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Estimate the parameters of Exponential-Rayleigh distribution, by using Bayesian method

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Abstract
In this paper, point estimation method for parameters α and λ of the parameters of the Exponential-Rayleigh distribution which have been estimated by the use of a simulation technique by using two Bayesian estimation methods; the first Bayesian method of estimation consists of lindely approximation estimation method and the second Bayesian estimation method consists Tierney and Kadane approximation estimation method to estimate all the unknown parameters (α, λ) of Exponential-Rayleigh distribution. The Bayes estimate of the unknown parameter is obtained by using the approximation methods of lindley (1980) and Tierney and Kadane (1986). Comparisons between these two methods were made by employing mean squares error criterion. Applying a simulation technique with different sample sizes, these methods are being compared. Simulation procedure is used to generate some sample sizes and mean squares error measure, and when we compared between the above two methods we find that lindely approximation estimation method has the less mean squares error.
Keywords: Bayesian method, Exponential-Rayleigh distribution, Tierney and Kadane approximation Bayes estimation method, linedely approximation Bayes estimation method, Simulation technique, Mean squares error.

1. Introduction
Bayesian methods have become popular statistical procedures in areas from medicine to engineering (1). Initially, the most common objective priors were considered such as Jeffreys' prior, reference priors, maximum data information prior. Bayesian estimation methods are usually based on the idea that the parameters to be estimated can be random variables and $(t_1, t_2, \dots, t_n)$ be random variables (2,3). Bayesian estimation methods are usually based on the idea the parameters$(\theta_1, \theta_2, \dots, \theta_n)$ to be estimated can be random variables (4). Several researchers have worked on the Bayesian Estimation. Ferreira et al. (5) proposed to make Bayesian inferences for the parameters of the Lomax distribution using non-informative priors. Mazaal et al. (6) Compared Weibull Stress – Strength Reliability Bayesian Estimators for Singly Type II Censored Data under Different loss Functions Awatif. Al-Baldawi (7) employed non-informative priors to compare a few Bayesian estimation with the maximum likelihood

estimator for the Maxwell distribution. Iden and Sara (8) introduced the lindley approximation estimation method to estimate the logistic distribution's two parameters. Rasheed (9) used the Bayesian estimation under the quadratic loss function with no information before estimating the Maxwell distribution parameter. Alkanani and Salman (10) used Bayesian estimation and non- Bayesian estimation Methods for the Maxwell Boltzmann distribution parameter. Kalt and Hussein (11) used Bayesian estimation for Parameters of Modified Weibull distribution using Tierney and Kadane approximation method. Mohammed and Hussein (12) estimated methods for new mixture distribution with simulation and application.

The Exponential Rayleigh distribution is obtained based on mixed between cumulative distribution function of Exponential distribution and cumulative distribution function of Rayleigh distributions (13).

The probability density function of the Exponential-Rayleigh distribution is defined as follows (14):

$$f(t;\alpha,\lambda) = (\alpha + \lambda t)e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)} \quad , \; t \geq 0\,;\; \alpha,\lambda > 0 \tag{1}$$

The cumulative distribution function (CDF):

$$F(t) = 1 - e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)} \quad ,t \geq 0\,;\; \alpha,\lambda > 0 \tag{2}$$

And the survival function is:

$$S(t) = e^{-\left(\alpha t+\frac{\lambda}{2}t^2\right)} \quad ,t \geq 0\,;\; \alpha,\lambda > 0 \tag{3}$$

The hazard function is:

$$h(t) = (\alpha + \lambda t) \quad , \; t \geq 0\,;\; \alpha,\, \lambda > 0 \tag{4}$$

The aim of this paper showing how to derive and estimate the two parameters (scale) in Exponential-Rayleigh distribution by using two Bayesian estimation methods, linedely approximation method and Tierney and Kadane approximation method. Therefore, finding and estimating survival function. Finely compare between these two methods to find the best method.

The structure of this paper is as follows: In Section two; study of Bayesian estimation method. Section three; derive Exponential-Rayleigh distribution by using the linedely approximation method. Section four; derive Exponential-Rayleigh distribution by using the Tierney and Kadane approximation method. Section five; the simulation technique is discussed. Section six; the main results are discussed. Finally, Section seven presents the conclusions.

2. Materials and Methods

2.1. Bayesian Method

In the classical estimation Methods assuming that the parameters α or λ of any distribution was be constant and fixed, but it is known to us, then these methods were as classical methods (15). Now describing another approach to estimate the parameters which are called Bayesian methods, these methods of estimation are typically predicated on the idea that the parameters to be estimated can be random variables as opposed to fixed values (16,17). These Bayesian estimation methods are based on the previous information available on the anonymous parameter plus the information which come from the sample observations(18). A loss function is a measure of the amount of loss resulting from a decision to be made depends on while the decision to be made depends on θ (19). The Bayesian estimation method has received a lot of interest recently for analyzing failure time data which has mostly been proposed as an alternative to that of the traditional methods. The Bayesian estimate method employs both the available data and one's prior knowledge of the parameters (20,21). The non-informative prior in Bayesian estimation can be used when one's prior knowledge about the parameter is unavailable (22).

2.2. Lindely approximation estimation method

This method introduced by researcher lindely in 1980. The two unknown parameters for the Exponential-Rayleigh distribution are being estimated using the lindely approximation method (23,24).

lindely approximated the ratio of the integrals which as following form (25):

$$\frac{\int w(\theta)e^{L(\theta)}d\theta}{\int v(\theta)e^{L(\theta)}d\theta} \quad (5)$$

Where $\theta = (\theta_1, \theta_2, \dots \theta_N)$are parameters, $w(\theta)$ and $v(\theta)$ are any arbitrary functions for parameters.

$L(\theta)$ is the a logarithm of the likeihood function.

Suppose that $v(\theta)$ is the prior density function of parameters θ and let $w(\theta) = u(\theta) \cdot v(\theta)$.

From integrals equation (1) getting the posterior expectation which is as follows:

$$E[u(\theta)|t] = \frac{\int u(\theta)v(\theta)\, e^{L(\theta)+G(\theta)}d\theta}{\int v(\theta)\, e^{L(\theta)+G(\theta)}d\theta} = \frac{\int u(\theta)\, e^{L(\theta)+G(\theta)}d\theta}{\int\, e^{L(\theta)+G(\theta)}d\theta}$$. Where $G(\theta) = \log_e[v(\theta)]$

The likeihood function of Exponential-Rayleigh distribution is:

$$L(\lambda; t_{1,} t_2, \dots, t_n) = \prod_{i=1}^{n}(\alpha + \lambda t_i)\, e^{-\left(\alpha t_i + \frac{\lambda}{2} t_i{}^2\right)} \quad (6)$$

The natural algorithm of likeihood function is:

$$\ln L(\lambda; t_{1,} t_2, \dots, t_n) = \sum_{i=1}^{n}(\alpha + \lambda t_i) - \alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2}\sum_{i=1}^{n} t_i{}^2 \quad (7)$$

To apply this method, assuming the prior density function $g_1(\alpha)$ and $g_2(\lambda)$ for the parameters α and λ respectively, getting the posterior distribution. Assuming the prior density function for parameters α and λ is:

$$g_1(\alpha) = \begin{cases} \theta e^{-\theta\alpha} & \alpha > 0 \\ 0 & 0ther\ wise \end{cases} \quad (8)$$

$$g_2(\lambda) = \begin{cases} \theta e^{-\theta\lambda} & \lambda > 0 \\ 0 & 0ther\ wise \end{cases} \quad (9)$$

The joint prior density functions for parameters α and λ is:

$$g(\alpha, \lambda) = g_1(\alpha) \cdot g_2(\lambda) = \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda} \quad (10)$$

The natural algorithm of joint prior density functions for parameters α and λ is:

$$\ln g(\alpha, \lambda) = 2 \ln \theta - \theta\alpha - \theta\lambda \quad (11)$$

By using the reverse Bayes rule in the integration of the prior density function with the likelihood function we obtain the function of the posterior distribution of the parameters α and λ as follows (26):

$$H(\alpha, \lambda; t_{1,} t_2, \dots, t_n) = \prod_{i=1}^{n}(\alpha + \lambda t_i)\, e^{-\alpha\sum_{i=1}^{n} t_i - \frac{\lambda}{2}\sum_{i=1}^{n} t_i{}^2} \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda} \quad (12)$$

By employing the quadratic loss function, then the Bayes estimator as $\hat{\phi}_{Bayes}(\alpha, \lambda)$ for any function with respect to the parameters $\phi(\alpha, \lambda)$ is the posterior mean for this function.

The squared error loss function is given by following:

$$Loss(\hat{\alpha} - \alpha) = (\hat{\alpha} - \alpha)^2 \quad (13)$$

$$Loss(\hat{\lambda} - \lambda) = (\hat{\lambda} - \lambda)^2 \quad (14)$$

The Bayes estimators for α and λ for Exponential-Rayleigh distribution under squared error loss function is the posterior which given as follows:

$$\hat{\phi}_{Bayes} = E[\alpha, \lambda] = \frac{\int\int \phi(\alpha,\lambda) H(\alpha,\lambda; t_{1,} t_2, \dots, t_n) d\alpha\, d\lambda}{\int\int H(\alpha,\lambda; t_{1,} t_2, \dots, t_n) d\alpha\, d\lambda} \quad (15)$$

$$\hat{\phi}_{Bayes} = E[\alpha, \lambda] = \frac{\int\int \phi(\alpha,\lambda) \prod_{i=1}^{n}(\alpha+\lambda t_i)\, e^{-\alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2}\sum_{i=1}^{n} t_i^2}\, \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda}\, d\alpha\, d\lambda}{\int\int \prod_{i=1}^{n}(\alpha+\lambda t_i)\, e^{-\alpha \sum_{i=1}^{n} t_i - \frac{\lambda}{2}\sum_{i=1}^{n} t_i^2}\, \theta e^{-\theta\alpha} \cdot \theta e^{-\theta\lambda}\, d\alpha\, d\lambda} \quad (16)$$

Where the number of integrals is equal to the anonymous parameters.

The posterior density function is difficult to solve, the ratio of these integrals in Equation (15) does not seem to take a theoretical formula for the difficulty of calculating these integral (27). Then utilizing the lindely approximation to solve the posterior distribution as follows:

$$\mathrm{I}(x) = E(u; \alpha, \lambda) = \frac{\int_0^\infty \int_0^\infty u(\alpha,\lambda)\, e^{L(\alpha,\lambda;t_i)+G(\alpha,\lambda)} d\alpha d\lambda}{\int_0^\infty \int_0^\infty e^{L(\alpha,\lambda;t_i)+G(\alpha,\lambda)}\, d\alpha d\lambda} \quad (17)$$

Where: $u(\alpha, \lambda)$ is function for α andλ. $L(\alpha, \lambda; t_i)$ is natural logarithm of likeihood function. $G(\alpha, \lambda)$ is natural logorithm of joint prior density functions of α and λ.

Then, we can calculate $\mathrm{I}(x) = E(u; \alpha, \lambda)$ as follows:

$$\mathrm{I}(x) = u(\hat{\alpha}, \hat{\lambda}) + \frac{1}{2}\begin{bmatrix} (\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda \hat{P}_\lambda)\hat{\sigma}_{\lambda\lambda} + (\hat{u}_{\alpha\lambda} + 2\hat{u}_\alpha \hat{P}_\lambda)\hat{\sigma}_{\alpha\lambda} + (\hat{u}_{\lambda\alpha} + 2\hat{u}_\lambda \hat{P}_\alpha)\hat{\sigma}_{\lambda\alpha} \\ +(\hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha \hat{P}_\alpha)\hat{\sigma}_{\alpha\alpha} \end{bmatrix}$$

$$+\frac{1}{2} \left[(\hat{u}_\lambda \hat{\sigma}_{\lambda\lambda} + \hat{u}_\alpha \hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) + (\hat{u}_\lambda \hat{\sigma}_{\alpha\lambda} + \hat{u}_\alpha \hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})\right] \quad (18)$$

Then applying the lindely approach as in Equation (5) we get:

$$u(\hat{\alpha}, \hat{\lambda}) = \alpha,\ \hat{u}_\alpha = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial \alpha} = 1,\ \hat{u}_{\alpha\alpha} = 0,\ \hat{u}_\lambda = 0,\ \hat{u}_{\lambda\lambda} = 0,\ \hat{u}_{\lambda\alpha} = 0,\ \hat{u}_{\alpha\lambda} = 0$$

$$\mathrm{I}(x) = \hat{\alpha} + \frac{1}{2}\left[(2\hat{P}_\lambda)\hat{\sigma}_{\alpha\lambda} + (2\hat{P}_\alpha)\hat{\sigma}_{\alpha\alpha}\right] + \frac{1}{2}\left[(\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) + (\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})\right] \quad (19)$$

Now derivate $P = \ln g(\alpha, \lambda)$ with respect to α and λ in equation (11) we get:

$$\hat{P}_\lambda = \frac{\partial \ln g(\alpha,\lambda)}{\partial \lambda} = -\theta,\ \hat{P}_\alpha = \frac{\partial \ln g(\alpha,\lambda)}{\partial \alpha} = -\theta,\ \hat{L}_\alpha = \frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\alpha+\lambda t_i} - \sum_{i=1}^{n} t_i$$

$$\hat{L}_\lambda = \frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{t_i}{\alpha+\lambda t_i} - \frac{\sum_{i=1}^{n} t_i^2}{2},\ \hat{L}_{\lambda\lambda} = \frac{\partial^2 \ln L}{\partial \lambda^2} = -\sum_{i=1}^{n} \frac{t_i^2}{(\alpha+\lambda t_i)^2},$$

$$\hat{L}_{\alpha\alpha} = \frac{\partial^2 \ln L}{\partial \alpha^2} = -\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2},\ \hat{L}_{\lambda\alpha} = \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = -\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2},$$

$$\hat{L}_{\alpha\lambda} = \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2},\ \hat{L}_{\lambda\lambda\alpha} = \frac{\partial^3 \ln L}{\partial \lambda^2 \partial \alpha} = 2\sum_{i=1}^{n} \frac{t_i^2}{(\alpha+\lambda t_i)^3}$$

$$\hat{L}_{\lambda\lambda\lambda} = \frac{\partial^3 \ln L}{\partial \lambda^3} = 2\sum_{i=1}^{n} \frac{t_i^3}{(\alpha+\lambda t_i)^3},\ \hat{L}_{\lambda\alpha\lambda} = \frac{\partial^3 \ln L}{\partial \lambda \partial \alpha \partial \lambda} = 2\sum_{i=1}^{n} \frac{t_i^2}{(\alpha+\lambda t_i)^3}$$

$$\hat{L}_{\alpha\alpha\lambda} = \frac{\partial^3 \ln L}{\partial \alpha \partial \alpha \partial \lambda} = 2\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3},\ \hat{L}_{\alpha\lambda\lambda} = \frac{\partial^3 \ln L}{\partial \alpha \partial \lambda \partial \lambda} = 2\sum_{i=1}^{n} \frac{t_i^2}{(\alpha+\lambda t_i)^3}$$

$$\hat{L}_{\lambda\alpha\alpha} = \frac{\partial^3 \ln L}{\partial \lambda \partial \alpha \partial \alpha} = 2\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3},\ \hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 \ln L}{\partial \alpha^3} = 2\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^3}$$

$$\hat{L}_{\alpha\lambda\alpha} = \frac{\partial^3 \ln L}{\partial \alpha \partial \lambda \partial \alpha} = 2\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^3},\ \hat{\sigma}_{\lambda\alpha} = -\frac{1}{\hat{L}_{\lambda\alpha}} = \frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}}$$

$$\hat{\sigma}_{\alpha\alpha} = -\frac{1}{\hat{L}_{\alpha\alpha}} = \frac{1}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}},\ \hat{\sigma}_{\lambda\lambda} = -\frac{1}{\hat{L}_{\lambda\lambda}} = \frac{1}{\sum_{i=1}^{n} \frac{t_i^2}{(\alpha+\lambda t_i)^2}},\ \hat{\sigma}_{\alpha\lambda} = -\frac{1}{\hat{L}_{\alpha\lambda}} = \frac{1}{\sum_{i=1}^{n} \frac{t_i}{(\alpha+\lambda t_i)^2}}$$

To estimate the parameter α at case of squared error loss function (13), assuming that λ is information.

$$\hat{\alpha}_{LAE} = \hat{\alpha} + \left[(-\theta)\left(\frac{1}{\sum_{i=1}^{n} \frac{t_i^2}{(\alpha+\lambda t_i)^2}}\right) + (-\theta)\left(\frac{1}{\sum_{i=1}^{n} \frac{1}{(\alpha+\lambda t_i)^2}}\right)\right]$$

$$+\frac{1}{2}\left[\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(2\sum_{i=1}^{n}\frac{t_i{}^3}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)\right.\right.$$

$$\left.+2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right)$$

$$+\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\left(2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)\right.$$

$$\left.\left.+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right)\right]$$

$$\hat{\alpha}_{LAE}=\hat{\alpha}+\left[-\frac{\theta}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}-\frac{\theta}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right]+\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(\frac{\sum_{i=1}^{n}\frac{t_i{}^3}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right.$$

$$\left.+\frac{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)$$

$$+\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\left(\frac{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right) \quad (20)$$

To estimate the parameter λ at case of squared error loss function (14), assuming that α is information.

We choose: $(\hat{\alpha},\hat{\lambda})=\lambda$, $\hat{u}_\lambda=\frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\lambda}=1$, $\hat{u}_\alpha=0,\hat{u}_{\alpha\alpha}=0,\hat{u}_{\lambda\lambda}=0,\hat{u}_{\lambda\alpha}=0,\hat{u}_{\alpha\lambda}=0$

$$\mathrm{I}(x)=\hat{\lambda}+\frac{1}{2}\left[\left(2\hat{u}_\lambda\hat{P}_\lambda\right)\hat{\sigma}_{\lambda\lambda}+\left(2\hat{u}_\lambda\hat{P}_\alpha\right)\hat{\sigma}_{\lambda\alpha}\right]+\frac{1}{2}\left[(\hat{\sigma}_{\lambda\lambda})\left(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)+(\hat{\sigma}_{\alpha\lambda})\left(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad (21)$$

$$\hat{\lambda}_{LAE}=\hat{\lambda}+\left[(-\theta)\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}\right)+(-\theta)\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)\right]$$

$$+\frac{1}{2}\left[\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(2\sum_{i=1}^{n}\frac{t_i{}^3}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)\right.\right.$$

$$\left.+2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right)$$

$$+\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)\right.$$

$$\left.\left.+2\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right)+2\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)\right)\right]$$

$$\hat{\lambda}_{LAE}=\hat{\lambda}+\left[-\frac{\theta}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}-\frac{\theta}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\right]$$

$$+\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(\frac{\sum_{i=1}^{n}\frac{t_i^3}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right)$$

$$+\frac{1}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}\left(\frac{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i^2}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}}+\frac{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^3}}{\sum_{i=1}^{n}\frac{1}{(\alpha+\lambda t_i)^2}}\right) \quad (22)$$

2.4. Tierney and Kadane approximation Estimation method

In (1986), Tierney and Kadane the two researchers used an approximation method to calculate the ratio of integrals and according to the following formula (28):

$$\frac{\int w(\theta)e^{L(\theta)}d\theta}{\int v(\theta)e^{L(\theta)}d\theta} \quad (23)$$

Where: $\theta=(\theta_1,\theta_2,\dots\theta_m)$: represents the vector of parameters, to be estimated.$w(\theta)$ and $v(\theta)$: are optional functions in terms of the parameters.

$L(\theta)$: logarithm function of the Maximum likeihood.

When $v(\theta)$ is the prior density function of parameters θ and let $w(\theta)=\varphi(\theta)\cdot v(\theta)$.

Which is a function in terms of the parameter θ where:

$$E[\varphi(\theta)|t]=\frac{\int \varphi(\theta)\, e^{(L(\theta)+\rho(\theta))}d\theta}{\int\, e^{(L(\theta)+\rho(\theta))}d\theta}=\frac{\int \varphi(\theta)\, e^{(\Lambda(\theta))}d\theta}{\int\, e^{(\Lambda(\theta))}d\theta} \quad (24)$$

Where:

$\rho(\theta)=\ln v(\theta)\ ,\ \Lambda(\theta)=L(\theta)+\rho(\theta)=L(\theta)+\ln v(\theta)$

Tierney and Kadane were managed to arrive at an estimate value $E[\varphi(\theta)|t]$ in equation (24) by using Taylor's Series to approximate the maximum of likeihood the parameterθ, using independent computations for the integration of the denominator and the integration of the numerator separately and then the division of the output (29), taking into account the following two possibilities:

$$\mathrm{T}=\frac{L(\theta/t)+\ln v(\theta)}{n} \quad (25)$$

$$\mathrm{T}^*=\frac{ln\,\varphi(\theta)L(\theta/t)+ln\,v(\theta)}{n} \quad (26)$$

Find the posterior distribution of the function by Based on equation (25) and (26), then the Equation (24) becomes as follows:

$$E[\varphi(\theta)|t]=\frac{\int e^{(n\mathrm{T}^*)}d\theta}{\int\, e^{(n\mathrm{T}))}d\theta} \quad (27)$$

Then the approximate Bayes estimator by using the Tierney and Kadane method of the function $\phi(\theta)$ is as follows:

$$\hat{\phi}_{Byse}=E[\varphi(\theta)|t]=\left[\frac{|H^*|}{|H|}\right]^{\frac{1}{2}}\exp(n\{T^*(\hat{\alpha}^*{}_{max})-T(\,\hat{\alpha}_{max})\}) \quad (28)$$

Where:

$\hat{\phi}_{Byse}(t)$: Estimation of the function $\phi(\theta)$ based on the method of researchers Tierney and Kadane

$\hat{\theta},\hat{\theta}^*$: The values that maximize T and T^* respectively.

H,H^*: Negative inverse matrix Hessian for both,T , T^*at $\hat{\theta}$and $\hat{\theta}^*$ respectively of class $m\times m$, m the number of parameters to be estimated.

As a result, it needs to find values T and T^* in Equations (25, 26) as follows:

$$\mathrm{T}=\frac{\ln[h(\theta;t_1,t_2,\dots,t_n)]}{n} \quad (29)$$

$$\mathrm{T}^*=\frac{\ln[\phi(\theta)h(\theta;t_1,t_2,\dots,t_n)]}{n} \quad (30)$$

Estimation of parameter $(\alpha)$when the parameter (λ) is knows. Assume that $\phi(\alpha,\lambda)=\alpha$ in equation (27) and then

$\widehat{\alpha}_{Byse}=E(\alpha|t)$.

Where $\hat{\alpha}_{Byse}$ denote the Bayes estimator of α according to Tierney and Kadane approximation.

Then

$$\hat{\alpha}_{Byse}=\frac{\int_0^\infty \phi(\alpha)\, e^{\mathrm{I}(\alpha,\lambda)}\, e^{\ln(v)} d\alpha}{\int_0^\infty e^{\mathrm{I}(\alpha,\lambda)} e^{\ln(v)} d\alpha}=\frac{\int_0^\infty \phi(\alpha)\, e^{\mathrm{I}(\alpha,\lambda)}\, v\, d\alpha}{\int_0^\infty e^{\mathrm{I}(\alpha,\lambda)}\, v\, d\alpha}$$, Where $v=g_1(\alpha)$

$$\hat{\alpha}_{Byse}=\frac{\theta\ e^{-\frac{\lambda}{2}\sum_{i=1}^n t_i{}^2}\int_0^\infty \alpha\ \prod_{i=1}^n(\alpha+\lambda t_i)\ e^{-\alpha\sum_{i=1}^n t_i} e^{-\theta\alpha}\, d\alpha}{\theta\ e^{-\frac{\lambda}{2}\sum_{i=1}^n t_i{}^2}\int_0^\infty \prod_{i=1}^n(\alpha+\lambda t_i)\ e^{-\alpha\sum_{i=1}^n t_i} e^{-\theta\alpha}\, d\alpha} \quad (31)$$

So

$$\mathrm{T}=\frac{ln}{n}\left[\prod_{i=1}^n(\alpha+\lambda t_i)\ e^{-\alpha\sum_{i=1}^n t_i}\ e^{-\theta\alpha}\right]=\frac{\sum_{i=1}^n \ln(\alpha+\lambda t_i)}{n}-\frac{\alpha\sum_{i=1}^n t_i}{n}-\frac{\theta\alpha}{n} \quad (32)$$

$$\mathrm{T}^*=\frac{ln}{n}\left[\alpha\ \prod_{i=1}^n(\alpha+\lambda t_i)\ e^{-\alpha\sum_{i=1}^n t_i}\ e^{-\theta\alpha}\right]=\frac{\ln(\alpha)}{n}+\frac{\sum_{i=1}^n \ln(\alpha+\lambda t_i)}{n}-\frac{\alpha\sum_{i=1}^n t_i}{n}-\frac{\theta\alpha}{n} \quad (33)$$

To find α_{max} and $\alpha^*{}_{max}$ which are the values that maximize T and T^*respectively:

$$\frac{\partial \mathrm{T}}{\partial\alpha}=\frac{1}{n}\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}-\frac{\sum_{i=1}^n t_i}{n}-\frac{\theta}{n}$$

$$\frac{\partial \mathrm{T}^*}{\partial\alpha}=\frac{1}{n\alpha}+\frac{1}{n}\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}-\frac{\sum_{i=1}^n t_i}{n}-\frac{\theta}{n}$$

$$\hat{\alpha}_{max}=\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}-\sum_{i=1}^n t_i-\theta$$

$$\hat{\alpha}^*{}_{max}=\frac{1}{\alpha}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}-\sum_{i=1}^n t_i-\theta$$

$\hat{\alpha}_{max}$ and $\hat{\alpha}^*{}_{max}$ are maximum since:

$$\frac{\partial^2\mathrm{T}}{\partial^2\alpha}=\frac{-1}{n}\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}<0 \quad \frac{\partial^2\mathrm{T}^*}{\partial^2\alpha}=\frac{-1}{n}\left(\frac{1}{\alpha^2}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}\right)<0$$

Necessary and sufficient is that second derivate respect to α is negative (30). The determinant of the negative of the inverse Hessian of T at $\hat{\alpha}_{max}$ and T^* at $\hat{\alpha}^*{}_{max}$ respectively:

$$H=-\left[\frac{\partial^2\mathrm{T}}{\partial^2\alpha}\right]^{-1}_{\alpha=\alpha_{max}}=-\left[\frac{-1}{n}\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}\right]^{-1}=\frac{n}{\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}} \quad (34)$$

$$H^*=-\left[\frac{\partial^2 T^*}{\partial^2\alpha}\right]^{-1}_{\alpha=\alpha^*{}_{max}}=-\left[\frac{-1}{n}\left(\frac{1}{\alpha^2}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}\right)\right]^{-1}=\frac{n}{\frac{1}{\alpha^2}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}} \quad (35)$$

$$\hat{\alpha}_{Byse}=\left[\frac{|H^*|}{|H|}\right]^{\frac{1}{2}}\exp(n\{T^*(\hat{\alpha}^*{}_{max})-T(\hat{\alpha}_{max})\})$$

$$\hat{\alpha}_{Byse}=\left[\frac{\frac{n}{\frac{1}{\alpha^2}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}}}{\frac{n}{\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}}}\right]^{\frac{1}{2}}\exp\left(n\left\{\frac{\ln(\alpha)}{n}+\frac{\sum_{i=1}^n\ln(\alpha+\lambda t_i)}{n}-\frac{\alpha\sum_{i=1}^n t_i}{n}-\frac{\theta\alpha}{n}\left(\frac{1}{\alpha}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}\sum_{i=1}^n t_i-\theta\right)\right.\right.$$
$$\left.\left.-\left(\frac{\sum_{i=1}^n\ln(\alpha+\lambda t_i)}{n}-\frac{\alpha\sum_{i=1}^n t_i}{n}-\frac{\theta\alpha}{n}\left(\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}-\sum_{i=1}^n t_i-\theta\right)\right)\right\}\right) \quad (36)$$

$$\hat{\alpha}_{Byse}=\left[\frac{\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}}{\frac{1}{\alpha^2}+\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)^2}}\right]^{\frac{1}{2}}\left(1+\alpha\sum_{i=1}^n\frac{1}{(\alpha+\lambda t_i)}-\alpha\sum_{i=1}^n t_i-\alpha\theta\right)exp\left(\frac{\frac{\sum_{i=1}^n ln(\alpha+\lambda t_i)}{\alpha}-}{\sum_{i=1}^n t_i-\theta}\right) \quad (37)$$

Estimation of parameter λ when the parameter α is knows. Assume that $\phi(\alpha,\lambda)=\lambda$ in Equation (27) and then $\hat{\lambda}=E(\lambda|t)$, Where $\hat{\lambda}_{Byse}$ denote the Bayes estimator of λ according to Tierney and Kadane approximation method.

$$\hat{\lambda}_{Byse}=\frac{\theta\ e^{-\alpha\sum_{i=1}^{n}t_i}\int_0^\infty \lambda\prod_{i=1}^{n}(\alpha+\lambda t_i)\ e^{-\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}d\lambda}{\theta e^{-\alpha\sum_{i=1}^{n}t_i}\int_0^\infty\prod_{i=1}^{n}(\alpha+\lambda t_i)\ e^{-\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}d\lambda} \tag{38}$$

And

$$\mathrm{T}=\frac{ln}{n}\left[\prod_{i=1}^{n}(\alpha+\lambda t_i)\ e^{-\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}\right]=\frac{\sum_{i=1}^{n}\ln(\alpha+\lambda t_i)}{n}-\frac{\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}{n} \tag{39}$$

$$\mathrm{T}^*=\frac{ln}{n}\left[\lambda\ \prod_{i=1}^{n}(\alpha+\lambda t_i)\ \ e^{-\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}\right]=\frac{\ln(\lambda)}{n}+\frac{\sum_{i=1}^{n}\ln(\alpha+\lambda t_i)}{n}-\frac{\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}{n} \tag{40}$$

To find λ_{max} and $\lambda^*{}_{max}$ which are the values that maximize T and T*respectively:

$$\frac{\partial\mathrm{T}}{\partial\lambda}=\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)}}{n}-\frac{\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}{n}$$

$$\frac{\partial\mathrm{T}^*}{\partial\lambda}=\frac{1}{n\lambda}+\frac{\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)}}{n}-\frac{\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}{n}$$

$$\hat{\lambda}_{max}=\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)}-\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)$$

$$\hat{\lambda}^*{}_{max}=\frac{1}{\lambda}+\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)}-\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)$$

$\hat{\lambda}_{max}\ and\ \hat{\lambda}^*{}_{max}$ is maximum since:

$\frac{\partial^2\mathrm{T}^*}{\partial^2\lambda}=\frac{-1}{n}\left(\frac{1}{\lambda^2}+\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}\right)<0$ and $\frac{\partial^2\mathrm{T}}{\partial^2\lambda}=\frac{-1}{n}\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}<0$

Necessary and sufficient is that second derivate respect to λ is negative.

The determinant of the negative of the inverse Hessian of T at $\hat{\lambda}_{max}$ and T* at $\hat{\lambda}^*{}_{max}$ respectively:

$$H=-\left[\frac{\partial^2\mathrm{T}}{\partial^2\lambda}\right]^{-1}{}_{\lambda=\lambda_{max}}=-\left[\frac{-1}{n}\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}\right]^{-1}=\frac{n}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}} \tag{41}$$

$$\hat{\lambda}_{Byse}=\left[\frac{\frac{n}{\frac{1}{\lambda^2}+\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}}{\frac{n}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}}\right]^{\frac{1}{2}}\exp\left(n\left\{\frac{\ln(\lambda)}{n}+\frac{\sum_{i=1}^{n}\ln(\alpha+\lambda t_i)}{n}-\frac{\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}{n}\begin{pmatrix}\frac{1}{\lambda}+\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)^2}\\-\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)\end{pmatrix}\right.\right.$$
$$\left.\left.-\left(\frac{\sum_{i=1}^{n}\ln(\alpha+\lambda t_i)}{n}-\frac{\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)}{n}\left(\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}-\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)\right)\right)\right\}\right) \tag{42}$$

$$\hat{\lambda}_{Byse}=\left[\frac{\frac{n}{\frac{1}{\lambda^2}+\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}}{\frac{n}{\sum_{i=1}^{n}\frac{t_i{}^2}{(\alpha+\lambda t_i)^2}}}\right]^{\frac{1}{2}}\left(1+\lambda\sum_{i=1}^{n}\frac{t_i}{(\alpha+\lambda t_i)}-\lambda\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)\right)\ \ exp\left(\frac{\sum_{i=1}^{n}ln(\alpha+\lambda t_i)}{\lambda}-\left(\frac{\sum_{i=1}^{n}t_i{}^2}{2}+\theta\right)\right) \tag{43}$$

3. Simulation technique

The simulation procedure is now generally used in many branches of statistics. They can be used to evaluate the behavior of models as well as for some random variables. A simulation is defined as a numerical scientific method that uses logical mathematical methods to describe the behavior of a certain approved system. Different techniques have been conducted.

- We select $n = 10,20,30,50$
- We select $\alpha = 0.25,0.5,0.75$
- We select $\lambda = 0.5,1,1.5$, These values were selected at random.
- N=represent the Replicate of Experiment then N=1000

- $F(t) = 1 - S(t) = 1 - e^{-\left(\alpha t + \frac{\lambda}{2}t^2\right)}$

$u = 1 - e^{-\left(\alpha t + \frac{\lambda}{2}t^2\right)}$

$1 - u = e^{-\left(\alpha t + \frac{\lambda}{2}t^2\right)}$

$\ln(1-u) = -\left(\alpha t + \frac{\lambda}{2}t^2\right)$

$\ln(1-u) + \alpha t + \frac{\lambda}{2}t^2 = 0$

$a = \frac{\lambda}{2}$, $b = \alpha$, $c = \ln(1-u)$

$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$t = \frac{-\alpha \pm \sqrt{\alpha^2 - 2\lambda \ln(1-u)}}{\lambda}$

- We find Absolut error value as follows:
  $\phi_\alpha = Min|\hat{\alpha}_i - \alpha|$, $\phi_\lambda = Min|\hat{\lambda}_i - \lambda|$
- We find mean squares errors for parameter(α) and parameter(λ).

$MSE[S(t)] = \sum_{I=1}^{N} \frac{[\hat{S}(t_i) - S(t_i)]^2}{N}$, $MSE[\alpha] = \sum_{I=1}^{N} \frac{[\hat{\alpha}_i - \alpha]^2}{N}$, $MSE[\lambda] = \sum_{I=1}^{N} \frac{[\hat{\lambda}_i - \lambda]^2}{N}$

4. Numerical Results

The numerical results for the simulation approach are exhibited in the following tables after the simulation procedure has been used to determine the parameter estimation, reliability function, and mean square error technique. With a 1000 iteration rate, the MATLAB software employed the Monte Carlo methodology for the simulation and the Newton-Raphson method for the numerical solution. As shown in table (1, 2, 3, 4, 5). Noting from the **Table 1** that Absolut error values of parameter $(\alpha)$for lindely approximation estimate values method are the best for the Tierney and Kadane approximation Estimation method. Noting from the **Table 2** that Absolut error values of parameter $(\lambda)$for lindely approximation estimate values method are the best for the Tierney and Kadane approximation Estimation method. Noting from the **Table 3** that mean square error (α) for lindely approximation estimate values method are the best for the Tierney and Kadane approximation Estimation method. Noting from the **Table 4**. that mean square error of survival function for lindely approximation estimate values method are the best for the Tierney and Kadane approximation Estimation method. Noting from the **Table 5** that mean square error of parameter (λ) for lindely approximation estimate values method are the best for the Tierney and Kadane approximation Estimation method

Table 1. Represent the estimate for parameter(α) of all Bayesian method with Absolut error values

α	λ	n	$\hat{\alpha}_{LAE}$	$\hat{\alpha}_{TKAE}$	$\emptyset$	Best
0.25	0.5	10	0.232955	0.36954	0.017045	1
0.25	0.5	20	0.24949	0.253797	0.00051	1
0.25	0.5	30	0.220153	0.25038	0.00038	2
0.25	0.5	50	0.250018	0.249896	1.77E-05	1
0.25	1	10	0.259541	0.358864	0.009541	1
0.25	1	20	0.248875	0.340926	0.001125	1
0.25	1	30	0.259211	0.251351	0.001351	2
0.25	1	50	0.25004	0.257901	3.95E-05	1
0.25	1.5	10	0.267571	0.199231	0.017571	1
0.25	1.5	20	0.247656	0.26011	0.002344	1
0.25	1.5	30	0.249608	0.248725	0.000392	1
0.25	1.5	50	0.250008	0.258761	7.72E-06	1
0.5	0.5	10	0.533735	0.757909	0.033735	1
0.5	0.5	20	0.50599	0.766192	0.00599	1
0.5	0.5	30	0.49971	0.500022	2.15E-05	2
0.5	0.5	50	0.499999	0.599998	8.39E-07	1
0.5	1	10	0.484835	0.445992	0.015165	1
0.5	1	20	0.503185	0.740497	0.003185	1
0.5	1	30	0.499308	0.495959	0.000692	1
0.5	1	50	0.500012	0.454503	1.22E-05	1
0.5	1.5	10	0.534858	0.697942	0.034858	1
0.5	1.5	20	0.495619	0.591782	0.004381	1
0.5	1.5	30	0.500308	0.481262	0.000308	1
0.5	1.5	50	0.499961	0.622744	3.91E-05	1
0.75	0.5	10	0.696886	0.379964	0.053114	1
0.75	0.5	20	0.749098	0.743101	0.000902	1
0.75	0.5	30	0.750103	0.749712	0.000103	1
0.75	0.5	50	0.745595	1.027192	0.004405	1
0.75	1	10	0.72881	1.082477	0.02119	1
0.75	1	20	0.750642	0.80692	0.000642	1
0.75	1	30	0.749779	0.91954	0.000221	1
0.75	1	50	0.750033	1.029041	3.27E-05	1
0.75	1.5	10	0.736458	0.729153	0.013542	1
0.75	1.5	20	0.750101	0.941367	0.000101	1
0.75	1.5	30	0.750304	0.741131	0.000304	1
0.75	1.5	50	0.749938	0.731215	6.23E-05	1

Table 2. Represent the estimated for parameter(λ) of all Bayesian method with Absolut error values:

α	λ	n	$\hat{\lambda}_{LAE}$	$\hat{\lambda}_{TKAE}$	$\emptyset$
0.25	0.5	10	0.509598	0.462907	0.009598
0.25	0.5	20	0.499007	0.443457	0.000993
0.25	0.5	30	0.500283	0.433694	0.000283
0.25	0.5	50	0.500042	0.469098	4.2E-05
0.25	1	10	1.026021	0.973806	0.026021
0.25	1	20	0.998455	1.032778	0.001545
0.25	1	30	0.999999	0.923849	5.62E-07
0.25	1	50	1.000025	1.116261	2.54E-05
0.25	1.5	10	1.484813	0.85661	0.015187
0.25	1.5	20	1.499029	1.676104	0.000971
0.25	1.5	30	1.500487	1.395173	0.000487
0.25	1.5	50	1.499997	1.467975	2.5E-06
0.5	0.5	10	0.48869	0.59474	0.01131
0.5	0.5	20	0.493391	0.500723	0.000723
0.5	0.5	30	0.49978	0.504399	0.00022
0.5	0.5	50	0.499963	0.508847	3.67E-05
0.5	1	10	1.016357	0.963378	0.016357
0.5	1	20	1.004066	1.010583	0.004066
0.5	1	30	1.000015	1.066669	1.47E-05
0.5	1	50	0.999995	1.110389	5.04E-06
0.5	1.5	10	1.543717	1.476047	0.023953
0.5	1.5	20	1.495319	1.536033	0.004681
0.5	1.5	30	1.500462	1.220021	0.000462
0.5	1.5	50	1.500013	1.543239	1.34E-05
0.75	0.5	10	0.52599	0.370504	0.02599
0.75	0.5	20	0.50068	0.551145	0.00068
0.75	0.5	30	0.499906	0.489555	9.44E-05
0.75	0.5	50	0.499992	0.499552	8.31E-06
0.75	1	10	1.0467	1.436493	0.0467
0.75	1	20	0.995807	0.783461	0.004193
0.75	1	30	1.000245	1.196793	0.000245
0.75	1	50	0.999999	0.990902	1.16E-06
0.75	1.5	10	1.545809	1.351652	0.045809
0.75	1.5	20	1.498723	1.640125	0.001277
0.75	1.5	30	1.500127	1.439778	0.000127
0.75	1.5	50	1.499983	1.298486	1.71E-05

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Table 3. Represent the MSE for parameter(α) of all Bayesian method

α	λ	n	$MSE_{\widehat{\alpha}_{LAE}}$	$MSE_{\widehat{\alpha}_{TKAE}}$	min
0.25	0.5	10	0.061014	0.07584	0.17413
0.25	0.5	20	4.11E-06	1.91E-05	4.11E-06
0.25	0.5	30	2.53E-07	1.5E-07	9.37E-08
0.25	0.5	50	7.47E-10	2.04E-08	7.47E-10
0.25	1	10	0.002126	0.042557	0.002126
0.25	1	20	8.36E-06	0.018076	8.36E-06
0.25	1	30	2.22E-07	0.004964	2.22E-07
0.25	1	50	2.49E-09	0.00025	2.49E-09
0.25	1.5	10	0.000907	0.002617	0.000907
0.25	1.5	20	1.24E-05	0.000428	1.24E-05
0.25	1.5	30	1.7E-07	4.21E-06	1.7E-07
0.25	1.5	50	8.12E-11	0.000194	8.12E-11
0.5	0.5	10	0.002175	0.151484	0.002175
0.5	0.5	20	4.75E-05	0.137914	4.75E-05
0.5	0.5	30	2.09E-07	1.4E-08	1.4E-08
0.5	0.5	50	7.9E-11	0.019993	7.9E-11
0.5	1	10	0.000705	0.01733	0.000705
0.5	1	20	1.62E-05	0.073333	1.62E-05
0.5	1	30	5.35E-07	1.63E-05	5.35E-07
0.5	1	50	4.62E-10	0.003877	4.62E-10
0.5	1.5	10	0.001242	0.058183	0.001242
0.5	1.5	20	3.03E-05	0.016625	3.03E-05
0.5	1.5	30	1.59E-07	0.000364	1.59E-07
0.5	1.5	50	4.42E-09	0.015201	4.42E-09
0.75	0.5	10	0.002838	0.309147	0.002838
0.75	0.5	20	2.4E-06	0.000105	2.4E-06
0.75	0.5	30	1.28E-07	1.64E-07	1.28E-07
0.75	0.5	50	7.23E-11	0.078898	7.23E-11
0.75	1	10	0.000452	0.135392	0.000452
0.75	1	20	1.31E-05	0.04142	1.31E-05
0.75	1	30	7.36E-08	0.061872	7.36E-08
0.75	1	50	1.18E-09	0.084636	1.18E-09
0.75	1.5	10	0.000532	0.147029	0.000532
0.75	1.5	20	1.74E-08	0.040548	1.74E-08
0.75	1.5	30	3.63E-07	0.000119	3.63E-07
0.75	1.5	50	4.55E-09	0.000387	4.55E-09

Table 4. Represent the MSE for parameter(λ) of all Bayesian method

α	λ	n	$MSE_{\hat{\lambda}_{LAE}}$	$MSE_{\hat{\lambda}_{TKAE}}$	min
0.25	0.5	10	0.000858	0.031695	0.000858
0.25	0.5	20	1.5E-06	0.01405	1.5E-06
0.25	0.5	30	1.69E-07	0.006272	1.69E-07
0.25	0.5	50	2.85E-09	0.000979	2.85E-09
0.25	1	10	0.002652	0.171412	0.002652
0.25	1	20	5.36E-06	0.009031	5.36E-06
0.25	1	30	1.7E-07	0.005808	1.7E-07
0.25	1	50	1.43E-09	0.025105	1.43E-09
0.25	1.5	10	0.000791	0.72097	0.000791
0.25	1.5	20	1.1E-06	0.031248	1.1E-06
0.25	1.5	30	4.33E-07	0.239611	4.33E-07
0.25	1.5	50	3.21E-10	0.010335	3.21E-10
0.5	0.5	10	0.005259	0.009682	0.005259
0.5	0.5	20	4.92E-05	0.000478	4.92E-05
0.5	0.5	30	8.87E-08	0.012252	8.87E-08
0.5	0.5	50	1.48E-09	0.00043	1.48E-09
0.5	1	10	0.004654	0.002195	0.002195
0.5	1	20	3.26E-05	0.031042	3.26E-05
0.5	1	30	3.42E-07	0.012912	3.42E-07
0.5	1	50	8.28E-11	0.014289	8.28E-11
0.5	1.5	10	0.004483	0.092447	0.004483
0.5	1.5	20	3.71E-05	0.014191	3.71E-05
0.5	1.5	30	4.08E-07	0.11564	4.08E-07
0.5	1.5	50	2.63E-10	0.008968	2.63E-10
0.75	0.5	10	0.003688	0.031815	0.003688
0.75	0.5	20	2.55E-05	0.00287	2.55E-05
0.75	0.5	30	8.31E-08	0.000111	8.31E-08
0.75	0.5	50	8.9E-11	0.000294	8.9E-11
0.75	1	10	0.002613	0.233707	0.002613
0.75	1	20	2.49E-05	0.076475	2.49E-05
0.75	1	30	6.02E-08	0.038741	6.02E-08
0.75	1	50	5.13E-11	0.006712	5.13E-11
0.75	1	100	9.85E-12	0.027358	9.85E-12
0.75	1.5	10	0.002863	0.031182	0.002863
0.75	1.5	20	2.1E-06	0.143916	2.1E-06
0.75	1.5	30	1.06E-07	0.010499	1.06E-07
0.75	1.5	50	4.03E-10	0.07206	4.03E-10

Table 5. Represent the MSE for survival function methods of all Bayesian method.

α	λ	n	$S_{MSE_{LAE}}$	$S_{MSE_{TKAE}}$	min
0.25	0.5	10	1.79E-05	0.001218	1.78826E-05
0.25	0.5	20	6.69E-08	4.93E-05	6.68907E-08
0.25	0.5	30	0.000186	0.000428	0.00018582
0.25	0.5	50	2.42E-10	5.92E-05	2.42355E-10
0.25	1	10	3.46E-05	0.000936	3.45513E-05
0.25	1	20	3.87E-07	0.001142	3.87255E-07
0.25	1	30	6.94E-06	0.000106	6.9434E-06
0.25	1	50	2.22E-10	0.000422	2.21642E-10
0.25	1.5	10	8.16E-06	0.013263	8.15897E-06
0.25	1.5	20	3.59E-07	0.000367	3.59237E-07
0.25	1.5	30	1.46E-09	8.85E-05	1.46397E-09
0.25	1.5	50	2.01E-12	2.45E-06	2.00982E-12
0.5	0.5	10	5.21E-05	0.004192	5.21315E-05
0.5	0.5	20	9.98E-07	0.003911	9.97916E-07
0.5	0.5	30	1.57E-08	5.9E-07	1.57214E-08
0.5	0.5	50	2E-11	0.000716	2.00342E-11
0.5	1	10	4.54E-06	0.000492	4.54484E-06
0.5	1	20	1.58E-06	0.003329	1.57777E-06
0.5	1	30	2.07E-08	3.85E-05	2.06807E-08
0.5	1	50	3.8E-12	1.11E-05	3.79964E-12
0.5	1.5	10	0.000107	0.001286	0.000107102
0.5	1.5	20	1.55E-06	0.000418	1.55265E-06
0.5	1.5	30	8.5E-09	0.000836	8.49529E-09
0.5	1.5	50	5.85E-11	0.000875	5.84854E-11
0.75	0.5	10	5.55E-05	0.010901	5.55124E-05
0.75	0.5	20	1.69E-08	8.48E-06	1.69141E-08
0.75	0.5	30	2.77E-10	1.64E-06	2.77344E-10
0.75	0.5	50	1.12E-06	0.003646	1.12329E-06
0.75	1	10	1.16E-06	0.006332	1.16403E-06
0.75	1	20	2.29E-08	4.14E-05	2.29383E-08
0.75	1	30	3.09E-10	0.001051	3.09264E-10
0.75	1	50	4.93E-11	0.002993	4.92781E-11
0.75	1.5	10	1.98E-06	0.000167	1.98306E-06
0.75	1.5	20	3.31E-09	0.0011	3.31358E-09
0.75	1.5	30	2.99E-09	1.7E-05	2.98614E-09
0.75	1.5	50	9.37E-11	0.000217	9.37347E-11

5. Conclusion

In this paper, we have considered the Bayesian estimation of the unknown parameters of the two-parameter Exponential-Rayleigh distribution. Simulation procedure using Monte-Carlo technique and through previous tables show us that the estimated values and two distribution parameters are influenced by both (sample size, real value for estimator and estimation method), we find that the lindely approximation estimation method is the best for the parameters (α and λ). Bayes estimators were obtained using lindley approximation and Tierney and Kadane approximation while MLE were obtained using Newton-Raphson method, simulation study was conducted to examine and compare the lindely approximation estimation method with Tierney and Kadane approximation for different sample sizes with different values showing that the lindely approximation estimation method is the best when we use mean squares error for the parameter (α and λ). Noting that the lindely approximation estimation method is the best when we use mean squares error for survival function.

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Conflict of Interest

The authors claim they have no rival interests.

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