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Autoregressive Models Estimation Selection with Known Marginal Distribution

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Abstract

A time series is a sequence of observations recorded at regular intervals. Time series analysis has applications in diverse fields such as finance, stock prices, economics, environmental science, and social network data analysis. The recorded series is used to represent a measurable quantity or attribute, such as temperature readings, economic indicators, or other variables, depending on the context of the analysis. The idea of time series analysis is to identify patterns, trends, or underlying structures within the data, as well as to make predictions or forecasts about future values based on previous observations. Autoregressive (AR) models are widely used in modeling and forecasting data from time series. This work focuses on AR model parameter estimation, emphasizing the significance of the likelihood function by defining the marginal distribution of the AR process, which is getting by representing the AR process with random shocks and assuming the error terms in a time series have a normal distribution with a zero mean and variance σ^2. Some of the simulated experiments are designed to fit the model for different model orders and sample size to find model parameter estimation by likelihood function with marginal distribution. The results of Mean Squares Errors (MSE) and Mean Percentage Errors (MPE) indicate the significance and robust estimation of the AR –models parameters estimators that are computed theoretically.

Keywords: Autoregressive Time series Models, Marginal distribution of AR time series**,** Maximum likelihood Estimation**.**

1. Introduction

A time series is a collection of data points produced successively over time. If the set is continuous, the time series is considered continuous and discrete if the set it belongs to is also discrete (1). In the early 20^{th} century, time series analysis heavily relied on an autoregressive model, as researched by Yule in 1927 (1). The autoregressive model utilized regression analysis to estimate the values of the series at a particular time by applying a linear function of past values (2). Time series analysis employs this model to illustrate the relationship among time series data; refer to (2) for further details.

Generalized autoregressive (GAS) models are a type of observational time series model that is proposed in (3). This novel method offers a unified and consistent framework for adding time-varying parameters to a wide range of nonlinear models. Using multivariate point processes with time-varying parameters and novel model requirements, this method can result in a new formulation of observational models. We present detailed studies of the models along with experimental and simulation evidence. In (4), an optimal power spectrum estimator using the mixed ARMA (1,1) model for time series data that adheres to a normal distribution was conducted.

An extensively used model for analyzing time series data is the autoregressive (AR) model. The innovation noise in AR was traditionally described as a Gaussian in (5). But since many time series applications, like financial time series data, for example, are not Gaussian, the AR model with more broadly applicable substantial improvements is recommended. For the first time, an effective framework based on stochastic approximation expectation maximization (SAEM) combined with a Markov chain Monte Carlo (MCMC) approach is proposed to handle the problem of missing values in incomplete time series.

Authors in (6) constructed an autoregressive model incorporating an exogenous variable. They use two methods: the first method employed was the threshold approach, utilizing two proposed approaches to identify the optimal cut-off point for future forecasting and predictability in the time series; their goal was achieved through the threshold point indicator; and the second method involved using seasonal B-J models, which relied on the principles of the two approaches above to determine the most suitable seasonal model.

The enhancement of the estimate of a third-order autoregressive model by employing the LDR and WLSE estimation techniques was studied by researchers in (7). By creating a time series for the AR(3) model with normally and non-normally distributed error terms, the researcher in (7) found that enhancing the estimation of the autoregressive model using the LDR and WLSE methods is contingent upon the sample size for all considered error distributions. In (8), the best and most efficient artificial neural network models for solving linear and nonlinear time series behavior were considered. The researchers concluded that the ideal neural networks are the backpropagation network (BP) and the recurrent neural network (RNN) for solving time series, whether linear, quasi-linear, or nonlinear. The results showed an improvement over the modern methods of time series forecasting. Furthermore, the author in (9) combined ARMA models with EGARCH models to create a hybrid model: $\text{ARMA}(R, M)$–$\text{EGARCH}(Q, P)$. This hybrid model was used to analyze time series data on average temperatures. He determined that the optimal model is $\text{ARMA}(4{,}4)$–$\text{EGARCH}(3{,}3)$; see (9). In (10), researchers presented the fundamental genetic algorithm (CGA) to estimate the log-likelihood parameter function of a first-order moving average model. After comparing it based on mean squared error (MSE) with the intraday technique, they determined that CGA could yield more dependable outcomes. In (11), researchers presented a new ARIMA model and applied it to a monthly chemical sales dataset in the United States. After comparing it with other models, they found that the updated ARIMA model is more accurate. In (12) they conducted the first study with a small sample size that focused on predicting errors based on the concept of the Gaussian noise process. In (13), researchers studied monthly rainfall data from the Baghdad Meteorological Station to study the temporal behavior of the data series for many ARIMA models that have been tested. They verified the adequacy of the best one. They concluded that the seasonal ARIMA model for orders SARIMA(2,1,3)x(0,1,1) is the best.

The idea of this work is finding theoretically the estimation of the autoregressive model by likelihood function by defining the marginal distribution of the AR process when the error terms of the series are normal distribution with zero mean and constant variance. The rest of the paper is section 2 gives the theatrical properties of the AR model; section 3 presents the details of the maximum likelihood function for the AR model, at least the simulation and results given in section 4.

2. Autoregressive Time Series Model

The autoregressive time series model of order p may be written as (14-19):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \tag{1}$$

Where p is called the order of the AR model. ϕ_p of the linear combination are the model parameters and $a_t \sim N(0, \sigma_a^2)$. By using the backward shift operator, B, which is ($B^j Z_t = Z_{t-1}$),. Then the $AR(p)$ in **Equation 1** can be written as the following:

$Z_t = \phi_1 B Z_t + \phi_2 B^2 Z_t + \cdots + \phi_p B^p Z_t + a_t$. Which is simplified as below:

$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) Z_t = a_t$. Hence,

$$\phi(B) Z_t = a_t \tag{2}$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$. By dividing both sides of **Equation 2** on $\phi(B)$,we get: $Z_t = \frac{1}{\phi(B)} a_t = \phi(B)^{-1} a_t \Rightarrow Z_t = \Psi(B) a_t$, where $\phi(B)^{-1} = \Psi(B)$.

The parameters $\phi_1, \phi_2, \dots, \phi_p$ of an AR (p) process must satisfy certain conditions for the process to be stationary.

3. Likelihood Function of $AR(p)$ Model

As in (19-23), we suppose that the p^{th} order stationary autoregressive model is considered as follows:

$Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \cdots - \phi_p Z_{t-p} = a_t$. That is

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \tag{3}$$

The formula for the exact likelihood function is:

$$L = (2\pi\sigma_a^2)^{-\frac{n}{2}} \left| M_n^{(p,0)} \right|^{\frac{1}{2}} exp^{-\frac{S(\phi)}{2\sigma_a^2}} \tag{4}$$

where $S(\phi) = \sum_{i=1}^{p} \sum_{j=1}^{p} m_{ij}^{(p)} z_i z_j + \sum_{t=p+1}^{n} (Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \cdots - \phi_p Z_{t-p})^2$.

Also,

$$M_p^{(p,0)} = \{m_{ij}\}^{(p)} = \{\gamma_{|i-j|}\}^{-1} \sigma_a^2 = \begin{bmatrix} \gamma_0 & \gamma_1 \cdots\cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 \ \cdots & \gamma_{p-2} \\ \gamma_{p-1} & \gamma_{p-2} \ \cdots & \gamma_0 \end{bmatrix} \sigma_a^2 \tag{5}$$

Where $\gamma_0, \gamma_1, \dots\dots, \gamma_{p-1}$ are the theoretical covariance of the process and $\left| M_p^{(p,0)} \right| = \left| M_n^{(p,0)} \right|$

In our work, we have three cases for the likelihood functions. Therefore, for **Equation 4**, we have three cases as follows:

Maximum Likelihood for AR (1)

The general likelihood function is $L = (2\pi\sigma_a^2)^{-\frac{n}{2}} \left| M_1^{(1)} \right|^{\frac{1}{2}} exp^{-\frac{S(\phi_1)}{2\sigma_a^2}}$. where $M_1^{(1)} = m_{11}$ $\Rightarrow m_{11}^{(1)} = \gamma_0^{-1} \sigma_a^2 = \frac{1}{\gamma_0} \sigma_a^2 = \frac{\sigma_a^2}{(1-\phi_1^2)\sigma_a^2}$, and $\gamma_0 = (1 - \phi_1^2)\sigma_a^2$. $M_1^{(1)} = (1 - \phi_1^2) \Rightarrow \left| M_1^{(1)} \right| = |(1 - \phi_1^2)| = (1 - \phi_1^2)$. As a result, the exact likelihood function is

$$L(\phi_1) = (2\pi\sigma_a^2)^{-\frac{n}{2}} |1 - \phi_1^2|^{\frac{1}{2}} \exp\{\frac{-1}{2\sigma_a^2} [(1 - \phi_1^2) z_1^2 + \sum_{t=2}^{n} (z_t - \phi_1 z_{t-1})^2]\} \tag{6}$$

The log likelihood of **Equation 6** is :

$$lnL(\phi_1) = -\frac{n}{2}\ \ln(\ 2\pi\sigma_a^2) + \frac{1}{2}\ln(1-\phi_1^2) - \frac{1}{2\sigma_a^2}[(1-\phi_1^2)z_1^2 + \sum_{t=2}^{n}(z_t - \phi_1 z_{t-1})^2] \quad (7)$$

Differentiate both sides of **Equation 7** with respect to ϕ_1:

$\frac{\partial \text{ Ln L}(\phi_1)}{\partial\ \phi_1} = \frac{-\phi_1}{1-\phi_1^2} + \frac{\sum_{t=2}^{n}(z_t-\phi_1 z_{t-1})z_{t-1}}{\sigma_a^2}$. Therefore, $\frac{\partial \text{ Ln L}(\phi_1)}{\partial\ \phi_1} = \frac{-\phi_1}{1-\phi_1^2}\sigma_a^2 + \sum_{t=2}^{n} z_t\, z_{t-1} + \sum_{t=2}^{n}\phi_1\, z_{t-1}^2$.

Set $\frac{\partial \text{ Ln L}(\phi_1)}{\partial\ \phi_1} = 0$.

$$\frac{-\phi_1}{1-\phi_1^2}\sigma_a^2 + \sum_{t=2}^{n} z_t\, z_{t-1} + \sum_{t=2}^{n}\phi_1\, z_{t-1}^2 = 0 \quad (8)$$

Since $\frac{\phi_1}{1-\phi_1^2}\sigma_a^2 = \gamma_1$, where $\gamma_1 = \phi_1\gamma_0$ and $\gamma_{0=}\frac{\sigma_a^2}{1-\phi_1^2}$. Then, **Equation 8** become

$$-\gamma_1 + \sum_{t=2}^{n} z_t\, z_{t-1} + \sum_{t=2}^{n}\phi_1\, z_{t-1}^2 = 0 \quad (9)$$

substitute $\gamma_1 = \frac{\sum_{t=2}^{n} z_t z_{t-1}}{n-1}$, in **Equation 9**

$\frac{\sum_{t=2}^{n} z_t z_{t-1}}{n-1} + \sum_{t=2}^{n} z_t\, z_{t-1} + \sum_{t=2}^{n}\phi_1\, z_{t-1}^2 = 0 \Longrightarrow \left(1-\frac{1}{n-1}\right)\sum_{t=2}^{n} z_t\, z_{t-1} = \phi_1\sum_{t=2}^{n} z_{t-1}^2 \Longrightarrow$ $(\frac{n-2}{n-1})\ \sum_{t=2}^{n} z_t\, z_{t-1} = \phi_1\sum_{t=2}^{n} z_{t-1}^2$.Hence, $\phi_1 = \frac{(\frac{n-2}{n-1})\ \sum_{t=2}^{n} z_t z_{t-1}}{\sum_{t=2}^{n} z_{t-1}^2} \Longrightarrow\ \ \hat{\phi}_1 = (\frac{n-2}{n-1})\frac{\sum_{t=2}^{n} z_t z_{t-1}}{\sum_{t=2}^{n} z_{t-1}^2}$.

Maximum Likelihood for AR(2)

For $= 2$, the exact likelihood function is

$$L(\phi_2) = (2\pi\sigma_a^2)^{-\frac{n}{2}}\left|M_2^{(2)}\right|^{\frac{1}{2}}\exp\{-\frac{S(\phi_2)}{2\sigma_a^2}\} \quad (10)$$

Then, **Equation 5** be $S(\phi_2) = m_{11}^{(1)}z_1^2 + m_{22}^{(2)}z_2^2 + \sum_{t=3}^{n}(z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2})^2$. Where $m_{11}^{(1)} = 1-\phi_1^2$, and $m_{22}^{(2)} = \left\{\gamma_{|i-j|}\right\}^{-1}\sigma_a^2\ \Longrightarrow\ {\gamma_0}^{-1}\sigma_a^2 =\ \frac{\sigma_a^2}{(1-\phi_2^2)^{-1}\sigma_a^2}\ =\ 1-\phi_2^2$.

Hence, $S(\phi_2)\ = (1-\phi_1^2)z_1^2 + (1-\phi_2^2)\ z_2^2 + \sum_{t=3}^{n}(z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2})^2 \quad (11)$

Then $M_2^{(2)} = \left\{m_{22}^{(2)}\right\} = 1-\phi_2^2 \Longrightarrow \left|M_2^{(2)}\right| = |1-\phi_2^2| = 1-\phi_2^2 \quad (12)$

Substitute **Equations 11** and **12** in **Equation 10**

$$L(\phi_2) = (2\pi\sigma_a^2)^{-\frac{n}{2}}|1-\phi_2^2|^{\frac{1}{2}}\exp\{\frac{-(1-\phi_1^2)z_1^2 + (1-\phi_2^2)\ z_2^2 + \sum_{t=3}^{n}(z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2})^2}{2\sigma_a^2}\} \quad (13)$$

The log likelihood of **Equation 13** is

$$Ln\ L(\phi_2) =\ \frac{-n}{2}\ln(2\pi\sigma_a^2) - \frac{1}{2}\ln(1-\phi_2^2\ ) - \frac{\sum_{t=3}^{n}(z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2})^2}{2\sigma_a^2} \quad (14)$$

Differentiate both sides of **Equation 14** with respect to ϕ_1:

$\frac{\partial \text{ Ln L}(\phi_2)}{\partial\ \phi_1} = \frac{\sum_{t=3}^{n} z_t z_{t-1} - \phi_1\sum_{t=3}^{n} z_{t-1}^2 - \phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2}}{\sigma_a^2}$ .Set $\frac{\partial \text{ Ln L}(\phi_2)}{\partial\ \phi_1} = 0$

$\frac{\sum_{t=3}^{n} z_t z_{t-1} - \phi_1\sum_{t=3}^{n} z_{t-1}^2 - \phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2}}{\sigma_a^2} = 0$. Therefore,

$$\phi_1 = \frac{\sum_{t=3}^{n} z_t z_{t-1} - \phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2}}{\sum_{t=3}^{n} z_{t-1}^2} \quad (15)$$

Differentiate both sides of **Equation 14** with respect to ϕ_2:

$$\frac{\partial \text{ Ln L}(\phi_2)}{\partial\ \phi_2} = -\ \frac{1}{\phi_2} + \frac{\sum_{t=3}^{n} z_t z_{t-2} - \phi_1\sum_{t=3}^{n} z_{t-1}z_{t-2} - \phi_2\sum_{t=3}^{n} z_{t-2}^2}{\sigma_a^2}$$
$$= \frac{-\sigma_a^2 + \phi_2\sum_{t=3}^{n} z_t z_{t-2} - \phi_1\phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2} - \phi_2^2\sum_{t=3}^{n} z_{t-2}^2}{\phi_2\sigma_a^2}.$$

Set $\frac{\partial \text{ Ln L}(\phi_2)}{\partial\ \phi_2} = 0$.

$$-\sigma_a^2 + \phi_2\sum_{t=3}^{n} z_t z_{t-2} - \phi_1\phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2} - \phi_2^2\sum_{t=3}^{n} z_{t-2}^2 = 0 \quad (16)$$

Substitute **Equation 15** in **Equation 16.**

$$-\sigma_a^2+\phi_2\sum_{t=3}^{n} z_t z_{t-2}-\left(\frac{\sum_{t=3}^{n} z_t z_{t-1}-\phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2}}{\sum_{t=3}^{n} z_{t-1}^2}\right)\phi_2\sum_{t=3}^{n} z_{t-1}z_{t-2}-\phi_2^2\sum_{t=3}^{n} z_{t-2}^2=0.$$

After simplification ,

$$\phi_2=\frac{\sigma_a^2\sum_{t=3}^{n} z_{t-1}^2}{\sum_{t=3}^{n} z_t z_{t-2}\sum_{t=3}^{n} z_{t-1}^2-\sum_{t=3}^{n} z_{t-1}z_{t-2}\sum_{t=3}^{n} z_t z_{t-1}} \tag{17}$$

Hence, $\hat{\phi}_2=\frac{\sigma_a^2\sum_{t=3}^{n} z_{t-1}^2}{\sum_{t=3}^{n} z_t z_{t-2}\sum_{t=3}^{n} z_{t-1}^2-\sum_{t=3}^{n} z_{t-1}z_{t-2}\sum_{t=3}^{n} z_t z_{t-1}}$. Substitute **Equation 17** in **Equation 15**. Then:

$\phi_1=\frac{\sum_{t=3}^{n} z_t z_{t-1}-\left(\frac{\sigma_a^2\sum_{t=3}^{n} z_{t-1}^2}{\sum_{t=3}^{n} z_t z_{t-2}\sum_{t=3}^{n} z_{t-1}^2-\sum_{t=3}^{n} z_{t-1}z_{t-2}\sum_{t=3}^{n} z_t z_{t-1}}\right)\sum_{t=3}^{n} z_{t-1}z_{t-2}}{\sum_{t=3}^{n} z_{t-1}^2}$. After simplification,

$\phi_1=\sum_{t=3}^{n} z_t z_{t-1}\sum_{t=3}^{n} z_t z_{t-2}-\sum_{t=3}^{n} z_{t-1}z_{t-2}\sum_{t=3}^{n} z_t^2 z_{t-1}^2-\sigma_a^2\sum_{t=3}^{n} z_{t-1}z_{t-2}$.

Hence. $\hat{\phi}_1=\sum_{t=3}^{n} z_t z_{t-1}\sum_{t=3}^{n} z_t z_{t-2}-\sum_{t=3}^{n} z_{t-1}z_{t-2}\sum_{t=3}^{n} z_t^2 z_{t-1}^2-\sigma_a^2\sum_{t=3}^{n} z_{t-1}z_{t-2}$.

Maximum Likelihood for AR (3)

For $p=3$, the exact likelihood function will be:

$$L(\phi_3)=(2\pi\sigma_a^2)^{-\frac{n}{2}}\left|M_3^{(3)}\right|^{\frac{1}{2}}\exp\left\{-\frac{S(\phi_3)}{2\sigma_a^2}\right\} \tag{18}$$

The **Equation 5** will be

$S(\phi_3)=m_{11}^{(1)}z_1^2+m_{22}^{(2)}z_2^2+m_{33}^{(3)}z_3^2+\sum_{t=4}^{n}(z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2}-\phi_3 z_{t-3})^2$. So we have $m_{11}^{(1)}=1-\phi_1^2$, $m_{22}^{(2)}=1-\phi_2^2$, and $m_{33}^{(3)}=\left\{\gamma_{|i-j|}\right\}^{-1}\sigma_a^2\Rightarrow\frac{\sigma_a^2}{(1-\phi_3^2)^{-1}\sigma_a^2}=1-\phi_3^2$.

Hence,

$$S(\phi_3)=(1-\phi_1^2)z_1^2+(1-\phi_2^2)z_2^2+(1-\phi_3^2)z_3^2+\sum_{t=4}^{n}(z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2}-\phi_3 z_{t-3})^2 \tag{19}$$

Then $\left|M_3^{(3)}\right|=|1-\phi_3^2|=1-\phi_3^2$ (20)

By substitute **Equations 19** and **20** in **Equation 18**, the exact likelihood function will be

$$L(\phi_3)=(2\pi\sigma_a^2)^{-\frac{n}{2}}|1-\phi_3^2|^{\frac{1}{2}}\exp\left\{-\frac{(1-\phi_1^2)z_1^2+(1-\phi_2^2)z_2^2+(1-\phi_3^2)z_3^2+\sum_{t=4}^{n}(z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2}-\phi_3 z_{t-3})^2}{2\sigma_a^2}\right\} \tag{21}$$

The log likelihood of **Equation 21** is

$$Ln\ L(\phi_3)=\frac{-n}{2}\ln(2\pi\sigma_a^2)-\frac{1}{2}\ln(1-\phi_3^2)-\frac{\sum_{t=4}^{n}(z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2}-\phi_3 z_{t-3})^2}{2\sigma_a^2} \tag{22}$$

Differentiate both sides of **Equation 22** with respect to ϕ_1:

$\frac{\partial\ \mathrm{Ln\ L}(\phi_3)}{\partial\ \phi_1}=\frac{\sum_{t=4}^{n} z_t z_{t-1}-\phi_1\sum_{t=4}^{n} z_{t-1}^2-\phi_2\sum_{t=4}^{n} z_{t-1}z_{t-2}-\phi_3\sum_{t=4}^{n} z_{t-1}z_{t-3}}{\sigma_a^2}$. We make $\frac{\partial\ \mathrm{Ln\ L}(\phi_3)}{\partial\ \phi_1}=0$, so we have

$\frac{\sum_{t=4}^{n} z_t z_{t-1}-\phi_1\sum_{t=4}^{n} z_{t-1}^2-\phi_2\sum_{t=4}^{n} z_{t-1}z_{t-2}-\phi_3\sum_{t=4}^{n} z_{t-1}z_{t-3}}{\sigma_a^2}=0$. After simplification,

$$\phi_1=\frac{\sum_{t=4}^{n} z_t z_{t-1}-\phi_2\sum_{t=4}^{n} z_{t-1}z_{t-2}-\phi_3\sum_{t=4}^{n} z_{t-1}z_{t-3}}{\sum_{t=4}^{n} z_{t-1}^2} \tag{23}$$

Differentiate both sides of **Equation 22** with respect to ϕ_2:

$\frac{\partial\ \mathrm{Ln\ L}(\phi_3)}{\partial\ \phi_2}=\frac{\sum_{t=4}^{n} z_t z_{t-2}-\phi_1\sum_{t=4}^{n} z_{t-1}z_{t-2}-\phi_2\sum_{t=4}^{n} z_{t-2}^2-\phi_3\sum_{t=4}^{n} z_{t-2}z_{t-3}}{\sigma_a^2}$. We make $\frac{\partial\ \mathrm{Ln\ L}(\phi_3)}{\partial\ \phi_2}=0$, so we get $\frac{\sum_{t=4}^{n} z_t z_{t-2}-\phi_1\sum_{t=4}^{n} z_{t-1}z_{t-2}-\phi_2\sum_{t=4}^{n} z_{t-2}^2-\phi_3\sum_{t=4}^{n} z_{t-2}z_{t-3}}{\sigma_a^2}=0$. After simplification,

$$\phi_2=\frac{\sum_{t=4}^{n} z_t z_{t-2}-\phi_1\sum_{t=4}^{n} z_{t-1}z_{t-2}-\phi_3\sum_{t=4}^{n} z_{t-2}z_{t-3}}{\sum_{t=4}^{n} z_{t-2}^2} \tag{24}$$

Differentiate both sides of **Equation 22** with respect to ϕ_3. Then

$\frac{\partial \text{ Ln L}(\phi_3)}{\partial \phi_3} = -\frac{1}{\phi_3} - \frac{2\sum_{t=4}^{n}(-z_t z_{t-3} - \phi_1 z_{t-1} z_{t-3} - \phi_2 z_{t-2} z_{t-3} - \phi_3 z_{t-3}^2)}{2\sigma_a^2}$. Also, $\frac{\partial \text{ Ln L}(\phi_3)}{\partial \phi_3} = \frac{-\sigma_a^2 + \phi_3 \sum_{t=4}^{n} z_t z_{t-3} - \phi_1 \phi_3 \sum_{t=4}^{n} z_{t-1} z_{t-3} - \phi_2 \phi_3 \sum_{t=4}^{n} z_{t-2} z_{t-3} - \phi_3^2 \sum_{t=4}^{n} z_{t-3}^2}{\phi_3 \sigma_a^2}$. Set

$\frac{\partial \text{ Ln L}(\phi_3)}{\partial \phi_3} = 0$. Then $\frac{-\sigma_a^2 + \phi_3 \sum_{t=4}^{n} z_t z_{t-3} - \phi_1 \phi_3 \sum_{t=4}^{n} z_{t-1} z_{t-3} - \phi_2 \phi_3 \sum_{t=4}^{n} z_{t-2} z_{t-3} - \phi_3^2 \sum_{t=4}^{n} z_{t-3}^2}{\phi_3 \sigma_a^2} = 0$.

$$-\sigma_a^2 + \phi_3 \sum_{t=4}^{n} z_t z_{t-3} - \phi_1 \phi_3 \sum_{t=4}^{n} z_{t-1} z_{t-3} - \phi_2 \phi_3 \sum_{t=4}^{n} z_{t-2} z_{t-3} - \phi_3^2 \sum_{t=4}^{n} z_{t-3}^2 = 0 \qquad (25)$$

Substitute **Equation 23** in **Equation 24** to get that

$\phi_2 = \frac{\sum_{t=4}^{n} z_t z_{t-2} - [\frac{\sum_{t=4}^{n} z_t z_{t-1} - \phi_2 \sum_{t=4}^{n} z_{t-1} z_{t-2} - \phi_3 \sum_{t=4}^{n} z_{t-1} z_{t-3}}{\sum_{t=4}^{n} z_{t-1}^2}] \sum_{t=4}^{n} z_{t-1} z_{t-2} - \phi_3 \sum_{t=4}^{n} z_{t-2} z_{t-3}}{\sum_{t=4}^{n} z_{t-2}^2}$. Thus,

$$\phi_3 = \frac{\sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-1} z_{t-2} - \sum_{t=4}^{n} z_t z_{t-2}}{\sum_{t=4}^{n} z_{t-1} z_{t-3} \sum_{t=4}^{n} z_{t-1} z_{t-2} - \sum_{t=4}^{n} z_{t-2} z_{t-3}} \qquad (26)$$

Hence,, $\hat{\phi}_3 = \frac{\sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-1} z_{t-2} - \sum_{t=4}^{n} z_t z_{t-2}}{\sum_{t=4}^{n} z_{t-1} z_{t-3} \sum_{t=4}^{n} z_{t-1} z_{t-2} - \sum_{t=4}^{n} z_{t-2} z_{t-3}}$. Substitute **Equation 26** in **Equation 24** then

$$\phi_2 = \frac{\sum_{t=4}^{n} z_t z_{t-2} - \phi_1 \sum_{t=4}^{n} z_{t-1} z_{t-2} - [\frac{\sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-1} z_{t-2} - \sum_{t=4}^{n} z_t z_{t-2}}{\sum_{t=4}^{n} z_{t-1} z_{t-3} \sum_{t=4}^{n} z_{t-1} z_{t-2} - \sum_{t=4}^{n} z_{t-2} z_{t-3}}] \sum_{t=4}^{n} z_{t-2} z_{t-3}}{\sum_{t=4}^{n} z_{t-2}^2}.$$

That is $\phi_2 = \frac{\sum_{t=4}^{n} z_t z_{t-2} - \sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-2} z_{t-3}}{\sum_{t=4}^{n} z_{t-2} z_{t-3} \sum_{t=4}^{n} z_{t-2}^2}$ (27)

Hence, $\hat{\phi}_2 = \frac{\sum_{t=4}^{n} z_t z_{t-2} - \sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-2} z_{t-3}}{\sum_{t=4}^{n} z_{t-2} z_{t-3} \sum_{t=4}^{n} z_{t-2}^2}$. Substitute **Equations 26** and **27** in **Equation 23** to consider the following:

$$\hat{\phi}_1 = \frac{\sum_{t=4}^{n} z_t z_{t-1} - \sum_{t=4}^{n} z_t z_{t-2} \sum_{t=4}^{n} z_{t-1} z_{t-2} + \sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-2} z_{t-3}}{\sum_{t=4}^{n} z_{t-1}^2} - \frac{\sum_{t=4}^{n} z_t z_{t-1} \sum_{t=4}^{n} z_{t-1} z_{t-2} \sum_{t=4}^{n} z_{t-1} z_{t-3} + \sum_{t=4}^{n} z_t z_{t-2} \sum_{t=4}^{n} z_{t-1} z_{t-3}}{\sum_{t=4}^{n} z_{t-1}^2}$$

3.1. Marginal distribution of AR time series in case known error distribution

In (23-26), the general form for the random shock form is

$$Z_t = \Psi(B) a_t \qquad (28)$$

where a_t is a random error,

$\Psi(B) = 1 + \Psi_1 B + \Psi_2 B^2 + \cdots$,

and B is a back shift operator that is ($B^j Z_t = Z_{t-j}, j = 1,2,3,\cdots$.)

Then $Z_t = (1 + \Psi_1 B + \Psi_2 B^2 + \cdots) a_t$.

Using back shift operator, give us $Z_t = a_t + \Psi_1 a_{t-1} + \Psi_2 a_{t-2} + \cdots$.

Then $Z_t = \sum_{j=0}^{\infty} \Psi_j a_{t-j}$, where $\Psi_0 = 1$. Since

$$\varphi(B) Z_t = a_t \qquad (29)$$

Substituting **Equation 28** in **Equation 29**, we get

$\varphi(B)\Psi(B) a_t = a_t$

$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)(1 + \Psi_1 B + \Psi_2 B^2 + \cdots) = 1$

Z_t of $AR(1)$

$(1 - \phi_1 B)(1 + \Psi_1 B + \Psi_2 B^2 + \cdots) = 1$

$\Psi_1 = \phi_1$

$\Psi_2 = {\phi_1}^2$

$\vdots$

$\Psi_j = \phi_1^J$.

Hence,

$$\mathrm{Z_t} = \sum_{J=1}^{\infty} \phi_1^J \ \mathrm{a_{t-J}} \tag{30}$$

Z_t of $AR(2)$

$(1-\phi_1 B-\phi_2 B^2)(1+\Psi_1 B+\Psi_2 B^2+\cdots)=1$

$\Psi_1 = \phi_1$

$\Psi_2 = {\phi_1}^2 + \phi_2$

$\vdots$

$\Psi_j = \phi_1^J + \phi_1^{J-2}\phi_2$

Then, $,\mathrm{Z_t} = \sum_{J=2}^{\infty} (\phi_1^J + \phi_1^{J-2}\phi_2)\ \mathrm{a_{t-J}}$ where $\Psi_1 = \phi_1$ (31)

Z_t of $AR(3)$

$(1-\phi_1 B-\phi_2 B^2-\phi_3 B^3)(1+\Psi_1 B+\Psi_2 B^2+\cdots)=1$

$\Psi_1 = \phi_1$

$\Psi_2 = {\phi_1}^2 + \phi_2$

$\Psi_3 = {\phi_1}^3 + 2\phi_1\phi_2 + \phi_3$

$\vdots$

$\Psi_j = \phi_1^J + 2\phi_1^{J-2}\phi_2 + \phi_1^{J-3}\phi_3$

Then,

$\mathrm{Z_t} = \sum_{J=3}^{\infty}\left(\phi_1^J + 2\phi_1^{J-2}\phi_2 + \phi_1^{J-3}\phi_3\right)\mathrm{a_{t-J}}$ where $\Psi_1 = \phi_1$ and $\Psi_2 = {\phi_1}^2 + \phi_2$ (32)

Assuming that $\{a_t\}$ represents random errors that are uniformly distributed. It received a mean of zero and a variance of σ_a^2 that is $\mathrm{a_t} \sim \mathrm{i.i.d}\ \mathrm{N}(0, \sigma_a^2)$.

It finds the marginal distribution for $\{\mathrm{Z_t}\}$ by using the characteristic function if the data follow the autoregressive model $AR(p)$.

The $AR(p)$ model can be written in terms of random errors according to **Equations 30, 31, 32**, and $\Psi_{z_t}(s) = E\left(e^{isz_t}\right)$. For $\Psi_{z_t}(s)$, we have the following cases:

$\Psi_{z_t}(s)$ of AR(1)

$\Psi_{z_t}(s) = E\left[e^{isz_t}\right] \Rightarrow \Psi_{z_t}(s) = \mathrm{E}\left[e^{is\sum_{J=0}^{\infty}\phi_p^J\ \mathrm{a_{t-J}}}\right] \Rightarrow \Psi_{z_t}(s) =$ $\mathrm{E}\left[e^{is\mathrm{a_t} + \mathrm{is}\,\phi_1\ \mathrm{a_{t-1}} + \mathrm{is}\,\phi_1^2\ \mathrm{a_{t-2}} + \cdots}\right]$. So,

$\Psi_{z_t}(s) = \prod_{k=0}^{\infty} \Psi_{a_t}(\phi_1^k)$

When $\{a_t\} \sim N(0, \sigma_a^2)$, the character function for errors will be

$\Psi_{z_t}(s) = \prod_{k=0}^{\infty} e^{-\frac{1}{2}\sigma_a^2 s^2}(\phi_1^k)$, where $\Psi_{a_t}(s) = e^{-\frac{1}{2}\sigma_a^2 s^2}$

$\Psi_{z_t}(s) = \prod_{k=0}^{\infty} e^{-\frac{1}{2}\phi_1^{2k}\sigma_a^2 s^2} = e^{-\frac{1}{2}\sigma_a^2 s^2[1+\phi_1^2+\phi_1^4+\cdots]}$

$$\Psi_{z_t}(s) = e^{-\frac{1}{2}\sigma_a^2 s^2\left[\frac{1}{1-\phi_1^2}\right]} \tag{33}$$

Then, $Z_t \sim N\left(0, \frac{\sigma_a^2}{1-\phi_1^2}\right)$.

$\Psi_{z_t}(s)$ of AR(2)

Since $\mathrm{Z_t} = \sum_{J=2}^{\infty}(\phi_1^J + \phi_1^{J-1}\phi_2)\ \mathrm{a_{t-J}}$

Then, $\Psi_{z_t}(s) = E\left[e^{isz_t}\right] = \mathrm{E}\left[e^{is\sum_{J=2}^{\infty}(\phi_1^J+\phi_1^{J-2}\phi_2)\ \mathrm{a_{t-J}}}\right]$

$\Psi_{z_t}(s) =$

$\mathrm{E}\left[e^{is\left(({\phi_1}^2+\phi_2)a_{t-2}+({\phi_1}^3+\phi_1\phi_2)a_{t-3+\cdots}\right)}\right] = \mathrm{E}\left[e^{is({\phi_1}^2+\phi_2)a_{t-2}}\right].\ \mathrm{E}\left[e^{is\left(\phi_1({\phi_1}^2+\phi_2)\right)a_{t-3}}\right]\ldots.$

$$\Psi_{z_t}(s) = \prod_{k=0}^{\infty}\Psi_{a_t}\left({\phi_1}^{k-2}\left({\phi_1}^k + \phi_2\right)\right) \tag{34}$$

When $\{a_t\} \sim N(0, \sigma_a^2)$, since the character function for errors is

$\Psi_{a_t}(s) = e^{-\frac{1}{2}\sigma_a^2 s^2}$

Hence, $\Psi_{z_t}(s) = \prod_{k=2}^{\infty} e^{-\frac{1}{2}\left({\phi_1}^{k-2}\left({\phi_1}^{k}+\phi_2\right)\right)\sigma_a^2 s^2} \Rightarrow \Psi_{z_t}(s) = e^{-\frac{1}{2}\sigma_a^2 s^2\left[\frac{{\phi_1}^2+\phi_2}{1-\phi_1}\right]}$ (35)

Then, $Z_t \sim N\left(0, \frac{({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}\right)$

$\Psi_{z_t}(s)$ of AR(3)

Since $\mathrm{Z_t} = \sum_{J=3}^{\infty}\left(\phi_1^J + 2\phi_1^{J-2}\phi_2 + \phi_1^{J-3}\phi_3\right)\mathrm{a_{t-J}}$ .Then

$\Psi_{z_t}(s) = E\left[e^{isz_t}\right] = \mathrm{E}\left[e^{is\sum_{J=3}^{\infty}\left(\phi_1^J+2\phi_1^{J-2}\phi_2+\phi_1^{J-3}\phi_3\right)\mathrm{a_{t-J}}}\right]$

Then , $\Psi_{z_t}(s) = E\left[e^{is\left(({\phi_1}^3+2\phi_1\phi_2+\phi_3)a_{t-3}+({\phi_1}^4+2{\phi_1}^2\phi_2+\phi_1\phi_3)a_{t-2}+\cdots\right)}\right]$

Hence , $\Psi_{z_t}(s) = \prod_{k=0}^{\infty}\Psi_{a_t}\left({\phi_1}^{k-3}({\phi_1}^{k} + 2\phi_1\phi_2 + \phi_3\right)$ (36)

where $\{a_t\} \sim N(0, \sigma_a^2)$, The character function for errors is

$\Psi_{a_t}(s) = e^{-\frac{1}{2}\sigma_a^2 s^2}$

Hence , $\Psi_{z_t}(s) = \prod_{k=3}^{\infty} e^{-\frac{1}{2}\left(\frac{\left({\phi_1}^3+2\phi_1\phi_2+\phi_3\right)}{1-\phi_1}\right)\sigma_a^2 s^2}$

$\Psi_{z_t}(s) = e^{-\frac{1}{2}\sigma_a^2 s^2\left[\frac{{\phi_1}^3+2\phi_1\phi_2+\phi_3}{1-\phi_1}\right]}$ (37)

Then, $Z_t \sim N\left(0, \frac{({\phi_1}^3+2\phi_1\phi_2+\phi_3)\sigma_a^2}{1-\phi_1}\right)$.

3.2 Maximum likelihood function with marginal distribution

The Normal distribution function is

$f(z, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{(z-\mu)^2}{2\sigma_z^2}}$ where $z \sim N(\mu, \sigma_z^2)$

By using marginal distribution $\Psi_{z_t}(s)$ for z_t when $p = 1,2,$and 3 , where z_t is normal distribution by mean zero and variance σ_z^2(27-30).

The maximum likelihood function when $p = 1$ and $\sigma_z^2 = \frac{\sigma_a^2}{1-\phi_1^2}$

$L(z, \phi_1) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{(z-\mu)^2}{2\sigma_z^2}}$ where $z \sim N(\mu, \sigma_z^2)$

$L(z, \phi_1) = f(z_1, \phi_1).f(z_2, \phi_1).f(z_3, \phi_1)\cdots f(z_n, \phi_1) = \prod_{i=1}^{n} f(z_i, \phi_1)$

$L(z, \phi_1) = \left(2\pi.\frac{\sigma_a^2}{1-\phi_1^2}\right)^{\frac{-n}{2}} e^{-\frac{\sum_{i=1}^{n} z_i^2}{\frac{2\sigma_a^2}{1-\phi_1^2}}}$ (38)

The log likelihood of **Equation 38** is:

$$lnL(z, \phi_1) = \frac{-n}{2}ln\left(2\pi.\frac{\sigma_a^2}{1-\phi_1^2}\right) - \frac{\sum_{i=1}^{n} z_i^2}{\frac{2\sigma_a^2}{1-\phi_1^2}}$$

$lnL(z, \phi_1) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\frac{\sigma_a^2}{1-\phi_1^2}\right) - \frac{(1-\phi_1^2)\sum_{i=1}^{n} z_i^2}{2\sigma_a^2}$ (39)

Differentiate both sides of **Equation 39** with respect to ϕ_1.

$\frac{\partial \ln L(z,\phi_1)}{\partial \phi_1} = \frac{n\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma_a^2}\sum_{i=1}^{n} z_i^2$

$\frac{\partial \ln \mathrm{L}(z,\phi_1)}{\partial \phi_1} = \frac{n\phi_1\sigma_a^2+\phi_1(1-\phi_1^2)\sum_{i=1}^{n} z_i^2}{\sigma_a^2(1-\phi_1^2)}$ (40)

Make **Equation 40** equals to zero.

$n\phi_1\sigma_a^2 + \phi_1(1-\phi_1^2)\sum_{i=1}^{n} z_i^2 = 0 \Rightarrow n\sigma_a^2 + \sum_{i=1}^{n} z_i^2 = \phi_1^2\sum_{i=1}^{n} z_i^2$

Then, $\phi_1^2 = \frac{n\sigma_a^2+\sum_{i=1}^n z_i^2}{\sum_{i=1}^n z_i^2} \Longrightarrow \phi_1 = \sqrt{\frac{n\sigma_a^2+\sum_{i=1}^n z_i^2}{\sum_{i=1}^n z_i^2}} \Longrightarrow \hat{\phi}_1 = \sqrt{\frac{n\sigma_a^2+\sum_{i=1}^n z_i^2}{\sum_{i=1}^n z_i^2}}$

The maximum likelihood function when $p = 2$ and $\sigma_z^2 = \frac{({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}$

$L(z, \phi_1, \phi_2) = \frac{1}{\sqrt{2\pi\sigma_z^2}}\, e^{-\frac{(z-\mu)^2}{2\sigma_z^2}}$ where $z \sim N(\mu, \sigma_z^2)$

$L(z, \phi_1, \phi_2) = f(z_1, \phi_1, \phi_2).f(z_2, \phi_1, \phi_2).\cdots f(z_n, \phi_1, \phi_2) = \prod_{i=1}^n f(z_i, \phi_1, \phi_2)$

$$L(z, \phi_1, \phi_2) = \left(2\pi.\frac{({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}\right)^{\frac{-n}{2}} e^{-\frac{\sum_{i=1}^n z_i^2}{2\frac{({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}}} \quad (41)$$

The log likelihood of **Equation 41** is

$$lnL(z, \phi_1, \phi_2) = \frac{-n}{2} ln\left(2\pi.\frac{({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}\right) - \frac{\sum_{i=1}^n z_i^2}{\frac{2({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}}$$

$$lnL(z, \phi_1, \phi_2) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\frac{({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}\right) - \frac{\sum_{i=1}^n z_i^2}{\left(\frac{2({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}\right)}$$

$$lnL(z, \phi_1, \phi_2) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(({\phi_1}^2+\phi_2)\sigma_a^2\right) - \frac{-n}{2}\ln(1-\phi_1) - \frac{\sum_{i=1}^n z_i^2}{\left(\frac{2({\phi_1}^2+\phi_2)\sigma_a^2}{1-\phi_1}\right)} \quad (42)$$

Differentiate both sides of **Equation 42** with respect to ϕ_1.

$$\frac{\partial lnL(z,\phi_1,\phi_2)}{\partial \phi_1} = \frac{n\phi_1}{{\phi_1}^2+\phi_2} - \frac{n}{2(1-\phi_1)} + \frac{({\phi_1}^2+\phi_2)+(2\phi_1+2{\phi_1}^2)\sum_{i=1}^n z_i^2}{2\sigma_a^2({\phi_1}^2+\phi_2)^2}$$

By simplification,

$$\frac{\partial lnL(z,\phi_1,\phi_2)}{\partial \phi_1} = \frac{2n\phi_1\sigma_a^2(1-\phi_1) - n\sigma_a^2({\phi_1}^2-\phi_2) - \left((1-2\phi_1+2{\phi_1}^2)\right)\sum_{i=1}^n z_i^2}{2\sigma_a^2(1-\phi_1)({\phi_1}^2+\phi_2)} \quad (43)$$

Make **Equation 43** equals to zero.

$$2n\phi_1\sigma_a^2 + n{\phi_1}^2\sigma_a^2 + n\phi_2\sigma_a^2 - \left((1-3\phi_1+4{\phi_1}^2-3{\phi_1}^3)\right)\sum_{i=1}^n z_i^2 = 0$$

$$n\phi_2\sigma_a^2 = 2n\phi_1\sigma_a^2 - n{\phi_1}^2\sigma_a^2 + \left((1-3\phi_1+4{\phi_1}^2-3{\phi_1}^3)\right)\sum_{i=1}^n z_i^2$$

$$\phi_2 = \frac{2n\phi_1\sigma_a^2 - n{\phi_1}^2\sigma_a^2 + \left((1-3\phi_1+4{\phi_1}^2-3{\phi_1}^3)\right)\sum_{i=1}^n z_i^2}{n\sigma_a^2} \quad (44)$$

Differentiate both sides of **Equation 42** with respect to ϕ_2.

$$\frac{\partial lnL(z,\phi_1,\phi_2)}{\partial \phi_2} = \frac{-n}{2({\phi_1}^2+\phi_2)} + \frac{(1-\phi_1)\sum_{i=1}^n z_i^2}{2\sigma_a^2({\phi_1}^2+\phi_2)^2}$$

$$\frac{\partial lnL(z,\phi_1,\phi_2)}{\partial \phi_2} = \frac{-2n\sigma_a^2({\phi_1}^2+\phi_2)+(1-\phi_1)\sum_{i=1}^n z_i^2}{2\sigma_a^2({\phi_1}^2+\phi_2)^2} \quad (45)$$

Make **Equation 45** equals to zero,

$$-2n\sigma_a^2({\phi_1}^2+\phi_2) + (1-\phi_1)\sum_{i=1}^n z_i^2 = 0$$

$$\Longrightarrow \; -2n{\phi_1}^2\sigma_a^2 - 2n\sigma_a^2\phi_2 + \sum_{i=1}^n z_i^2 - \phi_1\sum_{i=1}^n z_i^2 = 0$$

Hence , $\phi_2 = \frac{=2n{\phi_1}^2\sigma_a^2 + \sum_{i=1}^n z_i^2 - \phi_1\sum_{i=1}^n z_i^2}{2n\sigma_a^2}$ (46)

Substitute **Equation 44** in **Equation 46**;

$$2n\sigma_a^2\left(2n\phi_1\sigma_a^2 - n{\phi_1}^2\sigma_a^2 + \left((1-3\phi_1+4{\phi_1}^2-3{\phi_1}^3)\right)\sum_{i=1}^n z_i^2\right)$$
$$= n\sigma_a^2\left(2n{\phi_1}^2\sigma_a^2 + \sum_{i=1}^n z_i^2 - \phi_1\sum_{i=1}^n z_i^2\right)$$

$$4n^2\phi_1(\sigma_a^2)^2 + n\,\sigma_a^2\sum_{i=1}^{n} z_i^2 - 5n\phi_1\sigma_a^2\sum_{i=1}^{n} z_i^2 = 0$$

After simplification,

$$\phi_1 = \frac{-\sum_{i=1}^{n} z_i^2}{4n\sigma_a^2 - 5\sum_{i=1}^{n} z_i^2} \Longrightarrow \hat{\phi}_1 = \frac{-\sum_{i=1}^{n} z_i^2}{4n\sigma_a^2 - 5\sum_{i=1}^{n} z_i^2} \tag{47}$$

Substitute **Equation 47** in Equation **46**

$$\phi_2 = \frac{=2n\sigma_a^2\left(\frac{-\sum_{i=1}^{n} z_i^2}{4n\sigma_a^2-5\sum_{i=1}^{n} z_i^2}\right)^2 + \sum_{i=1}^{n} z_i^2 - \left(\frac{-\sum_{i=1}^{n} z_i^2}{4n\sigma_a^2-5\sum_{i=1}^{n} z_i^2}\right)\sum_{i=1}^{n} z_i^2}{2n\sigma_a^2} \Longrightarrow$$

$$\phi_2 = \frac{\sigma_a^2(\sum_{i=1}^{n} z_i^2)^2 + 2\sigma_a^2\sum_{i=1}^{n} z_i^2 - 5(\sum_{i=1}^{n} z_i^2)^2}{\sigma_a^2(4n\sigma_a^2 - 5\sum_{i=1}^{n} z_i^2)}$$

Hence , $\hat{\phi}_2 = \frac{\sigma_a^2\left(\sum_{i=1}^{n} z_i^2\right)^2 + 2\sigma_a^2\sum_{i=1}^{n} z_i^2 - 5\left(\sum_{i=1}^{n} z_i^2\right)^2}{\sigma_a^2\left(4n\sigma_a^2 - 5\sum_{i=1}^{n} z_i^2\right)}$

The maximum likelihood function when $p = 3$ and $\sigma_z^2 = \frac{(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}$

$$L(z, \phi_1, \phi_2, \phi_3) = \frac{1}{\sqrt{2\pi\sigma_z^2}}\, e^{-\frac{(z-\mu)^2}{2\sigma_z^2}} \quad \text{where } z \sim N(\mu, \sigma_z^2)$$

$$L(z, \phi_1, \phi_2, \phi_3) = f(z_1, \phi_1, \phi_2, \phi_3).f(z_2, \phi_1, \phi_2, \phi_3).\cdots f(z_n, \phi_1, \phi_2, \phi_3) = \prod_{i=1}^{n} f(z_i, \phi_1, \phi_2, \phi_3)$$

$$L(z, \phi_1, \phi_2, \phi_3) = \left(2\pi.\frac{(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}\right)^{\frac{-n}{2}} e^{-\frac{\sum_{i=1}^{n} z_i^2}{2\frac{(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}}} \tag{48}$$

The log likelihood of **Equation 41** is

$$lnL(z, \phi_1, \phi_2, \phi_3) = \frac{-n}{2} ln\left(2\pi.\frac{(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}\right) - \frac{\sum_{i=1}^{n} z_i^2}{\frac{2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}}$$

$$lnL(z, \phi_1, \phi_2, \phi_3) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\frac{(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}\right) - \frac{\sum_{i=1}^{n} z_i^2}{\left(\frac{2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}\right)}$$

$$lnL(z, \phi_1, \phi_2, \phi_3) = \frac{-n}{2} ln(2\pi) - \frac{n}{2} ln\left((\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2\right)$$

$$+\frac{n}{2} ln(1-\phi_1) - \frac{\sum_{i=1}^{n} z_i^2}{\left(\frac{2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\sigma_a^2}{1-\phi_1}\right)} \tag{49}$$

Differentiate both sides of **Equation 49** with respect to ϕ_1.

$$\frac{\partial lnL(z, \phi_1, \phi_2, \phi_3)}{\partial\,\phi_1} = \frac{-n(3\phi_1{}^2 + 2\phi_2)}{2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)} + \frac{n}{2(1-\phi_1)} + \frac{\left((\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3) + (1-\phi_1)(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)\right)\sum_{i=1}^{n} z_i^2}{2\sigma_a^2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)^2}$$

By simplification,

$$\frac{\partial lnL(z, \phi_1, \phi_2, \phi_3)}{\partial\,\phi_1} = \frac{-n\sigma_a^2(3\phi_1{}^2 + 2\phi_2) + \sum_{i=1}^{n} z_i^2 + (1-\phi_1)(3\phi_1{}^2 + 2\phi_2)\sum_{i=1}^{n} z_i^2 + n\sigma_a^2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)}{2\sigma_a^2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3)} \tag{50}$$

Make Equati**on 50** equals to zero ,

$$-n\sigma_a^2(3\phi_1{}^2 + 2\phi_2) + \sum_{i=1}^{n} z_i^2 + (1-\phi_1)(3\phi_1{}^2 + 2\phi_2)\sum_{i=1}^{n} z_i^2 + n\sigma_a^2(\phi_1{}^3 + 2\phi_1\phi_2 + \phi_3) = 0$$

$$n\phi_3\sigma_a^2 = -(n\sigma_a^2 - 3\sum_{i=1}^{n} z_i^2)\phi_1{}^3 + (3n\sigma_a^2 + 3\sum_{i=1}^{n} z_i^2)\phi_1{}^2 - (2n\sigma_a^2 - 2\sum_{i=1}^{n} z_i^2)\,\phi_1\phi_2 - 2\,\phi_2\sum_{i=1}^{n} z_i^2$$

$$\phi_3 = \frac{-(n\sigma_a^2 - 3\sum_{i=1}^{n} z_i^2)\phi_1{}^3 + (3n\sigma_a^2 + 3\sum_{i=1}^{n} z_i^2)\phi_1{}^2 - (2n\sigma_a^2 - 2\sum_{i=1}^{n} z_i^2)\,\phi_1\phi_2 - 2\,\phi_2\sum_{i=1}^{n} z_i^2}{n\sigma_a^2} \tag{51}$$

Differentiate both sides of **Equation 49** with respect to ϕ_2. Then

$$\frac{\partial lnL(z,\phi_1,\phi_2,\phi_3)}{\partial\,\phi_2}=\frac{-n\phi_1}{\left({\phi_1}^3+\,2\phi_1\phi_2+\phi_3\right)}+\frac{\phi_1(1-\phi_1)\sum_{i=1}^{n}z_i^2}{\sigma_a^2\left({\phi_1}^3+\,2\phi_1\phi_2+\phi_3\right)^2}$$

$$\frac{\partial lnL(z,\phi_1,\phi_2,\phi_3)}{\partial\,\phi_2}=\frac{-n{\phi_1}^4\sigma_a^2-2n{\phi_1}^2\phi_2\sigma_a^2-n\phi_1\phi_3\sigma_a^2-\phi_1(1-\phi_1)\sum_{i=1}^{n}z_i^2}{\sigma_a^2(1-\phi_1)\left({\phi_1}^3+2\phi_1\phi_2+\phi_3\right)} \qquad (52)$$

Make **Equation 52** equals to zero ,

$$-n{\phi_1}^4\sigma_a^2-2n{\phi_1}^2\phi_2\sigma_a^2-n\phi_1\phi_3\sigma_a^2-\phi_1(1-\phi_1)\sum_{i=1}^{n}z_i^2=0$$

$$n\phi_1\phi_3\sigma_a^2=-n{\phi_1}^4\sigma_a^2-2n{\phi_1}^2\phi_2\sigma_a^2-\phi_1(1-\phi_1)\sum_{i=1}^{n}z_i^2$$

Hence , $\phi_3=\frac{-n{\phi_1}^3\,\sigma_a^2-2n\,\phi_1\phi_2\sigma_a^2-(1-\phi_1)\sum_{i=1}^{n}z_i^2}{2n\sigma_a^2}$ (53)

Differentiate both sides of Equation (49) with respect to ϕ_3. Then

$$\frac{\partial lnL(z,\phi_1,\phi_2,\phi_3)}{\partial\,\phi_3}=\frac{n}{2\left({\phi_1}^3+\,2\phi_1\phi_2+\phi_3\right)}+\frac{(1-\phi_1)\sum_{i=1}^{n}z_i^2}{2\sigma_a^2\left({\phi_1}^3+\,2\phi_1\phi_2+\phi_3\right)^2}$$

$$\frac{\partial lnL(z,\phi_1,\phi_2,\phi_3)}{\partial\,\phi_3}=\frac{-n{\phi_1}^3\sigma_a^2-2n\,\phi_1\phi_2\sigma_a^2-n\phi_3\sigma_a^2+(1-\phi_1)\sum_{i=1}^{n}z_i^2}{2\sigma_a^2\left({\phi_1}^3+2\phi_1\phi_2+\phi_3\right)} \qquad (54)$$

Make **Equation 54** equals to zero ,

$$-n{\phi_1}^3\sigma_a^2-2n\,\phi_1\phi_2\sigma_a^2-n\phi_3\sigma_a^2+(1-\phi_1)\sum_{i=1}^{n}z_i^2=0$$

$$-n{\phi_1}^3\sigma_a^2-n\phi_3\sigma_a^2+(1-\phi_1)\sum_{i=1}^{n}z_i^2=2n\,\phi_1\phi_2\sigma_a^2$$

$$\phi_2=\frac{-n{\phi_1}^3\sigma_a^2-n\phi_3\sigma_a^2+(1-\phi_1)\sum_{i=1}^{n}z_i^2}{2n\phi_1\sigma_a^2} \qquad (55)$$

Substitute **Equation 55** in **Equation 54**;

$$\phi_3=\frac{n\phi_3\sigma_a^2-(1-\phi_1)\sum_{i=1}^{n}z_i^2-(1-\phi_1)\sum_{i=1}^{n}z_i^2}{2n\sigma_a^2}$$

After simplification,

$$\phi_3=\frac{2\phi_1\sum_{i=1}^{n}z_i^2-2\sum_{i=1}^{n}z_i^2}{n\sigma_a^2} \qquad (56)$$

Substitute **Equation 56** in **Equation 55**;

$$\phi_2=\frac{-n{\phi_1}^3\sigma_a^2-\phi_1\sum_{i=1}^{n}z_i^2+2\sum_{i=1}^{n}z_i^2+\sum_{i=1}^{n}z_i^2-\phi_1\sum_{i=1}^{n}z_i^2}{2n\phi_1\sigma_a^2}$$

$$\phi_2=\frac{-n{\phi_1}^3\sigma_a^2-3\phi_1\sum_{i=1}^{n}z_i^2+3\sum_{i=1}^{n}z_i^2}{2n\phi_1\sigma_a^2} \qquad (57)$$

Substitute **Equation 57** in **Equation 55**;

$$-6n{\phi_1}^2\sigma_a^2\sum_{i=1}^{n}z_i^2+6n\phi_1\sigma_a^2\sum_{i=1}^{n}z_i^2+\,2n^2\phi_1\phi_3(\sigma_a^2)^2-2n\phi_1\sigma_a^2\sum_{i=1}^{n}z_i^2+2n{\phi_1}^2\sigma_a^2\sum_{i=1}^{n}z_i^2=0$$

By simplification,

$$2n{\phi_1}^2\sigma_a^2\sum_{i=1}^{n}z_i^2+4\mathrm{n}\phi_1\sigma_a^2\sum_{i=1}^{n}z_i^2=\,2n^2\phi_1\phi_3(\sigma_a^2)^2\Longrightarrow\phi_3=\frac{2\phi_1\sum_{i=1}^{n}z_i^2+2\sum_{i=1}^{n}z_i^2}{n\sigma_a^2}$$

Hence, $\phi_1=\frac{n\phi_3\sigma_a^2-2\sum_{i=1}^{n}z_i^2}{2\sum_{i=1}^{n}z_i^2}$ (58)

Substitute **Equation 58** in **Equation 56**;

$$\phi_3=\frac{2n\phi_3\sigma_a^2\sum_{i=1}^{n}z_i^2-4(\sum_{i=1}^{n}z_i^2)^2-2\sum_{i=1}^{n}z_i^2}{n\sigma_a^2}$$

$$\phi_3\,n\sigma_a^2-2n\phi_3\sigma_a^2\sum_{i=1}^{n}z_i^2=-4(\sum_{i=1}^{n}z_i^2)^2-2\sum_{i=1}^{n}z_i^2$$

$$\phi_3 = \frac{-4\left(\sum_{i=1}^{n} z_i^2\right)^2 - 2\sum_{i=1}^{n} z_i^2}{n\sigma_a^2\left(1-2\sum_{i=1}^{n} z_i^2\right)} \tag{59}$$

Therefore, $\widehat{\phi}_3 = \frac{-4\left(\sum_{i=1}^{n} z_i^2\right)^2 - 2\sum_{i=1}^{n} z_i^2}{n\sigma_a^2\left(1-2\sum_{i=1}^{n} z_i^2\right)}$

Substitute **Equation 59** in **Equation 58**;

$$\phi_1 = \frac{n\sigma_a^2\left(\frac{-4\left(\sum_{i=1}^{n} z_i^2\right)^2 - 2\sum_{i=1}^{n} z_i^2}{n\sigma_a^2\left(1-2\sum_{i=1}^{n} z_i^2\right)}\right) - 2\sum_{i=1}^{n} z_i^2}{2\sum_{i=1}^{n} z_i^2} \Longrightarrow \phi_1 = \frac{-2}{1-2\sum_{i=1}^{n} z_i^2} \tag{60}$$

Hence , $\hat{\phi}_1 = \frac{-2}{1-2\sum_{i=1}^{n} z_i^2}$

Substitute **Equation 60** in **Equation 57**;

$$\phi_2 = \frac{-n\sigma_a^2\left(\frac{-2}{1-2\sum_{i=1}^{n} z_i^2}\right)^3 - 3\sum_{i=1}^{n} z_i^2\left(\frac{-2}{1-2\sum_{i=1}^{n} z_i^2}\right) + 3\sum_{i=1}^{n} z_i^2}{2n\sigma_a^2\left(\frac{-2}{1-2\sum_{i=1}^{n} z_i^2}\right)}$$

By simplification,

$$\phi_2 = \frac{8n\sigma_a^2 + 9\sum_{i=1}^{n} z_i^2\,(1-2\sum_{i=1}^{n} z_i^2)^2}{-4n\sigma_a^2(1-2\sum_{i=1}^{n} z_i^2)^2}$$

Hence,

$$\hat{\phi}_2 = \frac{8n\sigma_a^2 + 9\sum_{i=1}^{n} z_i^2\,(1-2\sum_{i=1}^{n} z_i^2)^2}{-4n\sigma_a^2(1-2\sum_{i=1}^{n} z_i^2)^2}$$

4. Simulation and Results.

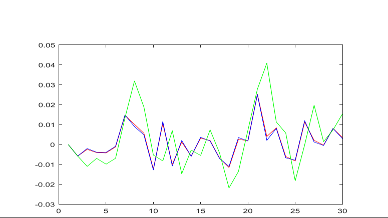
This study presents AR model estimation via maximum likelihood methods by defining marginal distribution of the given time series, for this some of simulated experiments designed ($r = 100, 500$, and 1000). Different initial values for the parameters $\phi_p$were assumed for each experiment. The results of these experiments are given in **Tables 1-9** for some statistical criteria, mean square error (MSE), and the absolute mean percentage error (MPE).

The first experiment was designed to calculate MLE and MAR for time series values for $AR\,(1)$, with ($\phi_1 = 0.1,\ 0.3, 0.5$, and -0.7), sample size ($n = 30,\ 50$, and 100), and number of replicate ($r\ = 100,\ 500$, and 1000), the results of this experiment are given by **Tables 1, 2**, and **3**. Regarding the MSE and MPE criteria, we can see from the given tables , the values of MSE and MPE decrease with increasing sample size n for each value of ϕ_p for MLE and MAR. The values of MPE of both methods become closer as the sample size increases for different values of ϕ_p. The MSE and MPE values decrease with decreasing replication for each value of ϕ_p. The values of MPE indicate MAR is better than MLE where most of the different model parameters and sample size n. Moreover, **Figures 1, 2,** and **3** represent the estimated model for the time series $AR(1)$ when $r = 100, 500, and\ 1000$, respectively, the red graph refers to the generated simulated $AR(1)$ model, the blue graph refers to the model estimated via exact likelihood, and the green graph refers to the estimated model by using the likelihood function via the marginal distribution of the time series. In **Figures 1, 2,** and **3**, we notice that all-time series are stable at different values of the sample size n and each number of replication r.

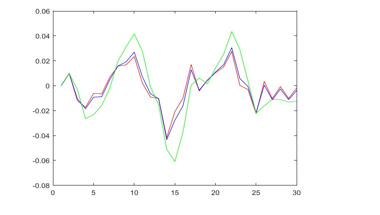
Table 1. $AR(1)$ Model for MLE and MAR with $r = 100$.

n	ϕ_p	MLE		MAR	
		MSE	MPE	MSE	MPE
30	$\phi_1 = 0.1$	1.5065e-32	46.9825	9.4843e-30	26.2206
	$\phi_1 = 0.3$	1.1237e-32	50.6085	9.6416e-31	30.6140
	$\phi_1 = 0.5$	1.9921e-33	48.6619	3.1081e-30	30.4533
	$\phi_1 = -0.7$	4.9416e-31	40.1620	4.1241e-30	31.8221
50	$\phi_1 = 0.1$	1.9921e-33	22.2352	1.6181e-30	11.9847
	$\phi_1 = 0.3$	0	21.6314	3.9045e-31	15.3154
	$\phi_1 = 0.5$	1.1205e-31	18.3909	1.0085e-30	14.0371
	$\phi_1 = -0.7$	2.1041e-32	16.3906	8.3717e-31	13.4952
100	$\phi_1 = 0.1$	1.2450e-34	7.3882	1.3466e-30	4.3022
	$\phi_1 = 0.3$	5.4907e-32	7.5252	7.9683e-33	4.6729
	$\phi_1 = 0.5$	2.8014e-32	7.7938	8.4165e-32	5.0796
	$\phi_1 = -0.7$	7.1715e-32	5.7299	1.9921e-31	4.8566

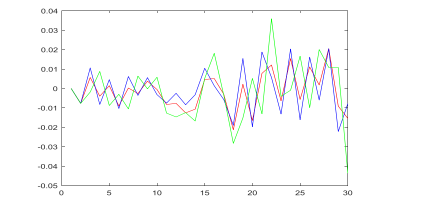
The results of **Table 1**; as shown in **Figure 1**.



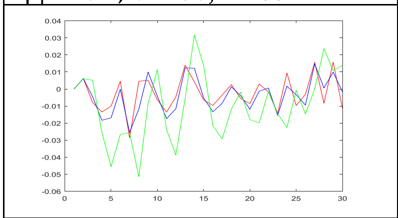
$\phi_1 = 0.1, n = 30$,r =100



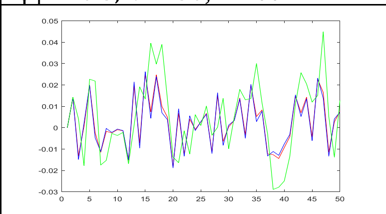
$\phi_1 = 0.3, n = 30$,r =100



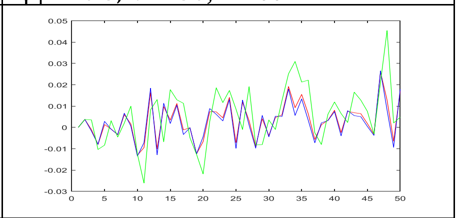
$\phi_1 = 0.5, n = 30$,r =100



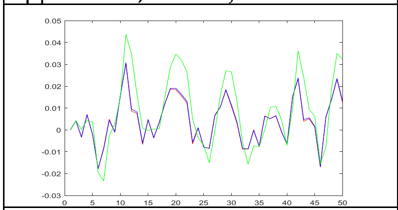
$\phi_1 = -0.7, n = 30$,r=100



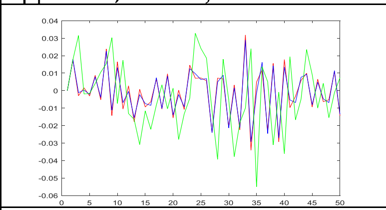
$\phi_1 = 0.1, n = 50$,r =100



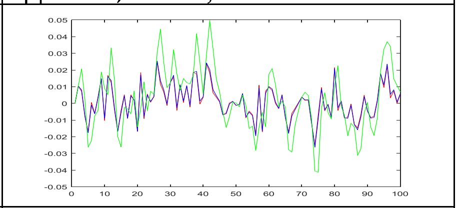
$\phi_1 = 0.3, n = 50$,r =100



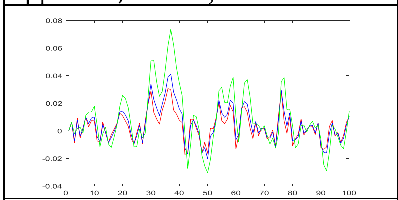
$\phi_1 = 0.5, n = 50$,r=100



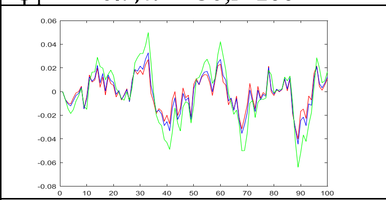
$\phi_1 = -0.7, n = 50$,r=100



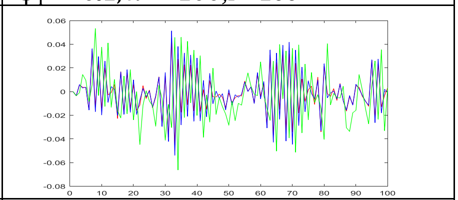
$\phi_1 = 0.1, n = 100$,r =100



$\phi_1 = 0.3, n = 100$,r =100



$\phi_1 = 0.5, n = 100$,r =100



$\phi_1 = -0.7, n = 100$,r =100

Figure 1. $AR(1)$ Model for MLE and MAR with r =100.

For number of replication $r = 500$, the results are given in **Table 2** and **Figure 2**.

Table 2. $AR(1)$ Model for MLE and MAR with $r = 500$.

n	ϕ_p	MLE		MAR	
		MSE	MPE	MSE	MPE
30	$\phi_1 = 0.1$	1.9921e-33	249.0715	6.3367e-28	156.3922
	$\phi_1 = 0.3$	2.5817e-30	240.5369	1.3884e-28	142.6665
	$\phi_1 = 0.5$	3.3487e-30	249.4303	5.0201e-28	159.3458
	$\phi_1 = -0.7$	8.6808e-31	189.4300	1.9144e-30	154.6298
50	$\phi_1 = 0.1$	7.1715e-32	105.8937	2.9528e-28	62.9162
	$\phi_1 = 0.3$	8.4165e-30	109.3878	6.6348e-29	62.0760
	$\phi_1 = 0.5$	1.1506e-29	104.1844	6.2410e-29	64.2770
	$\phi_1 = -0.7$	2.9407e-29	79.9037	5.6593e-28	59.7555
100	$\phi_1 = 0.1$	2.4359e-30	39.9870	2.6394e-28	21.2546
	$\phi_1 = 0.3$	1.9328e-29	38.2339	2.4543e-28	21.6598
	$\phi_1 = 0.5$	1.9574e-29	34.3012	4.8790e-29	22.1899
	$\phi_1 = -0.7$	6.7627e-29	28.4785	1.0327e-29	20.5642

The plot of the results in **Table 2**, as show in **Figure 2**.

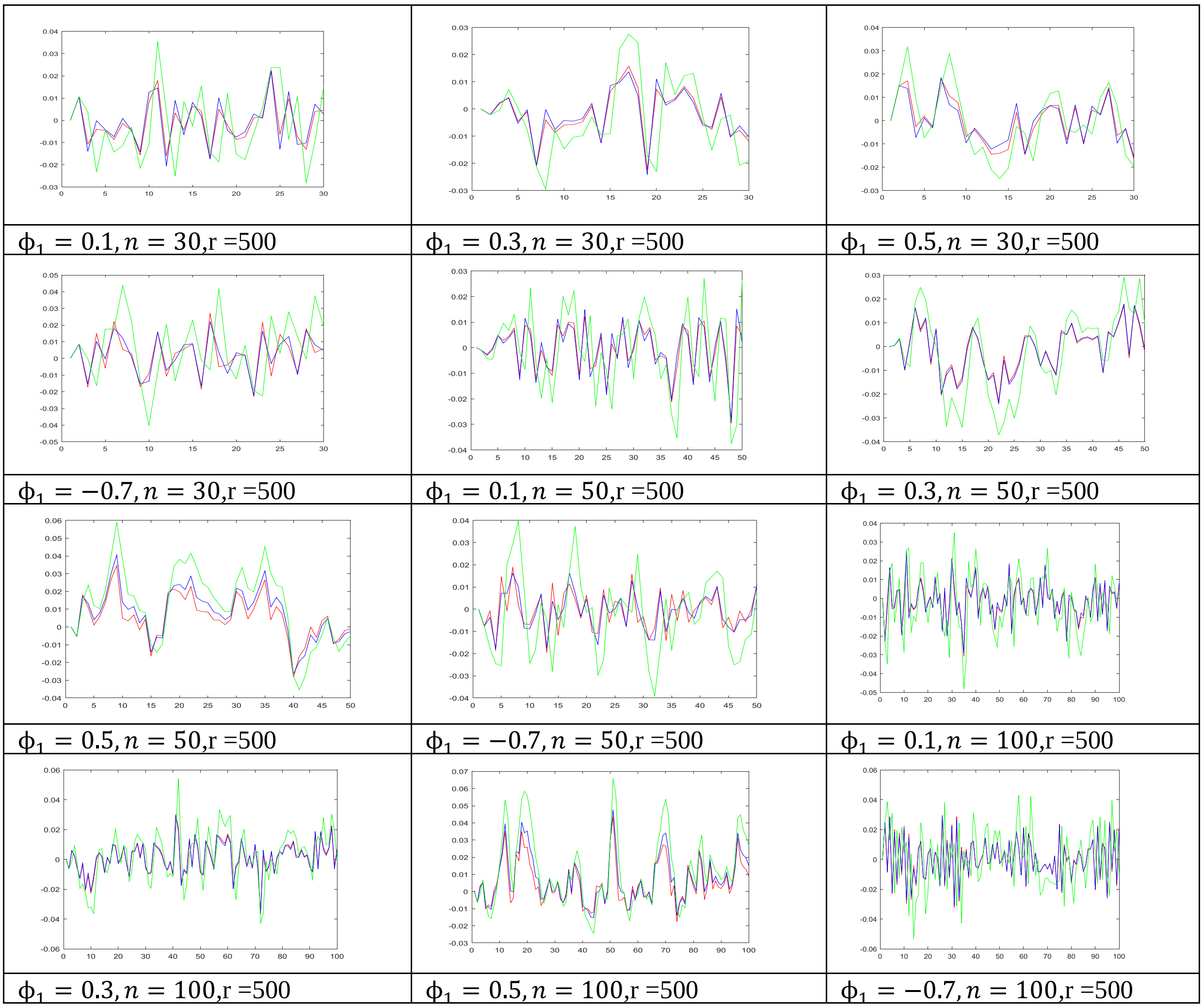


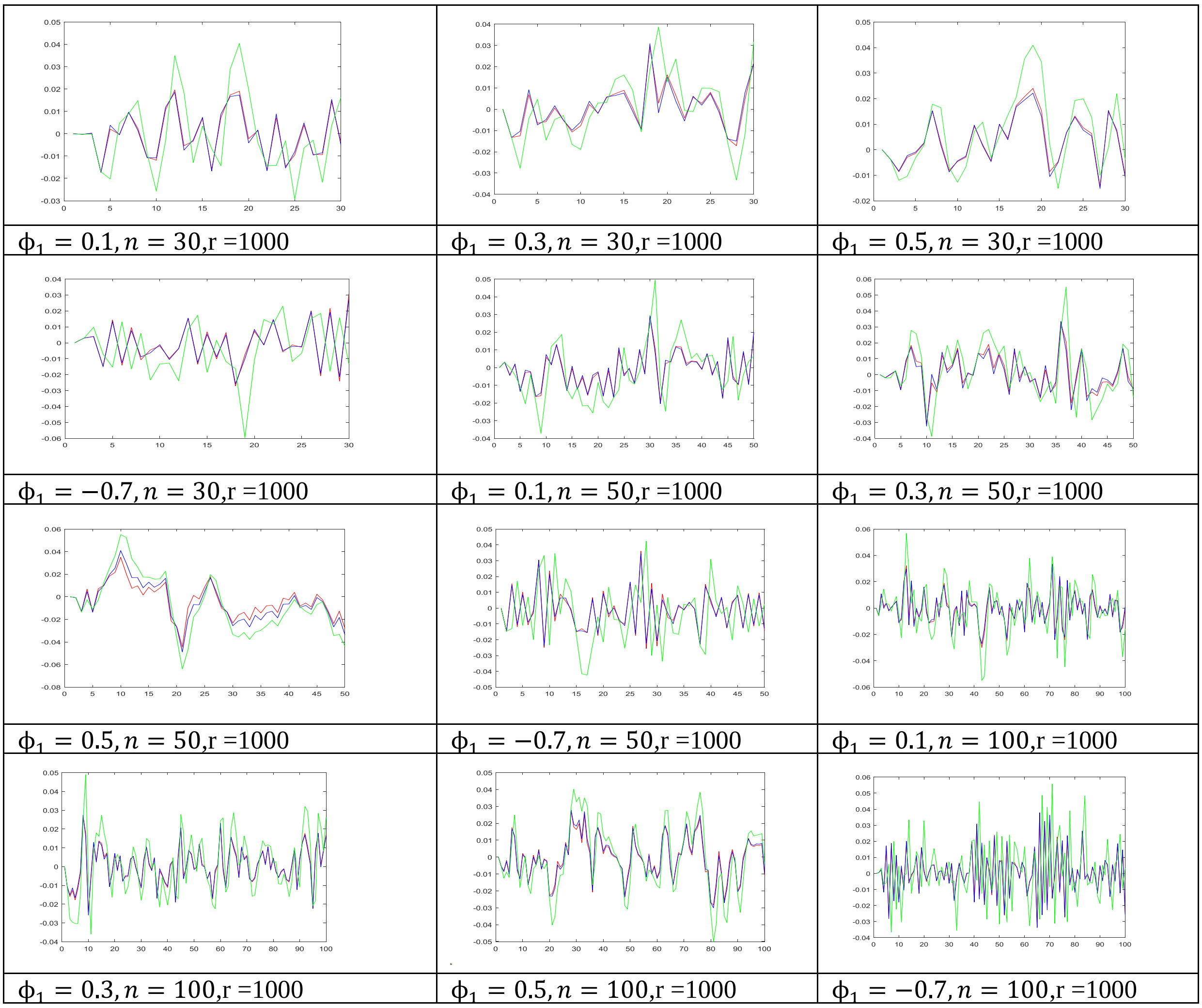
Figure 2. $AR(1)$ Model for MLE and MAR with $r = 500$.

For the number of replication, $r = 1000$, see the following **Table 3** and **Figure 3**:

Table 3. $AR(1)$ Model for MLE and MAR with r =1000.

n	ϕ_p	MLE		MAR	
		MSE	MPE	MSE	MPE
30	$\phi_1 = 0.1$	2.8766e-30	478.8193	2.3028e-28	290.8784
	$\phi_1 = 0.3$	3.1873e-30	494.8485	2.9650e-29	291.8135
	$\phi_1 = 0.5$	3.3537e-29	457.6450	2.3573e-28	290.2070
	$\phi_1 = -0.7$	7.6527e-29	386.0229	2.2490e-28	317.8166
50	$\phi_1 = 0.1$	8.9955e-31	233.6525	5.7334e-27	124.4451
	$\phi_1 = 0.3$	7.5749e-31	226.1102	1.3022e-27	124.1611
	$\phi_1 = 0.5$	2.2024e-30	210.4893	2.6904e-28	139.5043
	$\phi_1 = -0.7$	8.5978e-29	171.0632	5.4177e-28	127.1026
100	$\phi_1 = 0.1$	1.3466e-30	79.4712	1.3466e-30	42.9843
	$\phi_1 = 0.3$	1.5779e-29	76.0834	9.3883e-28	42.8568
	$\phi_1 = 0.5$	1.9623e-29	69.6642	2.3150e-27	43.1488
	$\phi_1 = -0.7$	1.1461e-27	59.9664	3.1794e-28	43.9696

Draw the results in **Table 3**, as shown in **Figure 3**.



$\phi_1 = 0.1, n = 30$,r =1000 | $\phi_1 = 0.3, n = 30$,r =1000 | $\phi_1 = 0.5, n = 30$,r =1000

$\phi_1 = -0.7, n = 30$,r =1000 | $\phi_1 = 0.1, n = 50$,r =1000 | $\phi_1 = 0.3, n = 50$,r =1000

$\phi_1 = 0.5, n = 50$,r =1000 | $\phi_1 = -0.7, n = 50$,r =1000 | $\phi_1 = 0.1, n = 100$,r =1000

$\phi_1 = 0.3, n = 100$,r =1000 | $\phi_1 = 0.5, n = 100$,r =1000 | $\phi_1 = -0.7, n = 100$,r =1000

Figure 3. $AR(1)$ Model for MLE and MAR with $r = 1000$.

The second experiment was designed to calculate MLE and MAR for time series values for $AR(2)$ with ($\phi_1 = 0.2,\ -0.3,\ 0.4,$ and 0.1), and ($\phi_2 = 0.4,\ 0.5,\ -0.5,$ and 0.3), sample size ($n = 30,\ 50,$ and 100), and number of replicate ($r = 100, 500,$ and 1000). The results of this experiment are given by **Tables 4, 5,** and **6. R**egarding the MSE and MPE criteria, we can see from the given tables that the values of MSE and MPE decrease with increasing sample size n for each value of ϕ_p for MLE and MAR. The values of MPE of both methods become closer as the sample size increases for different values of ϕ_p. The MSE and MPE values decrease with decreasing replication for each value of ϕ_p. The values of MPE indicate MAR is better than MLE where most of the different model parameters and sample size n. Moreover, **Figures 4, 5,** and **6** represent the estimated model for the time series $AR(2)$ when $r = 100, 500, and\ 1000$, respectively, the red graph refers to the generated simulated $AR(2)$ model, the blue graph refers to the model estimated via exact likelihood, and the green graph refers to the estimated model by using the likelihood function via the marginal distribution of the time series. In **Figures 4, 5,** and **6,** we notice that all-time series are stable at different values of the sample size n and each number of replication r.

The following table, as well as the figure that is related to it, represents the information for the second experiment in this paper.

For the number of replication $r = 100$, the results are given in **Table 4** and **Figure 4.**

Table 4. $AR(2)$ Model for MLE and MAR with $r = 100$.

n	ϕ_p	MLE		MAR	
		MSE	MPE	MSE	MPE
30	$\phi_1 = 0.2$	7.4211e-40	0.0030	1.2450e-34	1.4253
	$\phi_2 = 0.4$	6.6789e-41	0.0016	1.9921e-33	4.8494
	$\phi_1 = -0.3$	4.2745e-39	0.0111	0	1.4116
	$\phi_2 = 0.5$	4.7495e-40	0.0096	0	27.5439
	$\phi_1 = 0.4$	0	0.0024	0	1.4233
	$\phi_2 = -0.5$	2.9684e-41	0.0017	4.9802e-34	6.4939
	$\phi_1 = 0.1$	1.8553e-42	4.4416e-04	3.1126e-35	1.4428
	$\phi_2 = 0.3$	2.8988e-42	1.9749e-04	3.1126e-35	1.3447
50	$\phi_1 = 0.2$	2.6716e-40	0.0023	1.2450e-34	0.8658
	$\phi_2 = 0.4$	7.4211e-42	0.0013	1.2450e-34	1.4350
	$\phi_1 = -0.3$	1.8998e-37	0.0250	7.0034e-35	0.8381
	$\phi_2 = 0.5$	1.2159e-37	0.0228	2.0399e-30	107.9930
	$\phi_1 = 0.4$	1.1874e-40	0.0017	7.0034e-35	0.8637
	$\phi_2 = -0.5$	4.7495e-40	0.0015	3.1126e-35	0.4945
	$\phi_1 = 0.1$	2.9684e-41	6.1100e-04	7.0034e-35	0.8769
	$\phi_2 = 0.3$	1.8553e-42	2.4482e-04	3.1126e-33	3.6684
100	$\phi_1 = 0.2$	1.8998e-39	0.0015	7.0034e-35	0.4533
	$\phi_2 = 0.4$	2.9684e-41	8.5164e-04	7.9683e-3	7.7977
	$\phi_1 = -0.3$	3.0397e-38	0.0115	3.1126e-35	0.4305
	$\phi_2 = 0.5$	7.5992e-39	0.0103	7.9683e-31	30.1493
	$\phi_1 = 0.4$	4.7495e-40	8.6626e-04	0	0.4541
	$\phi_2 = -0.5$	7.4211e-42	6.1362e-04	1.7929e-32	7.8653
	$\phi_1 = 0.1$	0	3.2966e-04	3.1126e-35	0.4669
	$\phi_2 = 0.3$	2.6090e-43	2.3317e-05	3.1873e-32	7.6017

The plot of the results in **Table 4**, as shown in **Figure 4**.

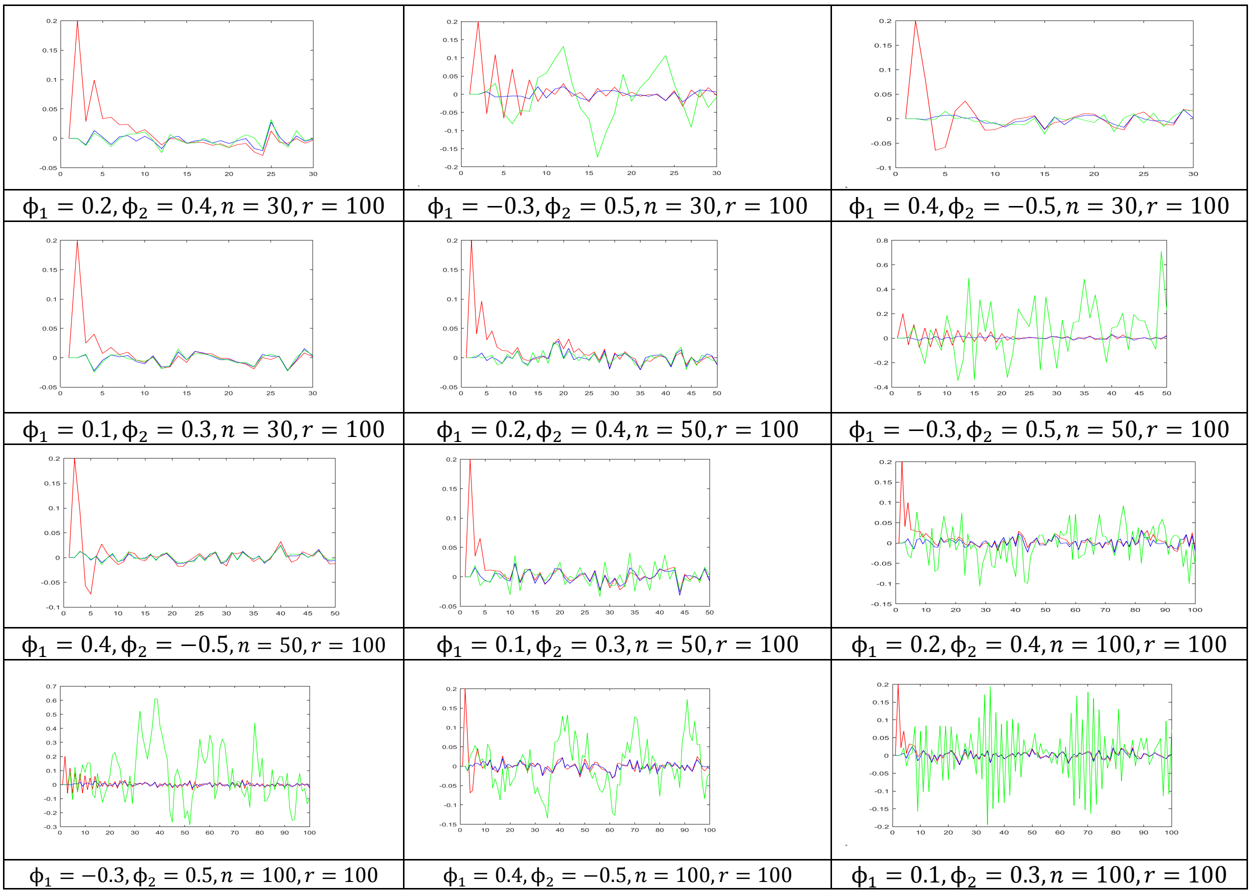


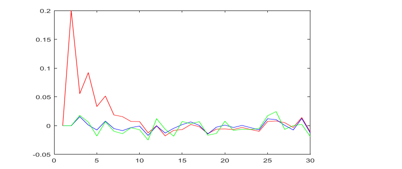
Figure 4. $AR(2)$ for MLE and MAR Model with $r = 100$.

For the number of replication $r = 500$; the results are given in **Table 5** and **Figure 5.**

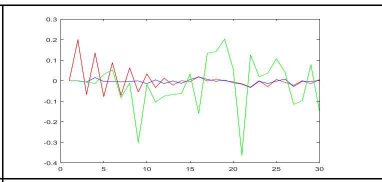
Table 5. $AR(2)$ Model for MLE and MAR with $r = 500$.

n	ϕ_p	MLE		MAR	
		MSE	MPE	MSE	MPE
30	$\phi_1 = 0.2$	9.6177e-39	0.0042	4.9802e-34	1.4365
	$\phi_2 = 0.4$	8.5787e-39	0.0034	4.9802e-34	5.1715
	$\phi_1 = -0.3$	1.5388e-37	0.0254	4.4822e-33	1.4125
	$\phi_2 = 0.5$	4.7495e-38	0.0221	1.4056e-29	99.4448
	$\phi_1 = 0.4$	7.5992e-39	0.0038	1.3151e-33	1.4287
	$\phi_2 = -0.5$	5.8181e-39	0.0032	4.0339e-32	14.2587
	$\phi_1 = 0.1$	2.4536e-40	5.1827e-04	1.5252e-33	1.4517
	$\phi_2 = 0.3$	4.1743e-42	1.2615e-04	1.7508e-33	0.9196
50	$\phi_1 = 0.2$	2.6716e-40	8.8243e-04	3.7663e-33	0.8814
	$\phi_2 = 0.4$	2.4536e-40	3.2758e-04	6.1007e-33	3.4285
	$\phi_1 = -0.3$	4.8635e-37	0.0083	2.8091e-33	0.8582
	$\phi_2 = 0.5$	2.0945e-37	0.0072	5.0997e-31	14.0768
	$\phi_1 = 0.4$	1.0716e-38	0.0020	1.1205e-33	0.8675
	$\phi_2 = -0.5$	2.4044e-39	0.0014	6.1007e-33	1.3755
	$\phi_1 = 0.1$	7.4211e-42	8.6381e-04	1.7508e-33	0.8805
	$\phi_2 = 0.3$	4.8991e-42	4.9186e-05	1.2450e-32	3.3063
100	$\phi_1 = 0.2$	2.6716e-38	0.0014	1.9454e-34	0.4573
	$\phi_2 = 0.4$	5.0166e-39	7.5191e-04	1.4522e-30	7.8931
	$\phi_1 = -0.3$	7.5992e-39	0.0099	7.7815e-36	0.4349
	$\phi_2 = 0.5$	0	0.0085	1.5618e-30	24.5102
	$\phi_1 = 0.4$	2.9684e-41	9.4607e-04	0	0.4552
	$\phi_2 = -0.5$	1.0686e-39	4.3304e-04	5.7571e-31	7.6444
	$\phi_1 = 0.1$	1.8553e-40	1.8933e-04	7.7815e-34	0.4687
	$\phi_2 = 0.3$	1.0436e-42	4.1176e-05	2.8686e-31	7.7983

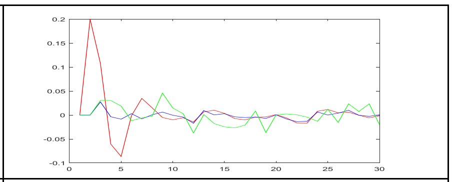
The plot of the results in **Table 5**, as shown in **Figure 5**.



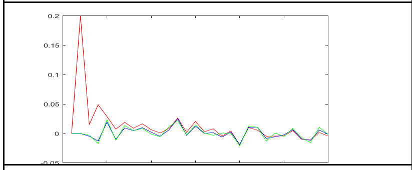
$\phi_1 = 0.2, \phi_2 = 0.4, n = 30, r = 500$



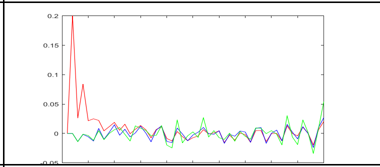
$\phi_1 = -0.3, \phi_2 = 0.5, n = 30, r = 500$



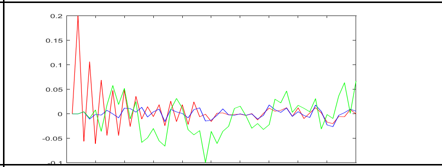
$\phi_1 = 0.4, \phi_2 = -0.5, n = 30, r = 500$



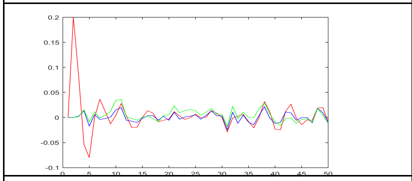
$\phi_1 = 0.1, \phi_2 = 0.3, n = 30, r = 500$



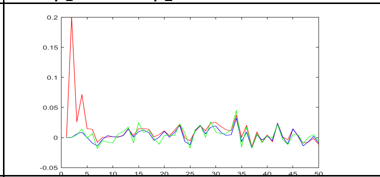
$\phi_1 = 0.2, \phi_2 = 0.4, n = 50, r$



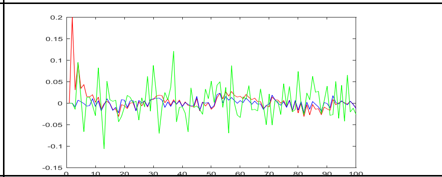
$\phi_1 = -0.3, \phi_2 = 0.5, n = 50, r = 500$



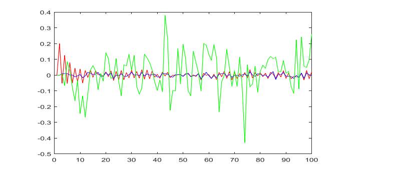
$\phi_1 = 0.4, \phi_2 = -0.5, n = 50, r = 500$



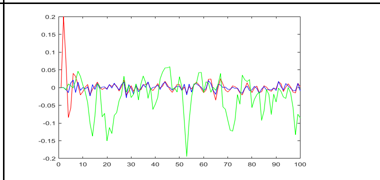
$\phi_1 = 0.1, \phi_2 = 0.3, n = 50, r = 500$



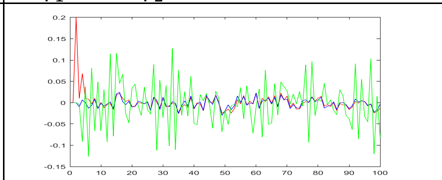
$\phi_1 = 0.2, \phi_2 = 0.4, n = 100, r = 500$



$\phi_1 = -0.3, \phi_2 = 0.5, n = 100, r = 500$



$\phi_1 = 0.4, \phi_2 = -0.5, n = 100, r = 500$



$\phi_1 = 0.1, \phi_2 = 0.3, n = 100, r = 500$

Figure 5. $AR(2)$ Model with $r = 500$.

For the number of replication $r = 1000$; the results are given in **Table 6** and **Figure 6.**

Table 6. $AR(2)$ Model for MLE and MAR with $r = 1000$.

n	ϕ_p	MLE		MAR	
		MSE	**MPE**	**MSE**	**MPE**
30	$\phi_1 = 0.2$	7.4211e-38	0.0046	4.4822e-33	1.4337
	$\phi_2 = 0.4$	1.0686e-39	0.0025	7.0034e-31	9.1800
	$\phi_1 = -0.3$	4.0200e-36	0.0275	7.0034e-33	1.4137
	$\phi_2 = 0.5$	8.3781e-37	0.0250	2.4989e-29	102.4699
	$\phi_1 = 0.4$	7.5992e-39	0.0028	4.4822e-33	1.4317
	$\phi_2 = -0.5$	9.0908e-39	0.0013	2.6345e-31	11.8544
	$\phi_1 = 0.1$	3.7569e-41	4.9649e-04	9.5324e-33	1.4550
	$\phi_2 = 0.3$	1.2627e-40	2.3499e-04	1.0653e-32	1.2305
50	$\phi_1 = 0.2$	8.6559e-38	0.0040	4.4822e-33	0.8680
	$\phi_2 = 0.4$	6.8392e-38	0.0029	5.2323e-32	1.6382
	$\phi_1 = -0.3$	8.3781e-37	0.0196	1.9921e-33	0.8484
	$\phi_2 = 0.5$	2.7357e-37	0.0176	6.1706e-29	72.0161
	$\phi_1 = 0.4$	2.1640e-38	0.0021	1.7508e-33	0.8681
	$\phi_2 = -0.5$	2.0846e-38	0.0013	8.9955e-33	1.5923
	$\phi_1 = 0.1$	1.6697e-39	7.1442e-04	2.8091e-33	0.8864
	$\phi_2 = 0.3$	8.9795e-40	3.6851e-04	1.2749e-31	3.8168
100	$\phi_1 = 0.2$	1.8998e-37	0.0015	1.1237e-32	0.4588
	$\phi_2 = 0.4$	2.0067e-38	0.0011	1.4522e-30	7.9779
	$\phi_1 = -0.3$	2.7357e-37	0.0110	7.9683e-33	0.4351
	$\phi_2 = 0.5$	1.0050e-36	0.0098	3.2638e-29	25.8946
	$\phi_1 = 0.4$	2.6716e-40	8.8139e-04	3.4317e-33	0.4591
	$\phi_2 = -0.5$	1.0716e-38	6.4366e-04	3.5140e-30	7.9920
	$\phi_1 = 0.1$	8.1817e-40	2.1775e-04	3.8130e-34	0.4734
	$\phi_2 = 0.3$	2.4536e-40	6.3309e-05	0	7.5106

The following Figure helps us show the results in **Table 6**.

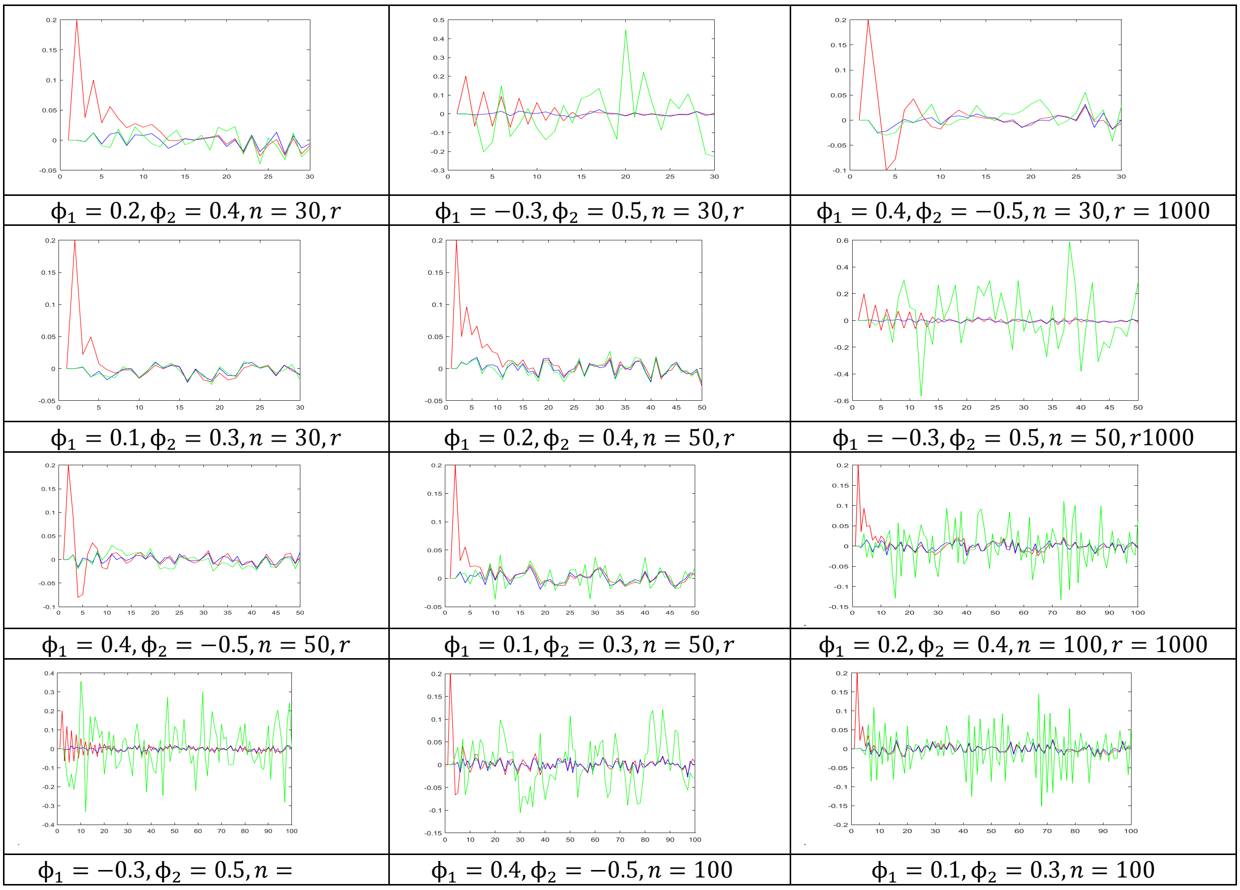


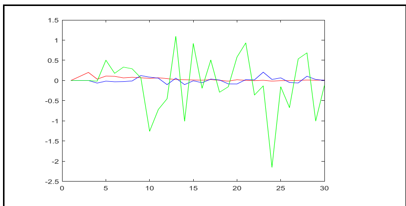
Figure 6. $AR(2)$ Model for MLE and MAR with $r = 1000$.

The third experiment is designed to calculate MLE and MAR for time series values for AR (3), with ($\phi_1 = 0.1$, 0.2, -0.3, and 0.4), ($\phi_2 = 0.3, -0.4$, 0.4, and 0.5), and ($\phi_3 = 0.4$, 0.6, 0.5, and -0.6), sample size ($n = 30$, 50, and 100), and the number replicate ($r = 100$, 500, and 1000). The results of this experiment are given by **Tables 7, 8,** and **9. R**egarding the MSE and MPE criteria, we can see from the given tables, the values of MSE and MPE decrease with increasing sample size n for each value of ϕ_p for MLE and MAR. The values of MPE of both methods become closer as the sample size increases for different values of ϕ_p. The MSE and MPE values decrease with decreasing replication for each value of ϕ_p. The values of MPE indicate MAR is better than MLE where most of the different model parameters and sample size n. Moreover, **Figures 7, 8,** and **9** represent the estimated model for the time series $AR(3)$ when $r = 100, 500, and\ 1000$, respectively, the red graph refers to the generated simulated $AR(3)$ model, the blue graph refers to the model estimated via exact likelihood, and the green graph refers to the estimated model by using the likelihood function via the marginal distribution of the time series. In **Figures 7, 8,** and **9,** we notice that all-time series are stable at different values of the sample size n and each number of replication r.

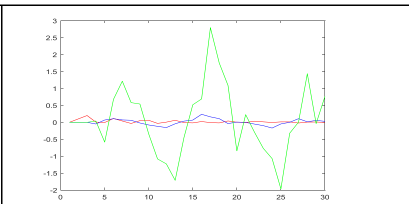
Table 7. $AR(3)$ Model for MLE and MAR with $r = 100$.

n	ϕ_p	MLE MSE	MLE MPE	MAR MSE	MAR MPE
30	$\phi_1 = 0.1$	1.7508e-35	0.6308	1.7929e-32	17.1093
	$\phi_2 = 0.3$	3.1873e-32	56.2287	0	6.3666
	$\phi_3 = 0.4$	4.9802e-34	6.0444	2.0399e-30	575.9633
	$\phi_1 = 0.2$	0	0.0214	4.9802e-34	16.2997
	$\phi_2 = -0.4$	3.1873e-32	38.5691	1.2450e-34	5.1487
	$\phi_3 = 0.6$	4.9802e-34	3.0948	4.5897e-30	440.9363
	$\phi_1 = -0.3$	4.8635e-37	0.2512	7.9683e-33	16.6317
	$\phi_2 = 0.4$	2.8686e-31	103.1897	0	5.6625
	$\phi_3 = 0.5$	1.9921e-33	9.6427	0	496.3091
	$\phi_1 = 0.4$	7.7815e-36	0.3762	4.9802e-34	16.6013
	$\phi_2 = 0.5$	3.1873e-32	35.2640	1.9921e-33	5.6163
	$\phi_3 = -0.6$	4.9802e-34	3.2239	4.5897e-30	491.2326
50	$\phi_1 = 0.1$	3.1126e-35	0.5560	1.9921e-33	10.6481
	$\phi_2 = 0.3$	0	26.8586	4.4822e-33	4.5445
	$\phi_3 = 0.4$	1.9921e-33	3.5300	1.8359e-29	256.8860
	$\phi_1 = 0.2$	0	0.0089	7.9683e-33	9.6300
	$\phi_2 = -0.4$	5.0997e-31	61.2589	4.9802e-34	3.0853
	$\phi_3 = 0.6$	1.2450e-34	4.8212	5.0997e-31	154.9406
	$\phi_1 = -0.3$	1.9454e-36	0.1122	4.4822e-33	9.8909
	$\phi_2 = 0.4$	1.1474e-30	70.7262	1.2450e-34	3.4879
	$\phi_3 = 0.5$	4.4822e-33	6.7510	4.5897e-30	181.0627
	$\phi_1 = 0.4$	4.3771e-36	0.2402	1.2450e-32	9.9961
	$\phi_2 = 0.5$	3.1873e-32	20.5466	4.9802e-34	3.6442
	$\phi_3 = -0.6$	3.1126e-35	2.0373	5.0997e-31	191.5910
100	$\phi_1 = 0.1$	3.1126e-35	0.3113	1.9921e-33	5.5194
	$\phi_2 = 0.3$	3.1873e-32	12.1245	4.9802e-34	2.6377
	$\phi_3 = 0.4$	1.9921e-33	1.8428	2.0399e-30	76.1240
	$\phi_1 = 0.2$	3.0397e-38	0.0094	4.4822e-33	5.0358
	$\phi_2 = -0.4$	3.1873e-32	19.0202	4.4822e-33	2.0113
	$\phi_3 = 0.6$	0	1.9054	0	51.8760
	$\phi_1 = -0.3$	1.2159e-37	0.0500	0	4.9543
	$\phi_2 = 0.4$	5.0997e-31	35.1164	1.2450e-34	1.8936
	$\phi_3 = 0.5$	4.4822e-33	3.5198	1.2749e-31	47.7893
	$\phi_1 = 0.4$	1.2159e-35	0.1258	4.4822e-33	5.1427
	$\phi_2 = 0.5$	1.2749e-31	13.6534	4.9802e-34	2.1598
	$\phi_3 = -0.6$	4.9802e-34	1.5854	1.1474e-30	57.2322

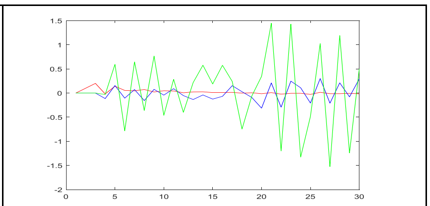
The following Figure shows the information in the table above.



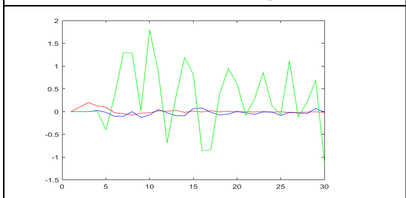
$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$



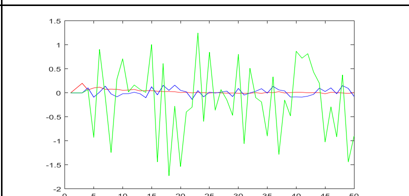
$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$



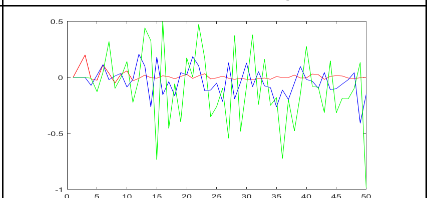
$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$



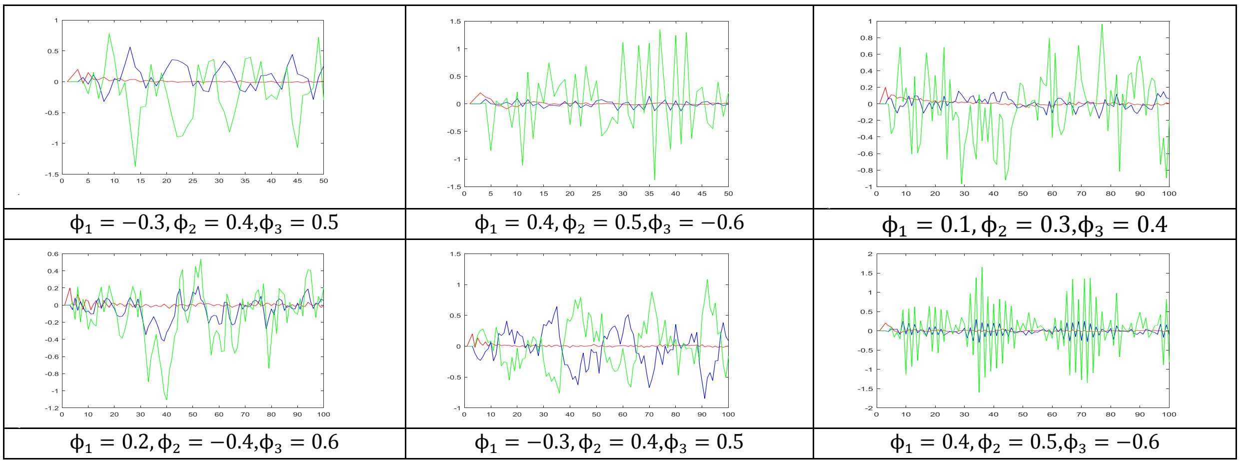
$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$



$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$



$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$



$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$ | $\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$ | $\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$

$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$ | $\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$ | $\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$

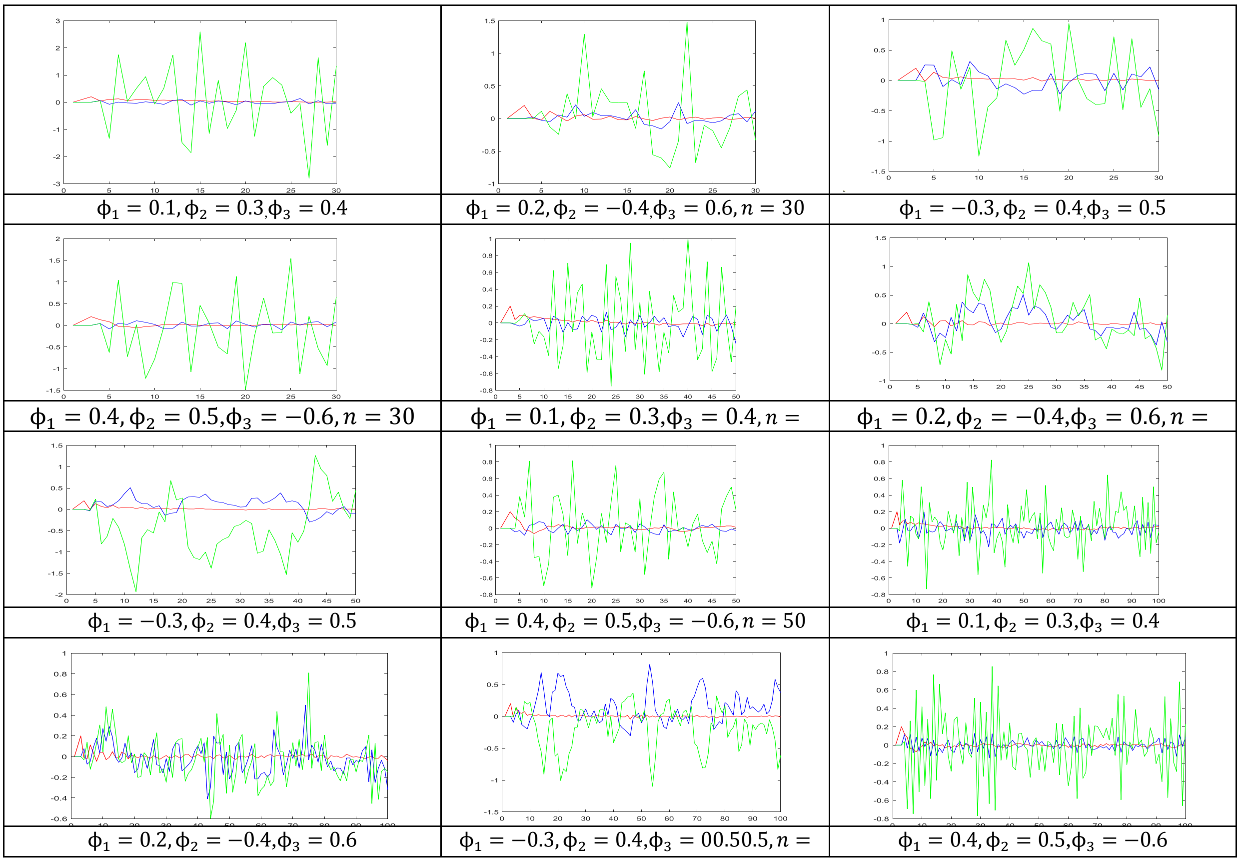
Figure 7. $AR(3)$ Model for MLE and MAR with $r = 100$.

For the number of replication $r = 500$, the results are given in **Table 8** and **Figure 8.**

Table 8. $AR(3)$ Model for MLE and MAR with $r = 500$.

n	ϕ_p	MLE MSE	MLE MPE	MAR MSE	MAR MPE
30	$\phi_1 = 0.1$	7.7815e-36	1.1129	2.4104e-31	18.8798
	$\phi_2 = 0.3$	1.1474e-30	41.6021	4.4946e-32	8.5553
	$\phi_3 = 0.4$	5.4907e-32	6.2432	2.4683e-28	852.8920
	$\phi_1 = 0.2$	1.1874e-38	0.0151	1.6136e-31	16.2684
	$\phi_2 = -0.4$	5.0997e-31	71.6615	4.9802e-34	4.9303
	$\phi_3 = 0.6$	1.2450e-34	5.4075	9.9954e-29	417.3141
	$\phi_1 = -0.3$	7.7815e-36	0.1940	1.9921e-31	16.5396
	$\phi_2 = 0.4$	3.1873e-30	115.0466	1.0085e-32	5.3600
	$\phi_3 = 0.5$	1.1205e-31	10.0319	1.3055e-28	462.5379
	$\phi_1 = 0.4$	7.0034e-35	0.3779	0	16.5729
	$\phi_2 = 0.5$	1.2749e-31	35.6797	6.1007e-33	5.4118
	$\phi_3 = -0.6$	7.0034e-33	3.0645	9.9954e-29	468.0983
50	$\phi_1 = 0.1$	2.8014e-34	0.3748	3.1873e-32	10.1978
	$\phi_2 = 0.3$	1.5618e-30	33.2431	2.8014e-32	3.8494
	$\phi_3 = 0.4$	1.7929e-32	3.5554	9.9954e-29	205.2482
	$\phi_1 = 0.2$	0	0.0131	2.1963e-31	9.8868
	$\phi_2 = -0.4$	8.1595e-30	78.4409	2.4403e-32	3.3887
	$\phi_3 = 0.6$	4.0339e-32	6.7044	1.2749e-29	174.1100
	$\phi_1 = -0.3$	2.8014e-34	0.2287	4.4822e-31	10.0690
	$\phi_2 = 0.4$	0	46.7047	3.1873e-32	3.6620
	$\phi_3 = 0.5$	1.2450e-32	4.6974	5.0997e-29	192.3506
	$\phi_1 = 0.4$	8.2192e-35	0.2531	1.7929e-32	10.0014
	$\phi_2 = 0.5$	1.2749e-31	19.8713	1.7929e-32	3.5618
	$\phi_3 = -0.6$	1.1237e-32	1.8715	1.3055e-28	185.5815
100	$\phi_1 = 0.1$	5.6222e-34	0.2117	2.4403e-32	5.1762
	$\phi_2 = 0.3$	1.3466e-30	14.8984	4.9802e-32	2.1664
	$\phi_3 = 0.4$	1.0085e-32	1.7558	2.4989e-29	57.2946
	$\phi_1 = 0.2$	1.8998e-39	0.0047	1.9921e-33	4.9021
	$\phi_2 = -0.4$	1.2749e-31	27.1229	1.2450e-32	1.7712
	$\phi_3 = 0.6$	4.9802e-34	2.3497	1.0327e-29	43.5548
	$\phi_1 = -0.3$	3.9394e-35	0.0650	1.4393e-31	4.9854
	$\phi_2 = 0.4$	8.1595e-30	32.7943	1.2450e-32	1.8957
	$\phi_3 = 0.5$	1.2749e-31	3.2813	6.2471e-30	47.7256
	$\phi_1 = 0.4$	7.7815e-36	0.1105	1.7978e-31	5.0245
	$\phi_2 = 0.5$	9.7612e-32	9.6529	6.1007e-33	1.9529
	$\phi_3 = -0.6$	8.9955e-33	0.9837	1.8359e-29	49.6866

The values in **Table 8** via the Figure are shown as follows:



$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$	$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6, n = 30$	$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$
$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6, n = 30$	$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4, n =$	$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6, n =$
$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$	$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6, n = 50$	$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$
$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$	$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 00.50.5, n =$	$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$

Figure 8. $AR(3)$ Model for MLE and MAR with $r\ = 500$**.**

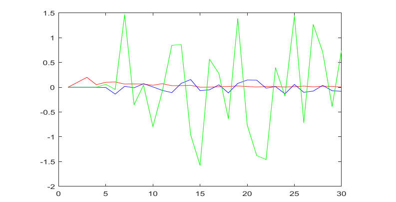
For the number of replication $r = 1000$, the results are given in **Table 9** and **Figure 9**:

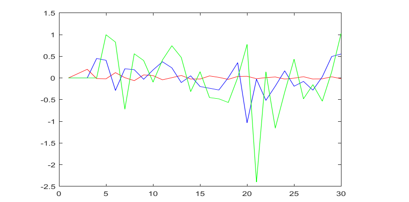
Table 9. $AR(3)$ Model for MLE and MAR with $r\ = 1000$**.**

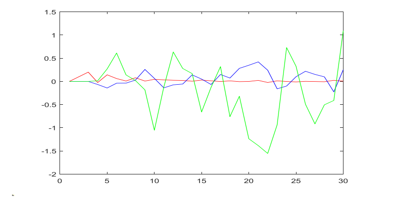
n	ϕ_p	**MLE**		**MAR**	
		MSE	**MPE**	**MSE**	**MPE**
30	$\phi_1 = 0.1$	7.7815e-34	0.6687	5.4234e-31	17.2401
	$\phi_2 = 0.3$	8.1595e-30	55.1572	1.7045e-31	6.3891
	$\phi_3 = 0.4$	6.0260e-32	5.8641	1.2749e-29	577.0794
	$\phi_1 = 0.2$	3.0397e-38	0.0045	1.1957e-30	16.9682
	$\phi_2 = -0.4$	1.4738e-28	240.8549	6.5863e-32	5.9919
	$\phi_3 = 0.6$	1.6136e-31	21.6782	7.3436e-29	531.7196
	$\phi_1 = -0.3$	2.3831e-35	0.1511	0	16.7239
	$\phi_2 = 0.4$	6.7444e-29	116.0243	3.1873e-32	5.6242
	$\phi_3 = 0.5$	6.4543e-31	10.5361	1.6523e-28	490.9766
	$\phi_1 = 0.4$	3.5455e-34	0.4434	1.5065e-30	17.0314
	$\phi_2 = 0.5$	3.8567e-30	30.9070	1.1205e-31	6.0854
	$\phi_3 = -0.6$	4.9802e-32	3.0517	9.9954e-29	542.2655
50	$\phi_1 = 0.1$	4.8635e-35	0.5620	1.4522e-30	10.7926
	$\phi_2 = 0.3$	2.8766e-30	27.3096	1.4393e-31	4.6473
	$\phi_3 = 0.4$	1.4393e-31	3.6123	7.7567e-28	263.9899
	$\phi_1 = 0.2$	2.0689e-36	0.0092	3.1126e-31	9.7663
	$\phi_2 = -0.4$	1.2749e-29	84.1834	0	3.1902
	$\phi_3 = 0.6$	1.1957e-30	6.7660	1.7454e-28	161.2271
	$\phi_1 = -0.3$	4.3771e-36	0.1259	8.3717e-31	10.0636
	$\phi_2 = 0.4$	1.1474e-28	64.8504	4.9802e-32	3.6429
	$\phi_3 = 0.5$	7.5749e-31	6.3882	1.4738e-28	190.9959

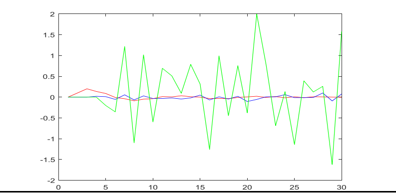
n	ϕ_p	MLE		MAR	
		MSE	MPE	MSE	MPE
	$\phi_1 = 0.4$	2.1448e-34	0.2248	1.6136e-31	10.0493
	$\phi_2 = 0.5$	2.0399e-30	21.1864	0	3.6217
	$\phi_3 = -0.6$	3.1126e-35	2.0627	2.4989e-29	189.5649
100	$\phi_1 = 0.1$	5.4646e-33	0.3224	7.1914e-31	5.5823
	$\phi_2 = 0.3$	8.1595e-30	12.4381	9.7612e-32	2.6766
	$\phi_3 = 0.4$	7.7815e-32	1.8716	1.2252e-28	77.4493
	$\phi_1 = 0.2$	2.8896e-36	0.0064	3.3666e-31	4.8274
	$\phi_2 = -0.4$	6.4543e-29	30.6033	1.7929e-32	1.6496
	$\phi_3 = 0.6$	3.5982e-32	2.4344	1.1474e-30	39.6038
	$\phi_1 = -0.3$	9.5324e-35	0.0737	4.9802e-32	4.9277
	$\phi_2 = 0.4$	5.3579e-29	26.3076	6.0260e-32	1.8042
	$\phi_3 = 0.5$	1.0471e-31	2.4699	5.0997e-29	44.6310
	$\phi_1 = 0.4$	4.9802e-34	0.1460	3.6306e-31	5.2147
	$\phi_2 = 0.5$	2.3028e-30	7.9169	5.4907e-32	2.2136
	$\phi_3 = -0.6$	3.1873e-32	0.9263	2.4683e-28	59.0183

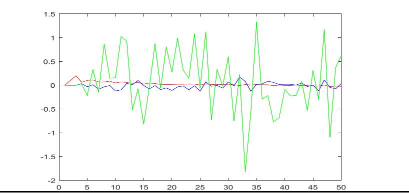
The results in the Table above are shown in the figures below.

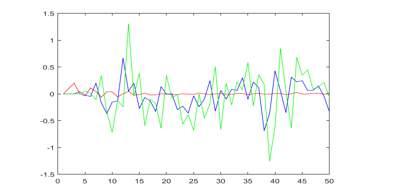

$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$

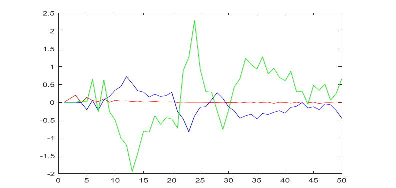

$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$

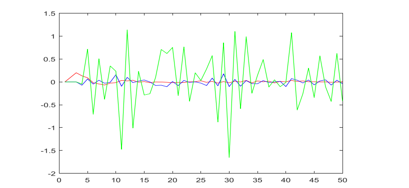

$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$

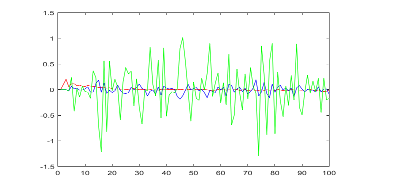

$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$

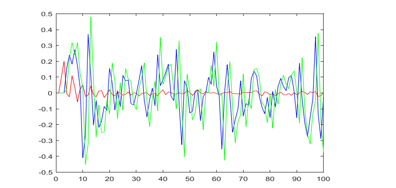

$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$

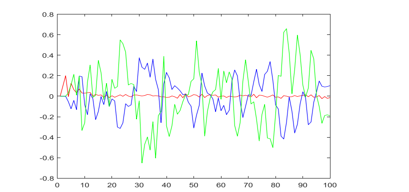

$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$


$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$


$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$


$\phi_1 = 0.1, \phi_2 = 0.3, \phi_3 = 0.4$


$\phi_1 = 0.2, \phi_2 = -0.4, \phi_3 = 0.6$


$\phi_1 = -0.3, \phi_2 = 0.4, \phi_3 = 0.5$

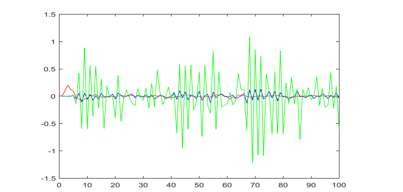

$\phi_1 = 0.4, \phi_2 = 0.5, \phi_3 = -0.6$

Figure 9. $AR(3)$ Model for MLE and MAR with $r = 1000$**.**

5. Conclusion

Recently, time series analysis is necessary for many fields and applications with huge and complex data. This paper presents the problem of autoregressive model parameter estimation with maximum likelihood function and model selection. The model parameter estimators are derived theoretically by defining the marginal distribution of the AR time series, where the distribution is got by random shocks representing the AR process with the assumption of a normal distribution of the white noise errors term. Here, AR model order is adopted for $p = 1,\ 2,$ and 3. Exact maximum likelihood function and maximum likelihood with time series marginal distribution are derived. To show the ability of the theoretical computing of the AR parameters, some of the experiments are implemented with simulation for the AR parameter estimation and model selection.

The experiments are designed for different values of model order (p) , sample size (n), and number of replications. The results show the efficiency of the model parameter estimation by computing mean squares errors, and mean percentage errors.

For different settings of ϕ_p and sample size n, the results show the stationarity of the predicted model with estimated parameters for different values of ϕ_p, sample size n, and replications. The values of MPE and MSE show the ability of MAR for model parameter estimation, where at most (MPE and MSE)-based MAR are better than MLE.

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Conflict of Interest

The authors assert that they do not have any conflicts of interest.

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