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New Lifetime Alpha Power Exponential Weibull Distribution: Structure and Properties

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Abstract

In statistics theory, adding a new parameter is considered one of the important things that help in producing statistical distributions more flexible and appropriate in data analysis. Alpha-power transformations are considered a modern technique that involves adding a shape parameter to generate new statistical distributions. In this paper, a new life continuous distribution of three parameters is presented by fitting the alpha power transformations family distribution with two parameters lifetime exponential Weibull distribution. The new model named alpha-power exponential Weibull distribution (APEWD) with three parameters$(\alpha, \Upsilon,$ and $\beta)$, where α and β are classified as scale parameters and Υ parameter is classified as a shape parameter. The cumulative, probability density, survival, hazard functions, and statistical properties of the proposed new model distribution were discussed and studied such as quantile function, moment about origin, moment generating function, Skewness, Kurtosis, factorial moments generating function, and characteristic function. To expand the probability density function for the new distribution, we took advantage of expanding the exponential function for ease of dealing with finding statistical properties.

Keywords: alpha power family, exponential Weibull distribution, survival function**,** moments about the origin, moment generating function

1. Introduction

Statistics distributions play a crucial role in analyzing data and making more accurate decisions. However, the world is constantly evolving, conditions change and new types of data and statistical challenges emerge. For this reason, discovering new statistical distributions represents a vital area of research. In this article, we will take a look at the importance of this process and how it can contribute to the development of the field of statistics. The idea of producing new distributions has gone through many stages over the past decades. The most important of these stages are combining distributions and creating families of distributions. The basic idea of this research is to apply the alpha power family to a statistical distribution resulting from mixing two distributions. (1) invented a new way to create statistical distributions through the survival function and applied it to find the three parameters of exponential–Weibull lifetime distribution. (2) used the survival mixed method to present Serial Weibull Rayleigh distribution. (3) and (4) involved the same technique with two stages to get three parameters of the new mixture distribution. (5) and (6) produced the inverse exponential Rayleigh distribution and estimated the distribution parameters with application. (7) created an exponential Weibull distribution with shape parameter Υ and scale parameter β, the cumulative distribution and probability density functions were as follows:

$$F(x) = 1 - e^{-(\Upsilon x + x^{\beta})} \tag{1}$$

$$f(x) = (\Upsilon + \beta x^{\beta-1}) e^{-(\Upsilon x + x^{\beta})} \tag{2}$$

Adding a parameter to distributions is considered one of the important things to produce new distributions that are more suitable for data analysis. Weighting, generalization, and exponentiated are among the common methods of adding a parameter to distributions that researchers have used to produce new distributions over the past decades (8-14) introduced a new method for generating a family of distributions by adding a scale parameter, which is called the alpha power transformation method (APT), and the cumulative distribution (CDF) and probability density (PDF) functions of (APT) family distribution are considered as the following formals:

$$G(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}, x > 0, \alpha > 0, and\ \alpha \neq 1 \\ F(x) \qquad\qquad\qquad \alpha = 1 \end{cases} \tag{3}$$

$$g(x) = \begin{cases} \dfrac{log(\alpha)}{\alpha - 1} \alpha^{F(x)} f(x) \quad x > 0, \alpha > 0, and\ \alpha \neq 1 \\ f(x) \qquad\qquad\qquad\qquad \alpha = 1 \end{cases} \tag{4}$$

(15) applied (APT) to introduce $\alpha -$ power inverted exponential distribution. (16) Employed the α- power technique to presenter Alpha-Power Pareto distribution. (17, (18), and (19) proposed a new family of generating lifetime distributions by extending the alpha power transformation method. (20) Presented a new alpha-power Teissier distribution. (21) Used the proposed method to produce αlpha power transformed Aradhana Distributions. (22) Explored a new probability distribution called $\alpha -$ power exponentiated inverse Rayleigh. (23) Proposed a newly generated family known as G-alpha power transformation distributions. (24) Came out with a new class called discrete alpha-power distribution. The technique of generalization was applied to output with a new collection of distribution alpha-power families (25), (26), (27), (28), (29), and (30). Statistically, power transformations can be considered one of the processes applied to create transformations of data so that they are monotonic by using the properties of power functions, which are common methods used through which the variance of the data is stabilized so that it is more similar to a normal distribution. The following movements in this paper include the mathematical construction of the basic functions of the new APEWD, each cdf, pdf, survival, and hazard functions, and the derivation of the statistical properties of the distribution is appended next.

2. Structure of New APEWD

For a random variable X > 0, and $\alpha, \beta > 0$ is the scale parameter, $\Upsilon, \beta > 0$ are shape parameters.

The (cdf) and (pdf) of the new (APEWD) are:

$$G(x) = \begin{cases} \dfrac{\alpha^{1-e^{-(\Upsilon x + x^{\beta})}} - 1}{\alpha - 1}, x > 0, \alpha, \Upsilon, \beta > 0, and\ \alpha \neq 1 \\ F(x) \qquad\qquad\qquad\qquad\qquad \alpha = 1 \end{cases} \tag{5}$$

$$g(x) = \begin{cases} \dfrac{\left(\Upsilon + \beta x^{\beta-1}\right) log(\alpha)}{\alpha - 1} \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} e^{-\left(\Upsilon x + x^{\beta}\right)} & x > 0, \alpha, \Upsilon, \beta > 0, and\ \alpha \neq 1 \\ f(x) & \alpha = 1 \end{cases} \quad (6)$$

$g(x)$ is actually a probability density function, since,$x > 0,\ \ \alpha, \Upsilon, \beta > 0, and\ \alpha \neq 1$, which implies that $g(x) > 0$.

Now, to prove that $\int_0^{\infty} f(x)\, dx = 1$,

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{\left(\Upsilon + \beta x^{\beta-1}\right) log(\alpha)}{\alpha - 1} \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} e^{-\left(\Upsilon x + x^{\beta}\right)}\, dx = \frac{\alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} - 1}{\alpha - 1} \Big|_0^{\infty}$$

$$= \left(\frac{\alpha - 1}{\alpha - 1} - \frac{1 - 1}{\alpha - 1}\right) = 1$$

Based on what was stated above, the survival and hazard functions can be defined as follows:

$$S(x) = 1 - G(x) = \begin{cases} 1 - \left(\dfrac{\alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} - 1}{\alpha - 1}\right), x > 0, \alpha, \Upsilon, \beta > 0, and\ \alpha \neq 1 \\ 1 - F(x) \qquad \alpha = 1 \end{cases} \quad (7)$$

$$h(x) = \frac{g(x)}{S(x)} = \begin{cases} \dfrac{\left(\Upsilon + \beta x^{\beta-1}\right) log(\alpha)\, \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} e^{-\left(\Upsilon x + x^{\beta}\right)}}{\alpha - \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}}}, x, \alpha > 0\ , \alpha \neq 1 \\ \dfrac{f(x)}{1 - F(x)} \qquad \alpha = 1 \end{cases} \quad (8)$$

2.1. The shapes of (APEWD)

Knowing the shape of the (APEWD) helps us understand the behavior and approach of distribution functions in dealing with data, in order to understand this mathematically, especially through the limit values of the probability density and hazard functions when $(x \to 0\ \& x \to \infty)$.

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \left(\frac{\left(\Upsilon + \beta x^{\beta-1}\right) log(\alpha)}{\alpha - 1} \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} e^{-\left(\Upsilon x + x^{\beta}\right)}\right) = \frac{\Upsilon\, log(\alpha)}{\alpha - 1}$$

$$\lim_{x \to \infty} g(x) = \frac{\alpha \log(\alpha)}{\alpha - 1} \lim_{x \to \infty} \left(\frac{\left(\Upsilon + \beta x^{\beta-1}\right)}{\alpha^{e^{-\left(\Upsilon x + x^{\beta}\right)}} e^{\left(\Upsilon x + x^{\beta}\right)}}\right)$$

Applied L'Hospital's Rule and as we continue to derive for x the result of the numerator will be equal constant as considered an integer value for β and as contained to derive the denominator is always contains the exponential part which equal to $\infty\ as\ (x \to \infty)$. The final result of the limit is a constant divide by ∞ which equal to zero $\lim_{x \to \infty} f(x) = 0$ (**Figures 1-4**).

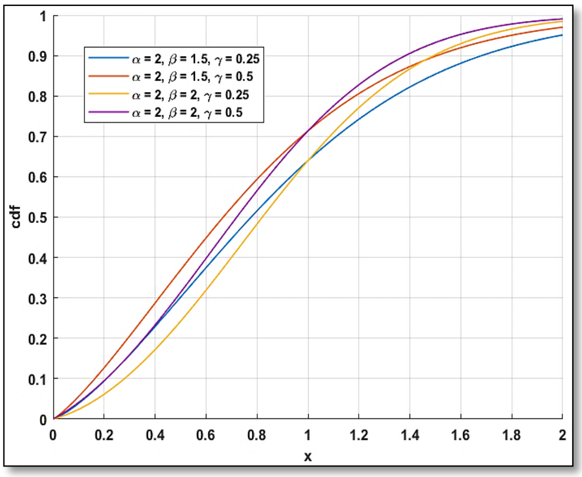

Figure 1. Plots of cdf for different values of $\alpha, \beta, \& \Upsilon$

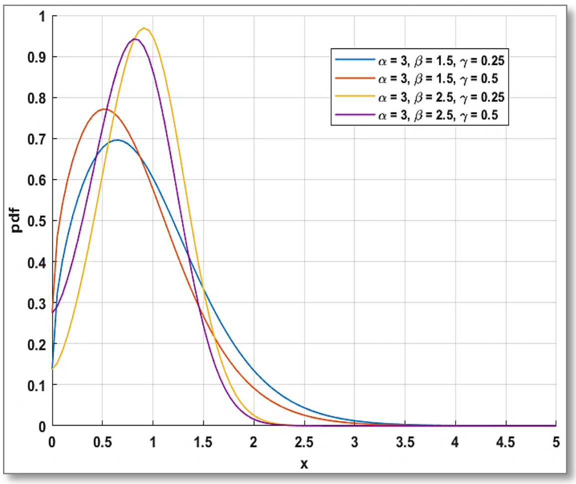

Figure 2. Plots of pdf for different values of $\alpha, \beta, \& \Upsilon$

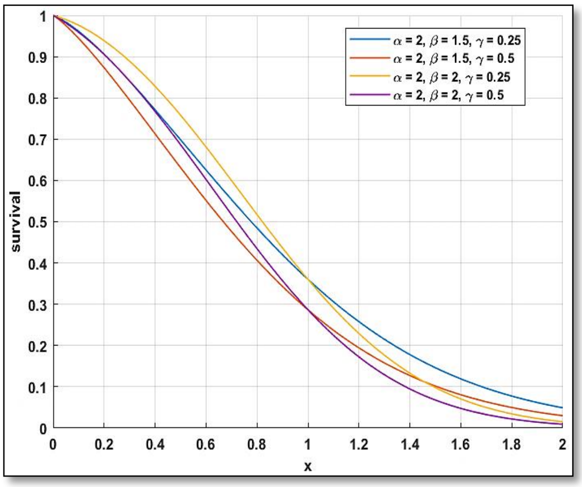

Figure 3. Plots of S(x) for different values of $\alpha, \beta, \& \Upsilon$

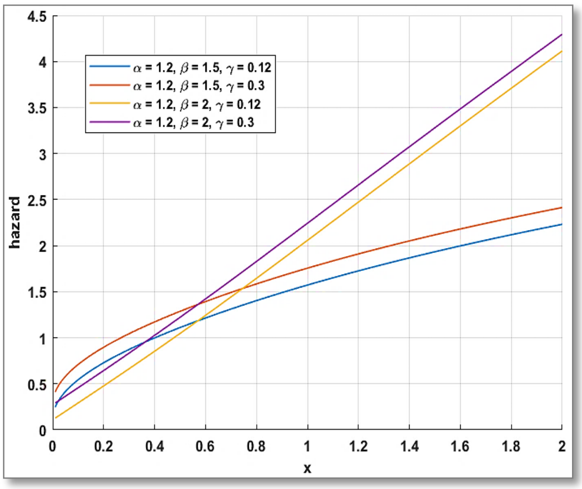

Figure 4. Plots of h(x) for different values of $\alpha, \beta, \& \Upsilon$

2.2 Expanding the probability density and cumulative functions

In order to easily deal with the cumulative and probability density functions, we use some mathematical formulas to expand the two functions to facilitate the process of finding the statistical properties of APEWD distribution.

$$let\ \alpha = e^{\log(\alpha)}, \text{then } \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} = \left(e^{log(\alpha)}\right)^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} = e^{log(\alpha) - log(\alpha)\, e^{-\left(\Upsilon x + x^{\beta}\right)}}$$

$$= e^{log(\alpha)} \cdot e^{-\,log(\alpha)\, e^{-\left(\Upsilon x + x^{\beta}\right)}} = \alpha \cdot e^{-\,log(\alpha)\, e^{-\left(\Upsilon x + x^{\beta}\right)}}$$

$$g(x) = \frac{\left(\Upsilon + \beta x^{\beta-1}\right) log(\alpha)}{\alpha - 1} \alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} e^{-\left(\Upsilon x + x^{\beta}\right)}$$

$$g(x) = \frac{\alpha\, log(\alpha)}{\alpha - 1} \left(\Upsilon + \beta x^{\beta-1}\right) e^{-\,log(\alpha)\, e^{-\left(\Upsilon x + x^{\beta}\right)}}\ e^{-\left(\Upsilon x + x^{\beta}\right)}$$

$$e^{-log(\alpha)\, e^{-\left(\Upsilon x + x^{\beta}\right)}} = \sum_{s=0}^{\infty} \frac{(-1)^s (log(\alpha))^s}{s!}\ e^{-s\left(\Upsilon x + x^{\beta}\right)}$$

$$g(x) = \frac{\alpha}{\alpha - 1} \sum_{s=0}^{\infty} \frac{(-1)^s (log(\alpha))^{s+1}}{s!} \left(\Upsilon + \beta x^{\beta-1}\right) e^{-(s+1)(\Upsilon x)} e^{-(s+1)\left(x^{\beta}\right)}$$

$$e^{-(s+1)\left(x^{\beta}\right)} = \sum_{b=0}^{\infty} \frac{(-1)^b (s+1)^b}{b!} x^{b\beta}$$

$$g(x) = \frac{\alpha}{\alpha - 1} \sum_{s=0}^{\infty} \sum_{b=0}^{\infty} \frac{(-1)^{s+b} (log(\alpha))^{s+1} \, (s+1)^b}{s! \, b!} x^{\beta b} \left(\Upsilon + \beta x^{\beta - 1} \right) e^{-(s+1)(\Upsilon x)}$$

$$g(x) = \frac{\alpha}{\alpha - 1} \sum_{s=0}^{\infty} \sum_{b=0}^{\infty} \frac{(-1)^{s+b} (log(\alpha))^{s+1} \, (s+1)^b}{s! \, b!} \left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1} \right) e^{-(s+1)(\Upsilon x)}$$

$$\text{let } \mathcal{G}_{s,b} = \frac{\alpha}{\alpha - 1} \sum_{s=0}^{\infty} \sum_{b=0}^{\infty} \frac{(-1)^{s+b} (log(\alpha))^{s+1} \, (s+1)^b}{s! \, b!}$$

$$g(x) = \mathcal{G}_{s,b} \cdot \left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1} \right) e^{-(s+1)(\Upsilon x)} \tag{9}$$

$$G(x) = \frac{\alpha^{1-e^{-\left(\Upsilon x + x^{\beta}\right)}} - 1}{\alpha - 1}$$

$$G(x) = \frac{1}{\alpha - 1} \left(\alpha \cdot \sum_{s=0}^{\infty} \frac{(-1)^s (log(\alpha))^s}{s!} \, e^{-s\left(\Upsilon x + x^{\beta}\right)} - 1 \right)$$

$$e^{-s x^{\beta}} = \sum_{w=0}^{\infty} \frac{(-s)^w}{w!} x^{w\beta}$$

$$G(x) = \frac{1}{\alpha - 1} \left(\alpha \sum_{s=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^{s+w} \, s^w \, (log(\alpha))^s}{s! \, w!} \, x^{w\beta} \, e^{-s\Upsilon x} - 1 \right)$$

$$\text{let } \xi_{s,w} = \sum_{s=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^{s+w} \, s^w \, (log(\alpha))^s}{s! \, w!}$$

$$G(x) = \frac{1}{\alpha - 1} \left(\alpha \, \xi_{s,w} \, x^{w\beta} \, e^{-s\Upsilon x} - 1 \right) \tag{10}$$

3. Mathematical and statistical properties of (APEWD)

3.1. The Mode

This is done by finding the point that the probability density function reaches its maximum value; therefore, the mode is calculated as follows:

$$g(x) = \mathcal{G}_{s,b} \cdot \left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1} \right) e^{-(s+1)(\Upsilon x)}$$

$$e^{-(s+1)(\Upsilon x)} = \sum_{z=0}^{\infty} \frac{(-1)^z ((s+1)\Upsilon)^z}{z!} \, x^z$$

$$g(x) = \sum_{z=0}^{\infty} \frac{(-1)^z ((s+1)\Upsilon)^z}{z!} \, \mathcal{G}_{s,b} \cdot \left(\Upsilon x^{z+\beta b} + \beta x^{z+\beta(b+1)-1} \right)$$

$$\frac{dg(x)}{dx} = \sum_{z=0}^{\infty} \frac{(-1)^z ((s+1)\Upsilon)^z}{z!} \mathcal{G}_{s,b} \left(\Upsilon (z+b) \, x^{z+b-1} + \beta \, (z+b+\beta-1) x^{z+b+\beta-2} \right) = 0$$

Divided both sides of the above equation by (x^{z+b-1})

$$\sum_{z=0}^{\infty} \frac{(-1)^z ((s+1)\Upsilon)^z}{z!} \mathcal{G}_{s,b} \cdot \Upsilon (z+b) + \sum_{z=0}^{\infty} \frac{(-1)^z ((s+1)\Upsilon)^z}{z!} \mathcal{G}_{s,b} \cdot \beta \, (z+b+\beta-1) x^{\beta-1} = 0$$

$$x = \left(\frac{- \sum_{s=0}^{\infty} \sum_{b=0}^{\infty} \sum_{z=0}^{\infty} \Upsilon (z+b)}{\sum_{s=0}^{\infty} \sum_{b=0}^{\infty} \sum_{z=0}^{\infty} \beta \, (z+b+\beta-1)} \right)^{\frac{1}{\beta-1}} \tag{11}$$

3.2. The Quantile function

The quantile function is considered very important from a theoretical and applied perspective. Theoretically, it is possible to find some statistical properties, such as Skewness and Kurtosis, and in application to generate data that is used in simulation.

$$G(x)=\frac{\alpha^{1-e^{-\left(\Upsilon x+x^{\beta}\right)}}-1}{\alpha-1}$$

$$Q(u)=G^{-1}(u)$$

$$u=\frac{\alpha^{1-e^{-\left(\Upsilon x+x^{\beta}\right)}}-1}{\alpha-1}$$

$$u(\alpha-1)=\alpha^{1-e^{-\left(\Upsilon x+x^{\beta}\right)}}-1$$

$$u(\alpha-1)+1=\alpha^{1-e^{-\left(\Upsilon x+x^{\beta}\right)}}$$

$$log(u(\alpha-1)+1)=\left(1-e^{-\left(\Upsilon x+x^{\beta}\right)}\right)\cdot\log(\alpha)$$

$$e^{-\left(\Upsilon x+x^{\beta}\right)}=1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)$$

$$\left(\Upsilon x+x^{\beta}\right)=-\log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)$$

$$x^{\beta}+\Upsilon x+log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)=0$$

To find the roots of the above nonlinear equation, which is represented by the values of x, special numerical methods are needed to find a solution to this nonlinear equation. Command syntax <fsolve> in MATLAB 2018a was used to find x with initial values of parameters $(\alpha,\Upsilon,\text{ and }\beta)$, and in order to understand the issue thoroughly, we assume some values for the parameter ($\beta=1{,}2$).

Let $\beta=1$, then we have

$$x+\Upsilon x+log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)=0$$

$$x(1+\Upsilon)=-log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)$$

$$x=\frac{-log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)}{(1+\Upsilon)}$$

Let $\beta=2$, then we have:

$$x^{2}+\Upsilon x+log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)=0$$

$$Q(u)=x=\frac{-\Upsilon\pm\sqrt{\Upsilon^{2}-4\,log\left(1-\left(\frac{log(u(\alpha-1)+1)}{log(\alpha)}\right)\right)}}{2}$$

The median point at which the cumulative distribution function equal to 0.5 $(u=0.5)$ is called the median and the median of (APEWD) is defined as:

$$G(x)=0.5$$

$$x^{\beta}+\Upsilon x+log\left(1-\left(\frac{log(0.5\alpha+0.5)}{log(\alpha)}\right)\right)=0$$

Let $\beta=1$, then we have

$$x=\frac{-log\left(1-\left(\frac{log(0.5\alpha+0.5)}{log(\alpha)}\right)\right)}{(1+\Upsilon)}$$

Let $\beta=2$, then we have:

$$Q(u)=x=\frac{-\Upsilon\pm\sqrt{\Upsilon^{2}-4\,log\left(1-\left(\frac{log(0.5\alpha+0.5)}{log(\alpha)}\right)\right)}}{2}$$

3.3. Moments about the origin

One of the most important distribution properties is moments because of their role in determining many other distribution properties, such as mean, variance, skewness, and kurtosis.

The moments of (APEWD) could be obtained as given:

$$M_r'=E(x^r)=\int_0^{\infty}x^r\cdot g(x)\,dx=\int_0^{\infty}x^r\cdot\mathcal{G}_{s,b}\cdot\left(\Upsilon x^{\beta b}+\beta x^{\beta(b+1)-1}\right)e^{-(s+1)(\Upsilon x)}dx$$

$$M_r'=\Upsilon\cdot\mathcal{G}_{s,b}\int_0^{\infty}x^{r+\beta b}\cdot\,e^{-(s+1)(\Upsilon x)}\,dx+\beta\cdot\mathcal{G}_{s,b}\int_0^{\infty}x^{\beta(b+1)-1}\cdot\,e^{-(s+1)(\Upsilon x)}\,dx$$

$$let\;y=(s+1)(\Upsilon x)\Rightarrow x=\frac{y}{\Upsilon(s+1)}\;\&\,d\,x=\frac{dy}{\Upsilon(s+1)}$$

$$M_r'=\Upsilon\cdot\mathcal{G}_{s,b}\int_0^{\infty}\left(\frac{y}{\Upsilon(s+1)}\right)^{r+\beta b}\cdot e^{-y}\,\frac{dy}{\Upsilon(s+1)}+\beta\cdot\mathcal{G}_{s,b}\int_0^{\infty}\left(\frac{y}{\Upsilon(s+1)}\right)^{r+\beta(b+1)-1}\cdot e^{-y}\,\frac{dy}{\Upsilon(s+1)}$$

$$M_r'=\frac{\alpha\Upsilon}{\alpha-1}\sum_{s=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{s+b}(log(\alpha))^{s+1}}{s!\,b!\cdot(s+1)^{r+b(\beta-1)+1}\cdot\Upsilon^{r+\beta b+1}}\int_0^{\infty}y^{r+\beta b}\cdot e^{-y}\,dy+\frac{\alpha\beta}{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{s+b}(log(\alpha))^{s+1}}{s!\,b!\,\cdot(s+1)^{r+\beta(b+1)-b}\cdot\Upsilon^{r+\beta(b+1)}}\int_0^{\infty}y^{r+\beta(b+1)-1}\cdot e^{-y}\,dy$$

$$M_r'=\psi_{s,b}\cdot\Gamma_{(r+\beta b+1)}+\varphi_{s,b}\cdot\Gamma_{(r+\beta(b+1))}\qquad(12)$$

$$\psi_{s,b}=\frac{\alpha\Upsilon}{\alpha-1}\sum_{s=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{s+b}(log(\alpha))^{s+1}}{s!\,b!\cdot(s+1)^{r+b(\beta-1)+1}\cdot\Upsilon^{r+\beta b+1}}$$

$$\varphi_{s,b}=\frac{\alpha\beta}{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{s+b}(log(\alpha))^{s+1}}{s!\,b!\,\cdot(s+1)^{r+\beta(b+1)-b}\cdot\Upsilon^{r+\beta(b+1)}}$$

Direct applied of (M_r'), it could be found $\mathrm{E}(\mathrm{x})=\mathrm{M}_1'$, $\mathrm{E}(\mathrm{x}^2)=\mathrm{M}_2'$, and var(x) as follows:

$$\mu_x=M_1'=\psi_{s,b}\cdot\Gamma_{(\beta b+2)}+\varphi_{s,b}\cdot\Gamma_{(1+\beta(b+1))}$$

$$var(x)=M_2'-(M_1')^2$$

$$var(x)=\left(\psi_{s,b}\cdot\Gamma_{(\beta b+3)}+\varphi_{s,b}\cdot\Gamma_{(2+\beta(j+1))}\right)-\left(\psi_{s,b}\cdot\Gamma_{(\beta b+2)}+\varphi_{s,b}\cdot\Gamma_{(1+\beta(b+1))}\right)^2$$

3.4. Coefficients of Skewness and Kurtosis

The Coefficients skewness $(C.S)$ and kurtosis $(C.K)$(**Table 1**) could be given through the following formulas:

$$M_3' = \psi_{s,b} \cdot \Gamma_{(\beta b+4)} + \varphi_{s,b} \cdot \Gamma_{(3+\beta(b+1))}$$

$$C.S = \frac{M_3'}{(M_2')^{\frac{3}{2}}} \tag{13}$$

$$M_4' = \psi_{s,b} \cdot \Gamma_{(\beta b+5)} + \varphi_{s,b} \cdot \Gamma_{(4+\beta(b+1))}$$

$$C.K = \frac{M_4'}{(M_2')^2} - 3 \tag{14}$$

Table 1. The first - fourth moments, variance, skewness, and kurtosis for the distribution

α	β	γ	μ_1'	μ_2'	μ_3'	μ_4'	K	S	var
2	1.5	2.5	0.3631	0.2222	0.1848	0.1918	0.8861	1.7645	0.0904
	0.5	1.5	0.4211	0.4233	0.6675	1.4417	5.0443	2.4234	0.2460
2.5	0.9	0.5	0.8354	1.3298	3.1145	9.6950	2.4825	2.0310	0.6320
	1.2	0.7	0.7261	0.8656	1.3971	2.8235	0.7686	1.7350	0.3383
3.5	0.2	0.8	0.7403	1.6480	5.6832	26.4787	6.7500	2.6864	1.0999
	1.4	2.8	0.3707	0.2230	0.1802	0.1814	0.6490	1.7113	0.0855

3.5. Characteristic Function

$$\phi_x(it) = E(e^{itx}) = \int_0^\infty e^{itx}\ g(x)\, dx = \int_0^\infty \mathcal{G}_{s,b} \cdot \left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1}\right) e^{-((s+1)\Upsilon - it)x}\, dx$$

$$\phi_x(it) = \mathcal{G}_{s,b}\left(\int_0^\infty \Upsilon x^{\beta b}\ e^{-((s+1)\Upsilon - it)x}\, dx + \int_0^\infty \beta x^{\beta(b+1)-1}\ e^{-((s+1)\Upsilon - it)x}\, dx\right)$$

$$Let\ ((s+1)\Upsilon - it)x = y, \text{and } x = \frac{y}{((s+1)\Upsilon - it)} \text{then } dx = \frac{dy}{((s+1)\Upsilon - it)}$$

$$\phi_x(it) = \mathcal{G}_{s,b}\left(\frac{\Upsilon}{((s+1)\Upsilon - it)^{\beta b+1}}\int_0^\infty y^{\beta b}\ e^{-yx}\, dy + \frac{\beta}{((s+1)\Upsilon - it)^{\beta(b+1)}}\int_0^\infty y^{\beta(b+1)-1}\ e^{-y}\, dy\right)$$

$$\phi_x(it) = \frac{\Upsilon \cdot \mathcal{G}_{s,b} \cdot \Gamma_{(\beta b+1)}}{((s+1)\Upsilon - it)^{\beta b+1}} + \frac{\beta \cdot \mathcal{G}_{s,b} \cdot \Gamma_{(\beta(b+1))}}{((s+1)\Upsilon - it)^{\beta(b+1)}} \tag{15}$$

3.6. Moment Generating Function

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}\ g(x)\, dx = \int_0^\infty \mathcal{G}_{s,b} \cdot \left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1}\right) e^{-((s+1)\Upsilon - t)x}\, dx$$

$$M_x(t) = \mathcal{G}_{s,b}\left(\int_0^\infty \Upsilon x^{\beta b}\ e^{-((s+1)\Upsilon - t)x}\, dx + \int_0^\infty \beta x^{\beta(b+1)-1}\ e^{-((s+1)\Upsilon - t)x}\, dx\right)$$

$$Let\ ((s+1)\Upsilon - t)x = y, and\ x = \frac{y}{((s+1)\Upsilon - t)}\ then\ dx = \frac{dy}{((s+1)\Upsilon - t)}$$

$$M_x(t) = \mathcal{G}_{s,b}\left(\frac{\Upsilon}{\left((s+1)\Upsilon - t\right)^{\beta b+1}}\int_0^\infty y^{\beta b}\ e^{-yx}\,dy + \frac{\beta}{\left((s+1)\Upsilon - t\right)^{\beta(b+1)}}\int_0^\infty y^{\beta(b+1)-1}\ e^{-y}\,dy\right)$$

$$M_x(t) = \frac{\Upsilon\cdot\mathcal{G}_{s,b}\cdot\Gamma_{(\beta b+1)}}{\left((s+1)\Upsilon - t\right)^{\beta b+1}} + \frac{\beta\cdot\mathcal{G}_{s,b}\cdot\Gamma_{(\beta(b+1))}}{\left((s+1)\Upsilon - t\right)^{\beta(b+1)}} \tag{16}$$

3.7. Factorial Moments Generating Function

$$\mathcal{M}_x(t) = E(t^x) = \int_0^\infty t^x\ g(x)\,dx = \int_0^\infty e^{x\ln t}\,\mathcal{G}_{s,b}\cdot\left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1}\right)e^{-\left((s+1)\Upsilon\right)x}\ dx$$

$$= \int_0^\infty \mathcal{G}_{s,b}\cdot\left(\Upsilon x^{\beta b} + \beta x^{\beta(b+1)-1}\right)e^{-\left((s+1)\Upsilon - \ln t\right)x}\ dx$$

$$Let\ \left((s+1)\Upsilon - \ln t\right)x = y, and\ x = \frac{y}{\left((s+1)\Upsilon - \ln t\right)}\ then\ dx = \frac{dy}{\left((s+1)\Upsilon - \ln t\right)}$$

$$\mathcal{M}_x(t) = \mathcal{G}_{s,b}\left(\frac{\Upsilon}{\left((s+1)\Upsilon - \ln t\right)^{\beta b+1}}\int_0^\infty y^{\beta b}\ e^{-yx}\,dy + \frac{\beta}{\left((s+1)\Upsilon - \ln t\right)^{\beta(b+1)}}\int_0^\infty y^{\beta(b+1)-1}\ e^{-y}\,dy\right)$$

$$\mathcal{M}_x(t) = \frac{\Upsilon\cdot\mathcal{G}_{s,b}\cdot\Gamma_{(\beta b+1)}}{\left((s+1)\Upsilon - \ln t\right)^{\beta b+1}} + \frac{\beta\cdot\mathcal{G}_{s,b}\cdot\Gamma_{(\beta(b+1))}}{\left((s+1)\Upsilon - \ln t\right)^{\beta(b+1)}} \tag{17}$$

4. Conclusion

Adding a new parameter, whether it is a shape or measurement parameter, gives convenience and flexibility to the distribution in terms of analysis and processing of the data. Based on the alpha-power family method for generating distributions, a new distribution called APEWD was presented. All the basic functions including cdf, pdf, survival, and hazard, statistical properties of this distribution such as moments, moment generating, factorial moments skewness, kurtosis … etc were presented and demonstrated using some mathematical formulas to facilitate dealing with the complexity in finding and discussing some properties.

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Conflict of Interest

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