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Generalization of Fixed Point Theorem for $\varphi -$ Contraction Mappings in the fuzzy $b -$ Metric Space

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Abstract

Let (X, W, T) be a fuzzy b- metric space, where X is a non-empty set, W is a fuzzy set on $X \times X \times (0, \infty)$ to $[0,1]$, and T is a continuous t-norm, and let a function $\varphi: [0,1] \to [0,1]$ satisfies the following conditions: The function φ is strictly decreasing and continuous, $\varphi(c) = 0$ If and only if c equals 1 and $\varphi\big(T(c, a)\big) = T\big(\varphi(c), \varphi(a)\big)$, where c and a in X. Which is called φ- function and use it to define φ- Contraction mappings of type I and II. In this research, we will complete the study of many authors about fixed point theory on fuzzy b-metric spaces as and generalize some results on fixed points theory on fuzzy metric spaces to fuzzy b- metric spaces with simplify different proofs. It was established many results on compact fuzzy metric spaces and complete fuzzy metric spaces. we generalize results of other results to fuzzy b-metric spaces by using φ– Contraction mappings of type I and II in both complete and compact fuzzy b-metric spaces to show existence of fixed points for this type of self-mapping.

Keywords : Complete fuzzy b- metric space, Compact fuzzy b- metric space, Fixed point.

1.Introduction

The definition of a fuzzy metric space was defined by (1) to (5) defined the completeness of the fuzzy metric space, Bakhtin in 1989 present the concept of b- metric spaces (6). After that, (7) generalized a b- metric spaces and a fuzzy metric spaces to fuzzy b- metric spaces in 2016. (8) used the continuous t- norm to modify the idea of fuzzy metric spaces, and showed in the completeness definition of Grabiec fails to make $\mathbb{R}$ complete, then they introduced another definition completeness for which $\mathbb{R}$, with the standard fuzzy metric induced by the Euclidean metric.

In 1922, (9) presented the Banach fixed point theorem, which is known an essential concept in fixed point theory. From that time , many results fixed points on different spaces and types of Contraction mappings have been proved, for more detail see (10-28). One of them, (3) established many results on the compact or complete fuzzy metric spaces.

In our work, we prove the existence a uniqueness of fixed point of a mapping on complete or compact fuzzy b- metric space (X, W, T) where X is a non-empty set, W is a fuzzy set on

$X \times X \times (0, \infty)$ to [0,1], and T is a continuous t-norm under φ- Contraction mapping of types I and II.

2. Preliminaries

In this section we will cover some basic definitions and results.

Definition 2.1 (29): A t- norm is a binary operation T:[0,1]×[0,1]→[0,1] for all $a, c, z \in [0,1]$, which satisfies the conditions:

1- Commutatively : $T(a, c) = T(c, a)$;

2- Associativity:$T(T(a, c), z) = T(a, T(c, z))$;

3- $T(a\ ,1) = a$ for each $0 \leq a \leq 1$,

4- $T(a, c) \leq T(z, e)$ for all $a, c, z, e \in [0,1]$ such that $a \leq z$ and $c \leq e$,

5- T is continuous .

The commutatively of (1), and conditions (2) , for all t-norm, T, and $0 \leq c \leq 1$:

$T(c, 0) = T(0, c) = 0$. (29)

e.g of a t-norm are $T_p = a.c$, $T_{min} = \min\{a, c\}$ (29) .

Definition 2.2 (30): Let T be a t-norm, and let T_n: [0,1] → [0,1]; $n \in \mathbb{N}$, be defined as

$T_1(a) = T(a, a)$, $T_{n+1}(a) = T(T_n(a), a)$; $a \in [0,1]$.

Remark 2.3 (29): Each t-norm T can be extended through associativity to an n-ary operation that takes the values of a $(a_1, \dots, a_n) \in [0,1]^n$ as

$T_{i=1}^{1} a_i = a_1$, $T_{i=1}^{n} a_i = T(T_{i=1}^{n-1} a_i, a_n) = T(a_1, \dots, a_n)$.

We say that a t-norm T is of H-type if the family $\{T_n(a)\}_{n\in\mathbb{N}}$ is equicontinuous at $a = 1$.

Remark 2.4 (29): In t-norm T can take any sequence $(c_n)_{n\in\mathbb{N}}$ form [0,1] can extend to countable infinite operation

$T_{i=1}^{\infty} c_i = \lim_{n\to\infty} T_{i=1}^{n} c_i$.

$\lim_{n\to\infty} T_{i=1}^{\infty} c_i$ exists since the sequence $(T_{i=1}^{n} c_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

The concept a fuzzy b-metric space is obtainable in (7).

Definition 2.5 (7): A 3-tuple (X, W, T) is called a fuzzy b- metric space (Fb-MS) if X is an arbitrary (nonempty) set, and T is a continues t- norm, W is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for all $a, c, z \in X$, $s, t > 0$:

- $W(c, a, t) > 0$,
- $W(c, a, t) = 1$ if and only if $c = a$,
- $W(c, a, t) = W(a, c, t)$,
- $W(a, c\ , s + t) \geq T\left(W(\mathrm{a}, \mathrm{z}, \frac{s}{b}), \mathrm{W}\left(z, c, \frac{t}{b}\right)\right)$,
- $W(c, a, .): [0, \infty) \to [0, 1]$ is continuous .

Another definition of Fb-MS is introduced in (1).

Definition 2.6 (1): A 3-tuple (X, W, T) is called a (Fb-MS) if X is an arbitrary (nonempty) set, and T is a continues t- norm, W is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for all $a, c, z \in X$, $s, t > 0$:

- $W(a, c, t) > 0$,
- 2 - $W(c, a, t) = 1$ if and only if $a = c$,
- $W(a, c, t) = G(c, a, t)$,
- $W(a, c\ , b(t + s)) \geq T(W(a, z, t), W(z, c, s))$,
- $W(c, a, .): [0, \infty) \to [0, 1]$ is continuous.

Let us give a simple application of a Fb-MS.

Example 2.7 [1, P.31]: Suppose $X = \mathbb{R}$, a function $W: X \times X \times [0, \infty)$ define by

$W(c,a,t)$=exp($\frac{-(c-a)^2}{\text{t}}$).Then (X,W,T_p) is a Fb-MS.

We now start with certain fundamental concepts.

Definition 2.8 (1): A sequence (c_n) in X is a converges to c if $W(c_n, c, t)$→1 as $n \to \infty$ for each $t > 0$. We can write $\lim_{n\to\infty} c_n = c$.

Definition 2.9 (1): If, for each $t > 0$ and $0 < \epsilon < 1$, there are $n_0 \in \mathbb{N}$ such that $W(c_n, c_m ,t) > 1-\epsilon$ for all $n, m \geq n_o$, then (c_n) is called a Cauchy sequence in X.

Proposition 2.10(1):A sequence (c_n) in X said to be a Cauchy sequence if $\lim_{n\to\infty} W(c_n,c_m,t)$ =1 for each $t > 0$.

Definition 2.11 (1): If every Cauchy sequence in a Fb-MS (X,W,T) is convergent ,then Fb-MS is said to be complete.

In [3], give a concept of a compact fuzzy metric space. We will generalized to Fb-MS.

Definition 2.12: If there exists a convergent subsequence for each sequence in X, then a Fb-MS (X,W,T) is called compact.

In [2] proved the following propositions, which are main to in our results.

Proposition 2.13 (2): Suppose (c_n) be a sequence in a Fb-MS (X,W,T), and let T be of H-type. If there is $\lambda \in (0,\frac{1}{b})$ such that

$$W(c_n, c_{n+1},t) \geq W\left(c_{n-1}, c_n,\frac{t}{\lambda}\right) ; \quad n \in \mathbb{N}, t > 0$$

and there are $c_0,c_1 \in X$ and $v \in (0,1)$ such that

$$\lim_{n\to\infty} T_{i=n}^{\infty} W\left(c_0, c_1,\frac{t}{v^i}\right) =1 ; t > 0$$

Then the sequence (c_n) is a Cauchy.

Proposition 2.14 (2): Let (X,W,T) be a Fb-MS. If for some $\lambda \in (0,1)$ and $a,c \in X$, $W(a,c,t) \geq W\left(a,c,\frac{t}{\lambda}\right); t > 0$, then $a = c$.

3. Results

In this section. We establish several fixed point theorems for a class of self -mappings in complete fuzzy b-metric spaces and compact fuzzy b- metric spaces by using the φ −function. First we need the following Definitions.

Definition 3.1: Suppose (X,W,T) be a Fb-MS, a function φ: [0,1] → [0,1] satisfies:

- φ is strictly decreasing ,
- φ is continuous,
- $\varphi(c) = 0$ if and only if $c = 1$,
- $\varphi\left(T\,(a,c)\right) = T\left(\varphi\,(a),\varphi\,(c)\right)$.

Is called φ-function.

We use φ-function to define two types of Contraction mappings

Definition 3.2: Let (X,W,T) be a Fb-MS, g be a self –mapping on X, and φ: [0,1] → [0,1] be φ − function and k: $(0,\infty) \to (0,1)$ be a mapping then

1- g is called φ – contraction of type (I) if there is $0 < \lambda < \frac{1}{b}$ such that, for all $a,c \in X, t > 0$,

$$\varphi(W(ga,gc,t) \leq k(t)\varphi\left(W\left(a,c\,,\frac{t}{\lambda}\right)\right) \quad (1)$$

2- g is called φ – contraction of type (II) if

$$\varphi\left(W(ga,gc,t)\right) \leq k(t)\varphi\left(W(a,c,t)\right) \quad (2)$$

For all $a,c \in X$ and $t > 0$.

Theorem 3.3: Let (X,W,T) be a complete Fb-MS and let T be of H-type and g be a φ –

contraction mapping of type (I). If for $n \in \mathbb{N}$, there is $c_0 \in X$ and $0 < v < 1$ such that $\lim_{n\to\infty} T_{i=n}^{\infty} W\left(c_0, gc_0, \frac{t}{v^i}\right) = 1$; $t > 0$.

Then there is a unique fixed point for g.

Proof : Let $c_0 \in X$. Define $c_{n+1} = gc_n$ for each $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. To prove g has a fixed point we have two cases

Case 1 if [$W(c_n, c_{n+1}, t) = 1$]: Thus there is $n_0 \in \mathbb{N}$ then $c_{n_0} = c_{n_0+1}$ i.e , $c_{n_0} = gc_{n_0}$ then c_{n_0} is a fixed point of g.

Case 2 if [$W(c_n, c_{n+1}, t) \neq 1$]: hence $c_n \neq c_{n+1}$ for all n. From Equation (1) we get

$$\varphi\big(W(c_n, c_{n+1}, t)\big) = \varphi(W(gc_{n-1}, gc_n, t)$$

$$\leq k(t)\varphi\left(W\left(c_{n-1}, c_n, \frac{t}{\lambda}\right)\right) \quad (3)$$

$$< \varphi\left(W\left(c_{n-1}, c_n, \frac{t}{\lambda}\right)\right).$$

Claim: $W(c_n, c_{n+1}, t) \geq W\left(c_{n-1}, c_n, \frac{t}{\lambda}\right)$ for all $n \in \mathbb{N}$ and $t > 0$. Assume $W(c_n, c_{n+1}, t) < W\left(c_{n-1}, c_n, \frac{t}{\lambda}\right)$, since φ is strictly decreasing, hence $\varphi\left(W\left(c_{n-1}, c_n, \frac{t}{\lambda}\right)\right) < \varphi\big(W(c_n, c_{n+1}, t)\big)$ which is contradiction with Equation (3), so $W(c_n, c_{n+1}, t) \geq W\left(c_{n-1}, c_n, \frac{t}{\lambda}\right)$ through Proposition (2.13) we hence (c_n) is a Cauchy sequence. As X is a complete Fb-MS, thus $c_n \to c$. Let $\sigma_1 \in (\lambda b, 1)$ and σ_2=1– σ_1. By the Definition ((2.5), part 4)).

$$W(gc, c, t) \geq T\left(W\left(gc, gc_n, \frac{t\sigma_1}{b}\right), W\left(gc_n, c, \frac{t\sigma_2}{b}\right)\right)$$

Since φ is strictly decreasing and by Definition ((3.1) , part (4)) implies that

$$0 \leq \varphi(W(gc, c, t)) \leq \varphi\left(T\left(W\left(gc, gc_n, \frac{t\sigma_1}{b}\right), W\left(gc_n, c, \frac{t\sigma_2}{b}\right)\right)\right)$$

$$= T\left(\varphi\left(W\left(gc, gc_n, \frac{t\sigma_1}{b}\right)\right), \varphi\left(W\left(gc_n, c, \frac{t\sigma_2}{b}\right)\right)\right)$$

By Equation (1) we get

$$0 \leq \varphi(W(gc, c, t) \leq T\left(\left(k\left(\frac{t\sigma_1}{b}\right)\varphi\left(W(c, c_n, \frac{t\sigma_1}{b\,\lambda}\right)\right), \varphi\left(W\left(gc_n, c, \frac{t\sigma_2}{b}\right)\right)\right)$$

As $\to \infty$, thanks to the continuity of T, W, φ and by definition ((3.1) , part (3)) $0 \leq \varphi\big(W(gc, c, t)\big) \leq T\left(\left(k\left(\frac{t\sigma_1}{b}\right)\varphi(1)\right), \varphi\,(1)\right) = 0$

We have $\varphi\big(W(gc, c, t)\big) = 0$ implies that $W(gc, c, t) = 1$, hence $g(c) = c$

Assume that g has another fixed point say z

Assume $W(c, z, t) < W\left(c, z, \frac{t}{\lambda}\right)$, since φ is strictly decreasing, then $\varphi(W(c, z, t)) < \varphi\left(W\left(c, z, \frac{t}{\lambda}\right)\right)$, which is contradiction with equation (1), we get $W(c, z, t) \geq W\left(c, z, \frac{t}{\lambda}\right)$. There for by Proposition (2.14) $c = z$ ■

In the following finding, we established the presence of a fixed point in a compact Fb-MS.

Theorem 3.4: Suppose (X, W, T) be a compact Fb-MS and g be continuous φ- contraction of type (II). Then there is a fixed point that is unique to g in X.

Proof : Assume $c_0 \in X$. Define $c_{n+1} = gc_n$ with $n \in \mathbb{N} \cup \{0\}$

Now, There are two cases:
Case 1: [If $g^n c_0 = g^{n+1} c_0$ for some] , so c_n be a fixed point of g .
Case 2: [If $g^n c_0 \neq g^{n+1} c_0$ for every n]: Since X be a compact Fb-MS, then there is a subsequence $\{g^{k(n)} c_0\}$ of $\{g^n c_0\}$ such that $g^{k(n)} c_0$ convergent to c as $n \to \infty$. According to continuity of g , we have

$lim_{n\to\infty} c_{k(n)+1} = g(c)$ and $lim_{n\to\infty} c_{k(n)+2} = g^2(c)$

$$\varphi\big(W(c, gc, t)\big) = \lim_{n\to\infty} \varphi\left(W\big(c_{k(n)}, c_{k(n)+1}, t\big)\right) = \lim_{n\to\infty} \varphi\left(W\big(c_{k(n)+1}, c_{k(n)+2,} t\big)\right) \tag{4}$$
$$= \varphi(W(gc, g^2 c, t)$$

As φ is continuous, we get
But Equation (4) contradiction with equation (2), as
$\varphi\big(W(gc, g^2 c, t)\big) \leq k(t)\varphi\big(W(c, gc, t)\big) < \varphi\big(W(c, gc, t)\big)$
Furthermore, Assume that g has two fixed point say z and p
$\varphi(W(z, c, t)) = \varphi(W(gz, gc, t) \leq k(t)\varphi(W(z, c, t) < \varphi(W(z, c, t)$
This lead to a contradiction. Therefore $z = c$ ■
Remark :The above theorem needs only condition(2) in the Definition (3.1).

4. Conclusion

Based on the results, we will complete the study of many authors about fixed point theory on fuzzy b- metric spaces, by using $\varphi -$ Contraction mappings of type I and II to show existence of fixed points for this type of self-mapping.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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