution. Also, Jabar et al. (21), introduced the truncated inverse generalized Rayleigh distribution. Moreover, Mahmood and Iden (22) introduced the comparison between the modified weighted Pareto distribution and other distributions. In addition, Pedro (23) presented a new distribution, the Fréchet distribution. Finally, Lamyaa and Iden (24) introduced a new distribution in 2023, which is a mixture of exponential and Rayleigh distribution. In this paper, we introduce a new mixture of continuous distributions, which we call the 'Shanker-Weibull distribution', by mixing both the 'Shanker distribution' and the 'Weibull distribution' based on the survival function for both distributions. Section 2 describes how we mixed the 'Shanker distribution' with the 'Weibull distribution' to obtain the 'Shanker-Weibull distribution' (SHWD) and introduces the basic statistical functions such as the probability density function, the commutative density function, the survival function and the hazard rate function. Finally, Section 3 presents the statistical and mathematical properties of the new distribution, such as mode, median, moments around the origin, mean, variance, skewness coefficient, kurtosis coefficient, characteristic function, moment generation function, factorial moment generation function and mean time to failure.

2. Materials and Methods

2.1. Structure of New Shanker Weibull distribution

Equations (1and 2) represent the pdf and cdf of the Shanker distribution. In addition, the pdf, cdf, and the survival function of standard Weibull distribution are(25)

$$f_W(v) = \lambda v^{\lambda-1} e^{-v^{\lambda}}, \lambda > 0, v > 0 \quad (4)$$

$$F_W(v) = 1 - e^{-v^{\lambda}}$$

$$S_w(v) = e^{-v^{\lambda}} \quad (5)$$

Let Z, and V are independent random variables and $X = min\,(Z, V)$

$S(x) = p_r(X > x) = p_r(min(Z, V) > x)$,

because Z and V are independent random variables then :

$$S_{(SHWD)}(x) = p_r(Z > x).p_r(V > x) = \big(1 - p_r(Z \le x)\big)\cdot\big(1 - p_r(V \le x)\big)$$

$$= \big(1 - F_{sh}(z)\big)\cdot\big(1 - F_w(v)\big) = S_{sh}(x) \cdot S_w(x)$$

From equations (3) and (5) we get:

$$S_{(SHWD)}(x) = \frac{(\beta^2+1)+\beta x}{\beta^2+1} e^{-(\beta x + x^{\lambda})} \quad (6)$$

Where $x > 0$, and $\beta > 0$ is the scale parameter, $\lambda > 0$ is the shape parameter.

The (cdf) of the new $(SHWD)$ distribution is:

$$F(x) = 1 - \left(\frac{(\beta^2+1)+\beta x}{\beta^2+1} e^{-(\beta x + x^{\lambda})}\right) \quad (7)$$

Therefore, the pdf of the new $(SHWD)$ is:

$$f(x) = \left(\frac{(\beta^2+1)+\beta x}{\beta^2+1}\left(\beta + \lambda x^{\lambda-1}\right) - \frac{\beta}{(\beta^2+1)}\right) e^{-(\beta x + x^{\lambda})} \quad (8)$$

f(x) is a probability density function, since, $\beta > 0, \lambda > 0$, and $x > 0$, which implies that $f(x) > 0$.

Now, to prove that $\int_0^{\infty} f(x)\,dx = 1$,

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \left(\frac{\beta x + \beta^2 + 1}{\beta^2+1}\left(\beta + \lambda x^{\lambda-1}\right) - \frac{\beta}{\beta^2+1}\right) e^{-(\beta x + x^{\lambda})}\,dx$$

$$\int_0^{\infty} f(x)dx = \lim_{y\to\infty}\left(\int_0^{y} \left(\frac{\beta x + \beta^2 + 1}{\beta^2+1}\left(\beta + \lambda x^{\lambda-1}\right) - \frac{\beta}{\beta^2+1}\right) e^{-(\beta x + x^{\lambda})}\,dx\right)$$

$$= \lim_{y\to\infty}\frac{1}{\beta^2+1}\left(\int_0^{y} (\beta x + \beta^2 + 1)\left(\beta + \lambda x^{\lambda-1}\right) e^{-(\beta x + x^{\lambda})}\,dx - \int_0^{y} \beta\, e^{-(\beta x + x^{\lambda})}\,dx\right)$$

$u = (\beta x + \beta^2 + 1) \Rightarrow du = \beta dx$

$dv = \left(\beta + \lambda x^{\lambda-1}\right) e^{-(\beta x + x^{\lambda})} dx \Rightarrow v = -\,e^{-(\beta x + x^{\lambda})}$

$$\int_0^{\infty} f(x)\,dx = \lim_{y\to\infty}\frac{1}{\beta^2+1}\left(\left(-(\beta x + \beta^2 + 1)e^{-(\beta x + x^{\lambda})}\right)\Big|_0^{y} + \int_0^{y} \beta\, e^{-(\beta x + x^{\lambda})}\,dx - \int_0^{y} \beta\, e^{-(\beta x + x^{\lambda})} dx\right) = -\lim_{y\to\infty}\left(\frac{(\beta y + \beta^2 + 1)}{(\beta^2+1)e^{(\beta y + y^{\lambda})}} - \frac{\beta^2+1}{\beta^2+1}\right)$$

By using L'Hopital's - Role for the above limit,

$$\int_0^\infty f(x)\,dx = -\underset{y\to\infty}{Lim}\left(\frac{\beta}{(\beta^2+1)(\beta+\lambda y^{\lambda-1})e^{(\beta y+y^\lambda)}}-1\right) = -\frac{\beta}{\infty}+1 = 1$$

Based on what was stated above, the hazard function can be defined as follows:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\left((\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)-\beta\right)\frac{1}{(\beta^2+1)}e^{-\left(\beta x+x^\lambda\right)}}{(\beta x+\beta^2+1)\frac{1}{(\beta^2+1)}e^{-\left(\beta x+x^\lambda\right)}}$$

$$h(x) = \left(\beta+\lambda x^{\lambda-1}\right)-\frac{\beta}{(\beta x+\beta^2+1)} \tag{9}$$

2.1.1. The shapes of the new Shanker Weibull distribution

Knowing the shape of the (SHWD) helps us understand the behavior and approach of distribution functions in dealing with data, to understand this mathematically, especially through the limit values of the probability density and hazard functions when($x \to 0$ &$x \to \infty$).

$$\underset{x\to 0}{lim} f(x) = \underset{x\to 0}{lim}\left((\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)-\beta\right)\frac{1}{(\beta^2+1)}e^{-\left(\beta x+x^\lambda\right)} = \frac{\beta^3}{(\beta^2+1)}$$

$$\underset{x\to\infty}{lim} f(x) = \underset{x\to\infty}{lim}\left((\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)-\beta\right)\frac{1}{(\beta^2+1)}e^{-\left(\beta x+x^\lambda\right)}$$

$$= \underset{x\to\infty}{lim}\frac{(\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)}{(\beta^2+1)e^{\left(\beta x+x^\lambda\right)}} - \underset{x\to\infty}{lim}\frac{\beta}{(\beta^2+1)e^{\left(\beta x+x^\lambda\right)}}$$

$$= \underset{x\to\infty}{lim}\frac{(\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)}{(\beta^2+1)e^{\left(\beta x+x^\lambda\right)}} - \frac{\beta}{\infty} = \underset{x\to\infty}{lim}\frac{(\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)}{(\beta^2+1)e^{\left(\beta x+x^\lambda\right)}}$$

Applied L'Hospital's Rule:

$$\underset{x\to\infty}{lim} f(x) = \underset{x\to\infty}{lim}\frac{(\beta x+\beta^2+1)\left(\beta+\lambda x^{\lambda-1}\right)\left(\lambda(\lambda-1)x^{\lambda-2}\right)-\beta\left(\beta+\lambda x^{\lambda-1}\right)}{(\beta^2+1)(\beta+\lambda x^{\lambda-1})e^{\left(\beta x+x^\lambda\right)}}$$

As we continue to derive for x the result of the numerator will be equal constant as considered an integer value for x and as contained to derive the denominator it always contains the exponential part which is equal to ∞ as(x →∞). The final result of the limit is a constant divide by ∞ which equal to zero

$$\underset{x\to\infty}{\mathrm{Lim}}\, \mathrm{f(x)} = 0.$$

$$\underset{x\to 0}{\mathrm{Lim}}\, h(x) = \underset{x\to 0}{\mathrm{Lim}}\left(\left(\beta+\lambda x^{\lambda-1}\right)-\frac{\beta}{\beta x+\beta^2+1}\right) = \frac{\beta^3}{\beta^2+1}$$

$$\underset{x\to\infty}{\mathrm{Lim}}\, h(x) = \underset{x\to\infty}{\mathrm{Lim}}\left(\left(\beta+\lambda x^{\lambda-1}\right)-\frac{\beta}{\beta x+\beta^2+1}\right) = \infty$$

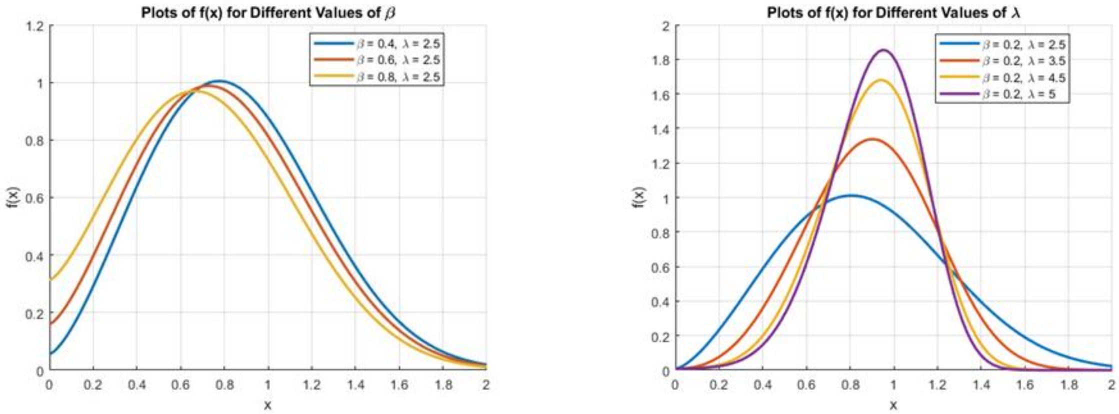

Figure 1. The shape of pdf for new Shanker Weibull distribution with different values of β and λ.

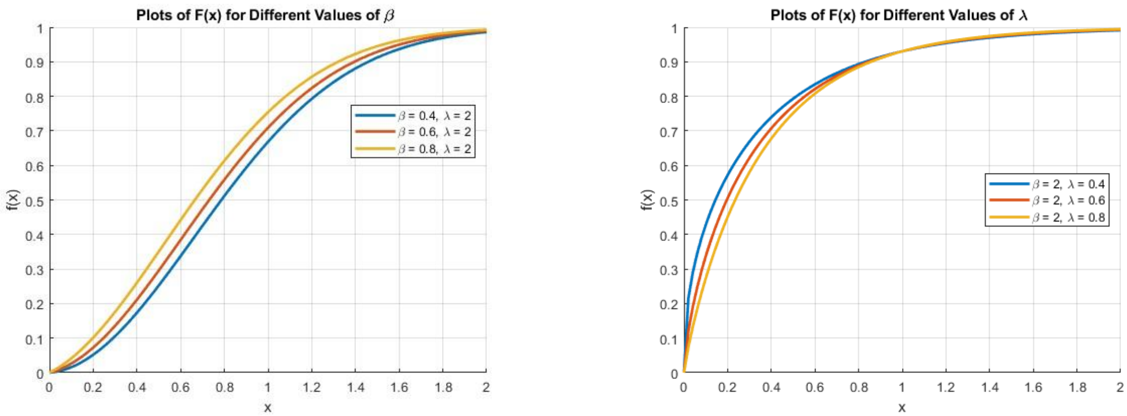

Figure 2. The shape of cdf for new Shanker Weibull distribution with different values of β andλ.

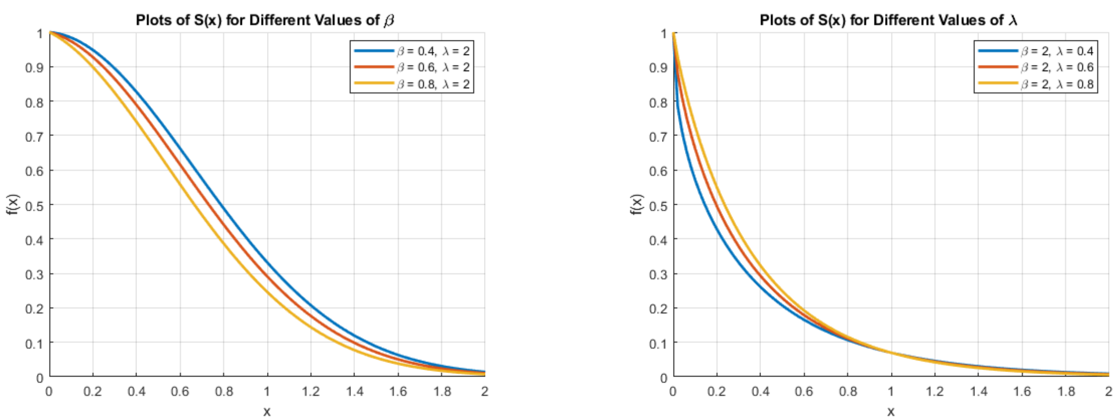

Figure 3. The shape of S(x) for new Shanker Weibull distribution with different values of λ andβ.

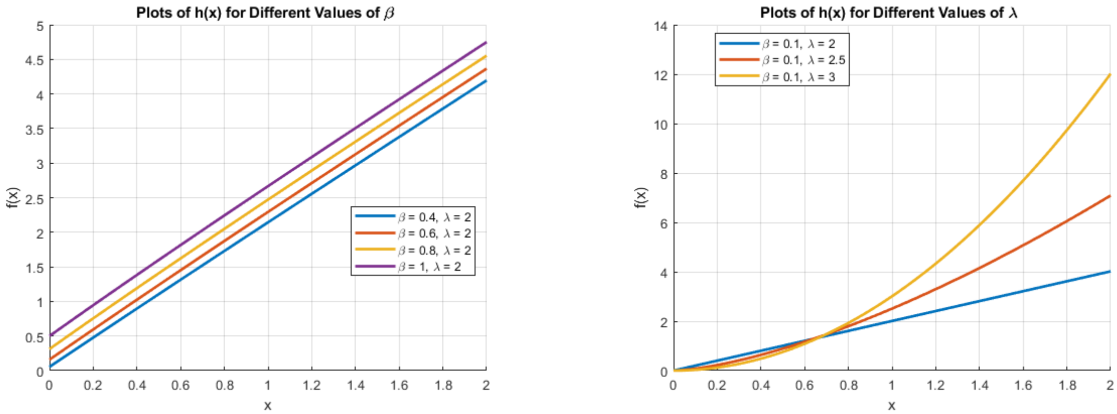

Figure 4. The shape of h(x) for new Shanker Weibull distribution with different values of λ andβ.

In the special case, if $\lambda = 1$, then the distribution becomes the Shanker exponential distribution, also if β=0, then the distribution becomes Weibull distribution.

2.2. Mathematical and statistical properties of SHWD

2.2.1. The Mode

The mod is defined as the point at which the probability density function achieves its maximum value, the mode of (SHWD) is obtained as follows:

$$f(x) = \left(\beta^2 x + \beta\lambda x^{\lambda} + \beta^3 + \beta^2\lambda x^{\lambda-1} + \lambda x^{\lambda-1}\right)\frac{1}{(\beta^2+1)}e^{-(\beta x + x^{\lambda})}$$

$$\frac{\partial f(x)}{\partial x} = \Big(\left(\beta^2 + \beta\lambda^2 x^{\lambda-1} + \beta^2\lambda(\lambda-1)x^{\lambda-2} + \lambda(\lambda-1)x^{\lambda-2}\right) - \left(\beta^2 x + \beta\lambda x^{\lambda} + \beta^3 + \beta^2\lambda x^{\lambda-1} + \lambda x^{\lambda-1}\right)\left(\beta + \lambda x^{\lambda-1}\right)\Big)\frac{1}{(\beta^2+1)}e^{-(\beta x + x^{\lambda})} = 0$$

Since that $\frac{e^{-(\beta x + x^{\lambda})}}{(\beta^2+1)} \neq 0$ for each selected value of β, λ, and x, This leads to the following:

$$\left(\beta^2 + \beta\lambda^2 x^{\lambda-1} + \beta^2\lambda(\lambda-1)x^{\lambda-2} + \lambda(\lambda-1)x^{\lambda-2}\right) - \left(\beta^2 x + \beta\lambda x^{\lambda} + \beta^3 + \beta^2\lambda x^{\lambda-1} + \lambda x^{\lambda-1}\right)\left(\beta + \lambda x^{\lambda-1}\right) = 0$$

This is a polynomial of degree $(\lambda - 2)$ and if $\lambda = 1$that will imply the following:

$$((\beta^2 + \beta) - (\beta^2 x + \beta x + \beta^3 + \beta^2 + 1)(\beta + 1)) = 0$$

$$(\beta^2 + \beta) -.(\beta^3 x + \beta^2 x + \beta^4 + \beta^3 + \beta + \beta^2 x + \beta x + \beta^3 + \beta^2 + 1) = 0$$

$$-(\beta^3 + 2\beta^2 + \beta)x = \beta^4 + 2\beta^3 + 1$$

$$x = \frac{-(\beta^4 + 2\beta^3 + 1)}{(\beta^3 + 2\beta^2 + \beta)}$$

2.2.2. Quantile Function

Let X be a continuous random variable distributed with "SHWD" then the Quantile function is defined as given:

Let $F(x) = u$, $u \sim U(0,1)$, then $\quad F(x) = U$, $x = F^{-1}(u)$

$$\left(1 - \frac{\beta x + \beta^2 + 1}{\beta^2 + 1} \cdot e^{-(\beta x + x^{\lambda})}\right) = u$$

$$-\left(1 - \frac{\beta x + \beta^2 + 1}{\beta^2 + 1} \cdot e^{-(\beta x + x^{\lambda})}\right) = u - 1$$

$$1 - u = \frac{\beta x + \beta^2 + 1}{\beta^2 + 1} \cdot e^{-(\beta x + x^{\lambda})} \text{ , } \lambda=1$$

For Lambert form (26-27)

$(1-u)(\beta^2+1) = (\beta x + \beta^2 + 1) \cdot e^{-(\beta x + x)}$

Where, $x = \frac{\beta b - (-\beta-1)(\beta^2+1)}{\beta(-\beta-1)}$

Hence the value of $b = (-\beta-1)x + \frac{(\beta^2+1)(-\beta-1)}{\beta}$

$$(1-u)(\beta^2+1) = \left(\beta\left(\frac{\beta b - (-\beta-1)(\beta^2+1)}{\beta(-\beta-1)}\right) + \beta^2 + 1\right) e^{\frac{(-\beta-1)((\beta b - (-\beta-1)(\beta^2+1))}{\beta(-\beta-1)}}$$

$$(1-u)(\beta^2+1) = \left(\frac{\beta b}{-\beta-1} - \frac{(-\beta-1)(\beta^2+1)}{(-\beta-1)} + \beta^2 + 1\right) e^{\frac{\beta b - (-\beta-1)(\beta^2+1)}{\beta}}$$

$$(1-u)(\beta^2+1)(-\beta-1) = \beta b \cdot e^{\frac{\beta b}{\beta}} \cdot e^{\frac{-(-\beta-1)(\beta^2+1)}{\beta}}$$

$$(1-u)(\beta^2+1)(-\beta-1) = \beta b \cdot e^{b} \cdot e^{\frac{-(-\beta-1)(\beta^2+1)}{\beta}}$$

$$(1-u)(\beta^2+1)(-\beta-1) e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}} = \beta b \cdot e^{b}$$

$$\frac{(1-u)(\beta^2+1)(-\beta-1)}{\beta} e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}} = b \cdot e^{b}$$

$$W\left(be^b = \frac{(1-u)(\beta^2+1)(-\beta-1)}{\beta} . e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right)$$

$$b = W\left(\frac{(1-u)(\beta^2+1)(-\beta-1)}{\beta} . e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right)$$

Since

$$b = (-\beta-1)x + \frac{(\beta^2+1)(-\beta-1)}{\beta}$$

$$(-\beta-1)x + \frac{(\beta^2+1)(-\beta-1)}{\beta} = W\left(\frac{(1-u)(\beta^2+1)(-\beta-1)}{\beta} . e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right)$$

$$(-\beta-1)x = W\left(\frac{(1-u)(\beta^2+1)(-\beta-1)}{\beta} . e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right) - \frac{(\beta^2+1)(-\beta-1)}{\beta}$$

$$x = \frac{W\left(\frac{(1-u)(\beta^2+1)(-\beta-1)}{\beta} \cdot e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right) - \frac{(-\beta-1)(\beta^2+1)}{\beta}}{(-\beta-1)}. \tag{10}$$

A special case from quantity function (10) is the median whene $u = \frac{1}{2}$.

$$x = \frac{W\left(\frac{\left(1-\frac{1}{2}\right)(\beta^2+1)(-\beta-1)}{\beta} \cdot e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right) - \frac{(-\beta-1)(\beta^2+1)}{\beta}}{(-\beta-1)}$$

$$x = \frac{W\left(\frac{\left(\frac{1}{2}\right)(\beta^2+1)(-\beta-1)}{\beta} \cdot e^{\frac{(-\beta-1)(\beta^2+1)}{\beta}}\right) - \frac{(-\beta-1)(\beta^2+1)}{\beta}}{(-\beta-1)}$$

2.2.3. Moments about the origin

One of the most important distribution properties is moments, as they play a role in the determination of many other distribution properties, such as (mean, variance, skewness, and kurtosis).

The moments of (SHWD) can be determined as follows:

$$M_r'(x)=\int_0^\infty x^r\,f(x)dx$$

$$=\int_0^\infty x^r\left(\beta e^{-(\beta x+x^\lambda)}+\lambda x^{\lambda-1}e^{-(\beta x+x^\lambda)}+\frac{\beta^2 x}{\beta^2+1}e^{-(\beta x+x^\lambda)}+\frac{\beta\lambda\,x^\lambda}{\beta^2+1}e^{-(\beta x+x^\lambda)}-\frac{\beta}{\beta^2+1}e^{-(\beta x+x^\lambda)}\right)dx$$

Let $e^{-x^\lambda}=\sum_{j=0}^\infty\frac{(-1)^j}{j!}x^{\lambda j}$ (28)

$$\int_0^\infty x^r\;f(x)\,dx=\sum_{j=0}^\infty\frac{(-1)^j}{j!}\left(\int_0^\infty\beta x^{r+\lambda j}\;e^{-\beta x}dx+\int_0^\infty\lambda x^{r+\lambda(j+1)-1}\,e^{-\beta x}dx+\int_0^\infty\frac{\beta^2}{\beta^2+1}x^{r+\lambda j+1}\;e^{-\beta x}dx+\int_0^\infty\frac{\beta\lambda}{\beta^2+1}x^{r+\lambda(j+1)}\;e^{-\beta x}dx-\int_0^\infty\frac{\beta}{\beta^2+1}x^{r+\lambda j}\,e^{-\beta x}dx\right)$$

Let $\beta x=y$, then $x=\frac{y}{\beta}$ and $dx=\frac{dy}{\beta}$, and since $\Gamma_{(\beta)}=\int_0^\infty x^{\beta-1}\,e^{-x}dx$.

$$M_r'(x)=\sum_{j=0}^\infty\frac{(-1)^j}{j!}\left(\frac{1}{\beta^{r+\lambda j}}\int_0^\infty y^{r+\lambda j}\;e^{-y}dy+\frac{\lambda}{\beta^{r+\lambda(j+1)}}\int_0^\infty y^{r+\lambda(j+1)-1}\,e^{-y}dy+\frac{\beta}{(\beta^2+1)\beta^{r+\lambda j+1}}\int_0^\infty y^{r+\lambda j+1}\;e^{-y}dy+\frac{\lambda}{(\beta^2+1)\beta^{r+\lambda(j+1)}}\int_0^\infty y^{r+\lambda(j+1)}\;e^{-y}dy-\frac{1}{(\beta^2+1)\beta^{r+\lambda j}}\int_0^\infty y^{r+\lambda j}\,e^{-y}dy\right)$$

$$M_r'(x)=\sum_{j=0}^\infty\frac{(-1)^j}{j!}\left(\frac{\Gamma_{(r+\lambda j+1)}}{\beta^{r+\lambda j}}+\frac{\lambda\cdot\Gamma_{(r+\lambda(j+1))}}{\beta^{r+\lambda(j+1)}}+\frac{\Gamma_{(r+\lambda j+1)}-\Gamma_{(r+\lambda j+2)}}{(\beta^2+1)\beta^{r+\lambda j}}+\frac{\lambda\cdot\Gamma_{(r+\lambda(j+1)+1)}}{(\beta^2+1)\beta^{r+\lambda(j+1)}}\right)\qquad(11)$$

As a direct result, it can be found$E(x)=M_1'(x)$, $E(x^2)=M_2'(x)$, and $var(x)$ as follows:

$$E(x)=\sum_{j=0}^\infty\frac{(-1)^j}{j!}\left(\frac{\Gamma_{(\lambda j+2)}}{\beta^{\lambda j+1}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+1)}}{\beta^{\lambda(j+1)+1}}+\frac{\Gamma_{(\lambda j+2)}-\Gamma_{(\lambda j+3)}}{(\beta^2+1)\beta^{\lambda j+1}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+2)}}{(\beta^2+1)\beta^{\lambda(j+1)+1}}\right)$$

$$E(x^2)=\sum_{j=0}^\infty\frac{(-1)^j}{j!}\left(\frac{\Gamma_{(\lambda j+3)}}{\beta^{\lambda j+2}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+2)}}{\beta^{\lambda(j+1)+2}}+\frac{\Gamma_{(\lambda j+3)}-\Gamma_{(\lambda j+4)}}{(\beta^2+1)\beta^{\lambda j+2}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+3)}}{(\beta^2+1)\beta^{r+\lambda(j+1)+2}}\right)$$

$$var(x)=M_2'(x)-\left(M_1'(x)\right)^2$$

2.2.4. Coefficients of Skewness and Kurtosis

Depending on the moment, the coefficient skewness $(C.S)$ and kurtosis $(C.K)$ can be found through the following formulas(29):

$$M_3'(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{\Gamma_{(\lambda j+4)}}{\beta^{\lambda j+3}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+3)}}{\beta^{\lambda(j+1)+3}}+\frac{\Gamma_{(\lambda j+4)}-\Gamma_{(\lambda j+5)}}{(\beta^2+1)\beta^{\lambda j+3}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+4)}}{(\beta^2+1)\beta^{r+\lambda(j+1)+3}}\right)$$

$$C.S=\frac{M_3'}{(M_2')^{\frac{3}{2}}}$$

$$M_4'(x)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\Gamma_{(\lambda j+5)}}{\beta^{\lambda j+4}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+4)}}{\beta^{\lambda(j+1)+4}}+\frac{\Gamma_{(\lambda j+5)}-\Gamma_{(\lambda j+6)}}{(\beta^2+1)\beta^{\lambda j+4}}+\frac{\lambda\cdot\Gamma_{(\lambda(j+1)+5)}}{(\beta^2+1)\beta^{\lambda(j+1)+4}}\right)$$

$$C.K=\frac{M_4'}{(M_2')^2}-3$$

Table 1. The first - fourth moments, variance, skewness, and kurtosis for the distribution

β	λ	μ_1'	μ_2'	μ_3'	μ_4'	K	S	var
2	0.5	0.3222	0.2719	0.3665	0.6762	6.1462	2.5849	0.1681
	0.3	0.2879	0.2668	0.3952	0.7962	8.1838	2.8673	0.1839
0.5	0.9	0.8598	1.5289	4.0855	14.5190	3.2110	2.1610	0.7897
	1.2	0.8255	1.1479	2.1874	5.2296	0.9687	1.7785	0.4664
2.5	2	0.3579	0.2177	0.1765	0.1739	0.6687	1.7376	0.0896
	3	0.3754	0.2303	0.1792	0.1620	0.0543	1.6218	0.0893

Since the values of skewness are positive, the distribution is skewed to the right according to the given data values. The flatness depends on the values of the given characteristics. Sometimes it is flattened for values above 3, and sometimes it is peaked for values below 3.

2.2.5. Characteristic Function

$$\phi_x(it)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}\ f(x)\,dx$$

$$=\int_0^{\infty}e^{itx}\left(1+\frac{\beta x}{\beta^2+1}\right)\left(\beta+\lambda x^{\lambda-1}\right)-\frac{\beta}{\beta^2+1}\Big)\ e^{-(\beta x+x^{\lambda})}\,dx$$

$$Let\ e^{-x^{\beta}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{\lambda n}$$

$$\phi_x(it)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_0^{\infty}\beta x^{\beta n}\ e^{-(\beta-it)x}dx+\int_0^{\infty}\lambda x^{\lambda n+\lambda-1}\ e^{-(\beta-it)x}dx\right.$$
$$+\int_0^{\infty}\left(\frac{\beta^2}{\beta^2+1}\right)x^{\lambda n+1}\ e^{-(\beta-it)x}dx+\int_0^{\infty}\left(\frac{\beta\lambda}{\beta^2+1}\right)x^{\lambda n+\lambda}\ e^{-(\beta-it)x}dx$$
$$\left.-\int_0^{\infty}\left(\frac{\beta}{\beta^2+1}\right)x^{\lambda n}\ e^{-(\beta-it)x}dx\right)$$

$$Let\ (\beta - it)x = y, and\ x = \frac{y}{\beta - it}\ then\ dx = \frac{dy}{\beta - it}$$

$$\phi_x(it) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\beta}{(\beta - it)^{\lambda n+1}} \int_0^{\infty} y^{\lambda n}\ e^{-y} dy + \frac{\lambda}{(\beta - it)^{\lambda n+\lambda}} \int_0^{\infty} y^{\lambda n+\lambda-1}\ e^{-y} dy \right.$$

$$+ \frac{\beta^2}{(\beta^2+1)(\beta - it)^{\lambda n+2}} \int_0^{\infty} y^{\lambda n+1}\ e^{-y} dy$$

$$+ \frac{\beta\lambda}{(\beta^2+1)(\beta - it)^{\lambda n+\lambda+1}} \int_0^{\infty} y^{\lambda n+\lambda}\ e^{-y} dy$$

$$\left. - \frac{\beta^2}{(\beta^2+1)(\beta - it)^{\lambda n+1}} \int_0^{\infty} y^{\lambda n}\ e^{-y} dy \right)$$

$$\phi_x(it) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\beta \cdot \Gamma_{(\lambda n+1)}}{(\beta - it)^{\lambda n+1}} + \frac{\lambda \cdot \Gamma_{(\lambda n+\lambda)}}{(\beta - it)^{\lambda n+\lambda}} + \frac{\beta^2 \cdot \Gamma_{(\lambda n+2)}}{(\beta^2+1)(\beta - it)^{\lambda n+2}} \right.$$

$$\left. + \frac{\beta\lambda \cdot \Gamma_{(\lambda n+\lambda+1)}}{(\beta^2+1)(\beta - it)^{\lambda n+\lambda+1}} - \frac{\beta^2 \cdot \Gamma_{(\lambda n+1)}}{(\beta^2+1)(\beta - it)^{\lambda n+1}} \right)$$

2.2.6. Moment Generating Function

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\ f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \left(1 + \frac{\beta x}{\beta^2+1} \right) \left(\beta + \lambda x^{\lambda-1} \right) - \frac{\lambda}{\lambda^2+1} \Big)\ e^{-(\beta x + x^{\lambda})}\, dx$$

$$Let\ \ e^{-x^{\beta}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{\lambda k}$$

$$M_x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\int_0^{\infty} \beta x^{\lambda k}\ e^{-(\beta - t)x} dx + \int_0^{\infty} \lambda x^{\lambda k+\lambda-1}\ e^{-(\beta - t)x} dx \right.$$

$$+ \int_0^{\infty} \left(\frac{\beta^2}{\beta^2+1} \right) x^{\lambda k+1}\ e^{-(\beta - t)x} dx + \int_0^{\infty} \left(\frac{\beta\lambda}{\beta^2+1} \right) x^{\lambda k+\lambda}\ e^{-(\beta - t)x} dx$$

$$\left. - \int_0^{\infty} \left(\frac{\beta}{\beta^2+1} \right) x^{\lambda k}\ e^{-(\beta - t)x} dx \right)$$

$$Let\ (\beta - t)x = y, and\ x = \frac{y}{\beta - t}\ then\ dx = \frac{dy}{\beta - t}$$

$$M_x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\beta}{(\beta - t)^{\lambda k+1}} \int_0^{\infty} y^{\lambda k} \; e^{-y} dy + \frac{\beta}{(\beta - t)^{\lambda k+\lambda}} \int_0^{\infty} y^{\lambda k+\lambda-1} \; e^{-y} dy \right.$$

$$+ \frac{\beta^2}{(\beta^2+1)(\beta - t)^{\lambda k+2}} \int_0^{\infty} y^{\lambda k+1} \; e^{-y} dy$$

$$+ \frac{\beta\lambda}{(\beta^2+1)(\beta - t)^{\lambda k+\lambda+1}} \int_0^{\infty} y^{\lambda k+\lambda} \; e^{-y} dy$$

$$\left. - \frac{\beta^2}{(\beta^2+1)(\beta - t)^{\lambda k+1}} \int_0^{\infty} y^{\lambda k} \; e^{-y} dy \right)$$

$$M_x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\beta \cdot \Gamma_{(\lambda k+1)}}{(\beta - t)^{\lambda k+1}} + \frac{\lambda \cdot \Gamma_{(\lambda k+\beta)}}{(\beta - t)^{\lambda k+\beta}} + \frac{\beta^2 \cdot \Gamma_{(\lambda k+2)}}{(\beta^2+1)(\beta - t)^{\lambda k+2}} \right.$$

$$\left. + \frac{\beta\lambda \cdot \Gamma_{(\lambda k+\lambda+1)}}{(\beta^2+1)(\beta - t)^{\lambda k+\lambda+1}} - \frac{\beta^2 \cdot \Gamma_{(\lambda k+1)}}{(\beta^2+1)(\beta - t)^{\lambda k+1}} \right)$$

2.2.7. Factorial Moments Generating Function

$$\mathcal{M}_x(t) = E(t^x) = \int_0^{\infty} t^x \; f(x) \, dx$$

$$= \int_0^{\infty} e^{x lnt} \left(1 + \frac{\beta x}{\beta^2+1}\right) \left(\beta + \lambda x^{\lambda-1}\right) - \frac{\beta}{\beta^2+1} \right) \; e^{-(\beta x + x^{\lambda})} \, dx$$

$$= \int_0^{\infty} \left(1 + \frac{\beta x}{\beta^2+1}\right) \left(\beta + \lambda x^{\lambda-1}\right) - \frac{\beta}{\beta^2+1} \right) \; e^{-\left((\beta - ln\, t)x + x^{\lambda}\right)} \, dx$$

$$Let \;\; e^{-x^{\beta}} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} x^{\lambda s}$$

$$\mathcal{M}_x(t) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\int_0^{\infty} \beta x^{\lambda s} \; e^{-(\beta - lnt)x} dx + \int_0^{\infty} \lambda x^{\lambda s+\lambda-1} \; e^{-(\beta - lnt)x} dx \right.$$

$$+ \int_0^{\infty} \left(\frac{\beta^2}{\beta^2+1}\right) x^{\lambda s+1} \; e^{-(\beta - lnt)x} dx + \int_0^{\infty} \left(\frac{\beta\lambda}{\beta^2+1}\right) x^{\lambda s+\lambda} \; e^{-(\beta - lnt)x} dx$$

$$\left. - \int_0^{\infty} \left(\frac{\beta}{\beta^2+1}\right) x^{\lambda s} \; e^{-(\beta - lnt)x} dx \right)$$

$$Let \; (\beta - lnt)x = y, and \; x = \frac{y}{\beta - lnt} \, then \; dx = \frac{dy}{\beta - lnt}$$

$$\mathcal{M}_x(t)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left(\frac{\beta}{(\beta-lnt)^{\lambda s+1}}\int_0^{\infty} y^{\lambda s}\ e^{-y}dy+\frac{\lambda}{(\beta-lnt)^{\lambda s+\lambda}}\int_0^{\infty} y^{\lambda s+\lambda-1}\ e^{-y}dy\right.$$
$$+\frac{\beta^2}{(\beta^2+1)(\beta-lnt)^{\lambda s+2}}\int_0^{\infty} y^{\lambda s+1}\ e^{-y}dy$$
$$+\frac{\beta\lambda}{(\beta^2+1)(\beta-lnt)^{\lambda s+\lambda+1}}\int_0^{\infty} y^{\lambda s+\lambda}\ e^{-y}dy$$
$$\left.-\frac{\beta}{(\beta^2+1)(\beta-lnt)^{\lambda s+1}}\int_0^{\infty} y^{\lambda s}\ e^{-y}dy\right)$$

$$\mathcal{M}_x(t)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left(\frac{\beta\cdot\Gamma_{(\lambda s+1)}}{(\beta-lnt)^{\lambda s+1}}+\frac{\lambda\cdot\Gamma_{(\lambda s+\lambda)}}{(\beta-lnt)^{\lambda s+\lambda}}+\frac{\beta^2\cdot\Gamma_{(\lambda s+2)}}{(\beta^2+1)(\beta-lnt)^{\lambda s+2}}\right.$$
$$\left.+\frac{\beta\lambda\cdot\Gamma_{(\lambda s+\lambda+1)}}{(\beta^2+1)(\beta-lnt)^{\lambda s+\lambda+1}}-\frac{\beta\cdot\Gamma_{(\lambda s+1)}}{(\beta^2+1)(\beta-lnt)^{\lambda s+1}}\right)$$

2.2.8. Mean time to failure

Let T denote the lifetime of a component, the$S_{(SHWD)}(x)=\Pr(T>x)$ [30]

$S_{(SHWD)}(x)=Pr(T>x)$

Since $S_{(SHWD)}(x)\to 0$ at $t\to\infty$, then is given by:

$$MTTE=\int_0^{\infty} S_{(SHWD)}(x)\,dx=\int_0^{\infty}(1+\frac{\beta x}{\beta^2+1})e^{-(\beta x+x^{\lambda})}\,dx$$

Recall that, $e^{-x^{\beta}}=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\ x^{\lambda j}$

$$MTTE=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\int_0^{\infty} x^{\lambda j}\cdot e^{-\beta x}dx+\frac{\beta}{\beta^2+1}\int_0^{\infty} x^{\lambda j+1}\cdot e^{-\beta x}dx\right)$$

Let $\beta x=y$ then $x=\frac{y}{\beta}$ and $dx=\frac{dy}{\beta}$

$$MTTE=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\lambda j}\cdot e^{-y}\frac{dy}{\beta}+\frac{\beta}{\beta^2+1}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\lambda j+1}\cdot e^{-y}\frac{dy}{\beta}\right)$$
$$=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{1}{\beta^{\lambda j+1}}\int_0^{\infty} y^{\lambda j}\cdot e^{-y}dy+\frac{1}{(\beta^2+1)\beta^{\lambda j+1}}\int_0^{\infty} y^{\lambda j+1}\cdot e^{-y}dy\right)$$
$$MTTE=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{1}{\beta^{\lambda j+1}}\cdot\Gamma_{(\lambda j+1)}+\frac{1}{(\beta^2+1)\beta^{\lambda j+1}}\cdot\Gamma_{(\lambda j+2)}\right)$$

3. Conclusion

A new statistical lifetime distribution was introduced, which is a mixture of Shanker and Weibull distribution depending on their survival functions and is called "Shanker-Weibull distribution". In addition, the shape of pdf, cdf and the hazard function were discussed. In addition, the basic statistical and mathematical properties of the new distribution such as mode, median, the rth moments around the origin and the moment generating function were described.

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Conflict of Interest

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