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Some Oscillatory Results of Nonlinear Neutral Differential Equation

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Abstract

In the last decades, functional differential equations have attracted the attention of many researchers; they were interested in the theory and its applications. The most common differential equations of functional type are advanced, neutral, and delay DEs. The theory of oscillatory DEs with retarded arguments has a paramount effect on the qualitative properties of DEs. It is essential to deduce conditions for oscillatory and non-oscillatory solutions. The objective of this paper is to obtain oscillatory conditions for differential equations with retarded arguments. So, the oscillatory behavior has been considered in the nonlinear differential equations (DEs) of neutral type with three delays. Some important conditions of all functions have been submitted. The sufficient conditions to secure the oscillatory property have been deduced. We dealt with special cases for delays to obtain some desired conditions for oscillation property. In fact, all new results and conditions innovate, and improved some oscillation properties that appeared in the literature. Some application examples with specific functions for important results have been illustrated and applied to all conditions. Two results with some different conditions have been obtained to get oscillatory behavior for DE. A new relationship between delays and other functions to get desired property has been formulated. Some application examples explained to ensure the importance of our results compared with other previous studies.

Keywords: property of oscillation, multiple delays, NDE, nonlinear case.

1. Introduction

The continuous development in the fields of science has led to the formulation of many physical laws, which often appear in the form of DEs when reformulated in mathematical form. We can also describe applied problems mathematically using DEs; thus, DEs play a critical role in solving practical problems (1, 2). The theory of DEs is a great tool in modeling many scientific problems in population dynamics, optimal control, and nonlinear problems (3). There are several methods for solving this kind of equation, such as the variational iteration method and the homotopy transforms analysis method (2, 4). Furthermore, some researchers considered some properties that describe all solutions, such as asymptotic behavior, oscillatory property, and stability to different kinds of DEs (5-8). In publications, different authors established some conditions for oscillation behavior and asymptotic property for nonlinear NDEs (9-17).

2. Materials and Methods

Consider nonlinear NDE with several delays:

$$\frac{d}{d\mathrm{t}}\left[\rho(\mathrm{t})\frac{d}{d\mathrm{t}}\Big(\mathfrak{x}(\mathrm{t})+\gamma_1(\mathrm{t})\mathfrak{x}\big(\tau(\mathrm{t})\big)\Big)\right]+\gamma_2(\mathrm{t})\hbar\big(\mathfrak{x}(\alpha(\mathrm{t}))\big)+\gamma_3(\mathrm{t})\big(\digamma(\mathfrak{x}(\sigma(\mathrm{t}))\big) = 0 \quad (1)$$

Supposing the below conditions hold:

H1: $\gamma_1(\mathrm{t}), \gamma_2(\mathrm{t}), \gamma_3(\mathrm{t}) \in C([\mathrm{t}_0,\infty),(0,\infty)), \gamma_1(\mathrm{t}) \leq \gamma$.

H2: $\rho(\mathrm{t}) \in C([t_0,\infty),(0,\infty)$ with $\pi(\mathrm{t}) = \int_{\mathrm{t}_0}^{\mathrm{t}} \frac{1}{\rho(s)} ds$ and $\int_{\mathrm{t}_0}^{\infty} \frac{1}{\rho(\mathrm{t})} d\mathrm{t} = \infty$.

H3: $\sigma(\mathrm{t}), \tau(\mathrm{t}), \alpha(\mathrm{t}) \in C([\mathrm{t}_0,\infty),\mathbb{R})$ with $\lim_{\mathrm{t}\to\infty}\sigma(\mathrm{t}) = \infty$, $\lim_{\mathrm{t}\to\infty}\tau(\mathrm{t}) = \infty$ and $\lim_{\mathrm{t}\to\infty}\alpha(\mathrm{t}) = \infty$.

The delays σ, τ, α are continuous strictly increasing and invertible.

H4: $\hbar(\mathfrak{x}), \digamma(\mathfrak{x}) \in C(\mathbb{R},\mathbb{R})$, $\hbar(\mathfrak{x}) \geq \phi_1\mathfrak{x} > 0$ and $\digamma(\mathfrak{x}) \geq \phi_2\mathfrak{x} > 0$ for $\mathfrak{x} \neq 0$ and ϕ_1, ϕ_2 are positive constants.

H5: Let $\mathcal{Z}(\mathrm{t}) = \mathfrak{x}(\mathrm{t}) + \gamma_1(\mathrm{t})\mathfrak{x}\big(\tau(\mathrm{t})\big)$ twice continuously differentiable in (1)

Definition 1.1 (18)

A nontrivial solution $\mathfrak{x}(\mathrm{t})$ to eq. (1) is called non-oscillatory if it is either eventually a positive solution or it is eventually a negative, if it doesn't, so $\mathfrak{x}(\mathrm{t})$ satisfies oscillatory property.

3. Results

Theorem1: Assume that H1- H5 are true with condition:

$$\int_{\mathrm{t}_0}^{\infty}\Gamma_1(\mathrm{t})dt = \int_{\mathrm{t}_0}^{\infty}\Gamma_2(\mathrm{t})d\mathrm{t} = \infty \quad (2)$$

$$\Gamma_1(\mathrm{t}) = \min\left\{\gamma_2(\mathrm{t}), \gamma_2\left(\alpha^{-1}(\tau(\alpha(\mathrm{t}))\right), \gamma_2\left(\sigma^{-1}(\tau(\sigma(\mathrm{t}))\right)\right\},$$

$$\Gamma_2(\mathrm{t}) = \min\left\{\gamma_3(\mathrm{t}), \gamma_3\left(\alpha^{-1}(\tau(\alpha(\mathrm{t}))\right), \gamma_3\left(\sigma^{-1}(\tau(\sigma(\mathrm{t}))\right)\right\}.$$

Then the oscillation property holds.

Proof: Let a solution $\mathfrak{x}(\mathrm{t})$ has non-oscillatory behaviour to eq. (1). So, without loss of generality, we suppose that the solution is eventually positive. It is obvious that $\mathcal{Z}(\mathrm{t}) > 0$. From equation (1), we get:

$$\frac{d}{d\mathrm{t}}\left[\rho(\mathrm{t})\frac{d}{d\mathrm{t}}\mathcal{Z}(\mathrm{t})\right] \leq 0$$

, so $\rho(\mathrm{t})\frac{d}{dt}\mathcal{Z}(\mathrm{t})$ is nonincreasing eventually.

$$\frac{d}{dt}\left[\rho(\mathrm{t})\frac{d}{dt}\big(\mathcal{Z}(\mathrm{t})\big)\right] = -\gamma_2(\mathrm{t})\hbar\left(\mathfrak{x}\big(\alpha(\mathrm{t})\big)\right) - \gamma_3(\mathrm{t})\digamma\big(\mathfrak{x}(\sigma(\mathrm{t}))\big) < 0$$

$\rho(\mathrm{t})\frac{d}{dt}\big(\mathcal{Z}(\mathrm{t})\big)$ is nonincreasing, we assume that $\frac{d}{dt}\big(\mathcal{Z}(t_1)\big) > 0, \forall \mathrm{t} \geq \mathrm{t}_1$

Otherwise, there exists $\mathrm{t}_2 \geq \mathrm{t}_1 \ni \frac{d}{dt}\big(\mathcal{Z}(\mathrm{t}_1)\big) < 0$, for $\mathrm{t} \geq \mathrm{t}_2$.

Then

$$\left[\rho(\mathrm{t})\frac{d}{dt}\big(\mathcal{Z}(\mathrm{t})\big) \leq \rho(\mathrm{t}_2)\frac{d}{dt}\big(\mathcal{Z}(\mathrm{t}_2)\big)\right], \quad \forall \mathrm{t} \geq \mathrm{t}_2.$$

$$\mathcal{Z}(\mathrm{t}) \leq \mathcal{Z}(\mathrm{t}_2)\, \rho(\mathrm{t}_2) \int_{\mathrm{t}_2}^{\mathrm{t}} \frac{ds}{\rho(s)} \quad (3)$$

When $\mathrm{t} \to \infty$ in (3), then $\mathcal{Z}(\mathrm{t}) \to -\infty$ contradiction.

So, we have $\frac{d}{dt}\big(\mathcal{Z}(\mathrm{t}_1)\big) > 0. \forall \mathrm{t} \geq \mathrm{t}_1$

We replace $\alpha^{-1}(\tau(\alpha(\mathrm{t})))$ instead of t in (1) once and $\sigma^{-1}(\tau(\sigma(\mathrm{t})))$ instead of t in (1) a second time.
So, we get:

$$\frac{d}{dt}\left[\rho(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\frac{d}{dt}\left(x(\alpha^{-1}(\tau(\alpha(\mathrm{t})))) + \gamma_1(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))x\left(\tau(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\right)\right)\right] + \gamma_2(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\hbar\left(x(\alpha(\alpha^{-1}(\tau(\alpha(\mathrm{t})))))\right) + \gamma_3(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\mathrm{F}\left(x(\sigma(\alpha^{-1}(\tau(\alpha(\mathrm{t})))))\right) = 0 \quad (4)$$

$$\frac{d}{dt}\left[\rho(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\frac{d}{dt}\left(x(\sigma^{-1}(\tau(\sigma(\mathrm{t})))) + \gamma_1(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))x(\tau(\sigma^{-1}(\tau(\sigma(\mathrm{t})))))\right)\right] + \gamma_2(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\hbar\left(x(\alpha(\sigma^{-1}(\tau(\sigma(\mathrm{t})))))\right) + \gamma_3(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\mathrm{F}\left(x(\sigma(\sigma^{-1}(\tau(\sigma(\mathrm{t})))))\right) = 0 \quad (5)$$

By multiplying equations (4) and (5) by γ and summing equations (1), (4), and (5), we get:

$$\frac{d}{dt}\left[\rho(\mathrm{t})\frac{d}{dt}\left(x(\mathrm{t}) + \gamma_1(\mathrm{t})x(\tau(\mathrm{t}))\right)\right] + \gamma_2(\mathrm{t})\hbar(x(\alpha(\mathrm{t}))) + \gamma_3(\mathrm{t})\mathrm{F}(x(\sigma(\mathrm{t}))) +$$
$$\frac{d}{dt}\left[\rho(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\frac{d}{dt}\left(x(\alpha^{-1}(\tau(\alpha(\mathrm{t})))) + \gamma_1(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))x\left(\tau(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\right)\right)\right] +$$
$$\gamma_2(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\hbar\left(x\left((\tau(\alpha(\mathrm{t})))\right)\right) + \gamma_3(\alpha^{-1}(\tau(\alpha(\mathrm{t}))))\mathrm{F}\left(x(\sigma(\alpha^{-1}(\tau(\alpha(\mathrm{t})))))\right) +$$
$$\frac{d}{dt}\left[\rho(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\frac{d}{dt}\left(x(\sigma^{-1}(\tau(\sigma(\mathrm{t})))) + \gamma_1(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))x(\tau(\sigma^{-1}(\tau(\sigma(\mathrm{t})))))\right)\right] +$$
$$\gamma_2(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\hbar\left(x(\alpha(\sigma^{-1}(\tau(\sigma(\mathrm{t})))))\right) + \gamma_3(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\mathrm{F}\left(x\left(\tau(\sigma(\mathrm{t}))\right)\right) = 0 \quad (6)$$

Substituting by $\Gamma_1(\mathrm{t})$ and $\Gamma_2(\mathrm{t})$ in (6) with deleting some terms:

$$\frac{d}{dt}\left[\rho(\mathrm{t})\frac{d}{dt}(\mathcal{Z}(\mathrm{t}))\right] + \Gamma_1(\mathrm{t})\hbar(x(\alpha(\mathrm{t}))) + \Gamma_2(\mathrm{t})\mathrm{F}(x(\sigma(\mathrm{t}))) +$$
$$\gamma\frac{d}{dt}\left[\rho(\alpha^{-1}(\tau(\alpha(\mathrm{t})))\frac{d}{dt}\left(\mathcal{Z}\left(\alpha^{-1}(\tau(\alpha(\mathrm{t})))\right)\right)\right] + \gamma\Gamma_1(\mathrm{t})\hbar\left(x\left((\tau(\alpha(\mathrm{t})))\right)\right) +$$
$$\gamma\frac{d}{dt}\left[\rho(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\frac{d}{dt}\left(\mathcal{Z}\left(\sigma^{-1}(\tau(\sigma(\mathrm{t})))\right)\right)\right] + \gamma\Gamma_2(\mathrm{t})\mathrm{F}\left(x\left(\tau(\sigma(\mathrm{t}))\right)\right) \leq 0 \quad (7)$$

From condition H4, we have:

$$\frac{d}{dt}\left[\rho(\mathrm{t})\frac{d}{dt}(\mathcal{Z}(\mathrm{t}))\right] + \phi_1\Gamma_1(\mathrm{t})x(\alpha(\mathrm{t})) + \phi_2\Gamma_2(\mathrm{t})x(\sigma(\mathrm{t}))$$
$$+\gamma\frac{d}{dt}\left[\rho(\alpha^{-1}(\tau(\alpha(\mathrm{t})))\frac{d}{dt}\left(\mathcal{Z}\left(\alpha^{-1}(\tau(\alpha(\mathrm{t})))\right)\right)\right] + \gamma\phi_1\Gamma_1(\mathrm{t})x\left((\tau(\alpha(\mathrm{t})))\right)$$
$$+\gamma\frac{d}{dt}\left[\rho(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\frac{d}{dt}\left(\mathcal{Z}\left(\sigma^{-1}(\tau(\sigma(\mathrm{t})))\right)\right)\right] + \gamma\phi_2\Gamma_2(\mathrm{t})x\left(\tau(\sigma(\mathrm{t}))\right) \leq 0$$

Or

$$\frac{d}{dt}\left[\rho(\mathrm{t})\frac{d}{dt}(\mathcal{Z}(\mathrm{t}))\right] + \gamma\frac{d}{dt}\left[\rho(\alpha^{-1}(\tau(\alpha(\mathrm{t})))\frac{d}{dt}\left(\mathcal{Z}\left(\alpha^{-1}(\tau(\alpha(\mathrm{t})))\right)\right)\right] +$$
$$\gamma\frac{d}{dt}\left[\rho(\sigma^{-1}(\tau(\sigma(\mathrm{t}))))\frac{d}{dt}\left(\mathcal{Z}\left(\sigma^{-1}(\tau(\sigma(\mathrm{t})))\right)\right)\right] + \phi_1\Gamma_1(\mathrm{t})\mathcal{Z}(\alpha(\mathrm{t})) + \phi_2\Gamma_2(\mathrm{t})\mathcal{Z}(\sigma(\mathrm{t})) \leq 0 \quad (8)$$

We integrate the inequality (8) from t_1 to t:

$$\phi_1 \int_{t_1}^{t} \phi_1 \Gamma_1(s)\, \mathcal{Z}(\alpha(s))ds + \phi_2 \int_{t_1}^{t} \Gamma_2(s)\, \phi_2 \mathcal{Z}(\sigma(s))ds$$

$$\le -\rho(t_1)\frac{d}{dt}\big(\mathcal{Z}(t_1)\big) + \rho(t)\frac{d}{dt}\big(\mathcal{Z}(t)\big)$$
$$-\gamma\rho(\alpha^{-1}\,\big(\tau\big(\alpha(t_1)\big)\big)\frac{d}{dt}\Big(\mathcal{Z}\,\Big(\alpha^{-1}\,(\tau\big(\alpha(t_1)\big))\Big)\Big)$$
$$+\gamma\rho(\alpha^{-1}\,\big(\tau\big(\alpha(t)\big)\big)\frac{d}{dt}\Big(\mathcal{Z}\,\Big(\alpha^{-1}\,(\tau\big(\alpha(t)\big))\Big)\Big)$$
$$-\gamma\rho(\sigma^{-1}\,(\tau(\sigma(t_1)))\frac{d}{dt}\Big(\mathcal{Z}(\sigma^{-1}\,(\tau(\sigma(t_1))))\Big)$$
$$+\gamma\rho(\sigma^{-1}\,(\tau(\sigma(t)))\frac{d}{dt}\Big(\mathcal{Z}\big(\sigma^{-1}\,(\tau(\sigma(t)))\big)\Big)$$

$\rho(t)\frac{d}{dt}\big(\mathcal{Z}(t)\big)$ is nonincreasing function, so as t goes to infinity, we conclude that:

$$\int_{t_1}^{t} \Gamma_1(\xi)\, d\xi < \infty \text{ and } \int_{t_1}^{t} \Gamma_2(\xi)\, d\xi < \infty$$

a contradiction !

Theorem2: Assume that H1- H5 hold, with condition (2) in addition to:

$$\int_{t_0}^{t} \frac{1}{\rho(\xi)} d\xi = \infty\,, \alpha^{-1}(\tau\big(\alpha(t)\big) \le \sigma^{-1}\,(\tau(\sigma(t)) \le t$$

$$\lim_{t\to\infty}\,\inf\int_{\delta(\alpha^{-1}\left(\tau^{-1}\left(\alpha(\alpha(t))\right)\right)}^{t} \phi_1 \Gamma_1(\xi)\frac{\pi\Big(\delta(\alpha^{-1}\Big(\tau^{-1}\big(\alpha(\alpha(\xi))\big)\Big)\Big)}{(1+2\gamma)} d\xi > \frac{1}{e} \tag{9}$$

Then the oscillation property holds.

Proof: Let a solution $\mathfrak{x}(t)$ has non-oscillatory behaviour to eq. (1). So, without loss of generality, we suppose that the solution is eventually positive. It is obvious that $\mathcal{Z}(t) > 0$. In the same steps as theorem (1), we get inequality (8). Let

$$¥(t) = \rho(t)\frac{d}{dt}\big(\mathcal{Z}(t)\big) + \gamma\rho(\alpha^{-1}\left(\tau\big(\alpha(t)\big)\right))\frac{d}{dt}\Big(\mathcal{Z}\left(\alpha^{-1}(\tau\big(\alpha(t)\big)\right)\Big) + \gamma\rho(\sigma^{-1}\,\left(\tau\big(\sigma(t)\big)\right)\frac{d}{dt}\Big(\mathcal{Z}\left(\sigma^{-1}\,(\tau\big(\sigma(t)\big)\right)\Big) \tag{10}$$

Substituting about $¥(t)$ in (8), we get:

$$\frac{d}{dt}\big(¥(t)\big) + \phi_1\Gamma_1(t)\mathcal{Z}(\alpha(t)) + \phi_2\Gamma_2(t)\mathcal{Z}(\sigma(t)) \le 0 \quad \text{for } t \ge t_2 \ge t_1$$

Or

$$\frac{d}{dt}\big(¥(t)\big) + \phi_1\Gamma_1(t)\mathcal{Z}(\alpha(t)) \le 0, \quad \forall t \ge t_2 \ge t_1 \tag{11}$$

But $\rho(t)\frac{d}{dt}\big(\mathcal{Z}(t)\big)$ is nonincreasing, so:

$$\rho(t)\frac{d}{dt}\big(\mathcal{Z}(t)\big) \le \rho(\xi)\frac{d}{d\xi}\big(\mathcal{Z}(\xi)\big), \text{for } t \ge \xi$$

Now, we divide last inequality by $\rho(\xi)$ and integrates it from t_1 to t, we have:

$$\rho(t)\frac{d}{dt}\big(\mathcal{Z}(t)\big)\int_{t_1}^{t}\frac{1}{\rho(\xi)}d\xi \le \mathcal{Z}(t) - \mathcal{Z}(t_1) \le \mathcal{Z}(t)$$

$$\pi(\mathrm{t})\rho(\mathrm{t})\frac{d}{d\mathrm{t}}\big(\mathcal{Z}(\mathrm{t})\big) \le \mathcal{Z}(\mathrm{t})$$

Or

$$\rho(\mathrm{t})\frac{d}{dt}\big(\mathcal{Z}(\mathrm{t})\big) \le \frac{\mathcal{Z}(\mathrm{t})}{\pi(\mathrm{t})} \tag{12}$$

But $\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big) \le \mathrm{t}$, so we have:

$$\begin{aligned}¥(\mathrm{t}) \le{}& \rho(\alpha^{-1}\left(\tau\big(\alpha(\mathrm{t})\big)\right))\frac{d}{d\mathrm{t}}\left(\mathcal{Z}\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right)\right)\\ &+ \gamma\rho(\alpha^{-1}\left(\tau\big(\alpha(\mathrm{t})\big)\right))\frac{d}{d\mathrm{t}}\left(\mathcal{Z}\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right)\right)\\ &+ \gamma\rho(\sigma^{-1}\left(\tau\big(\sigma(\mathrm{t})\big)\right)\frac{d}{d\mathrm{t}}\left(\mathcal{Z}\left(\sigma^{-1}\left(\tau\big(\sigma(\mathrm{t})\big)\right)\right)\right)\end{aligned}$$

And $\alpha^{-1}(\tau(\mathrm{t})) \le \sigma^{-1}(\tau(\mathrm{t}))$ So,

$$¥(\mathrm{t}) \le (1+2\gamma)\rho\left(\alpha^{-1}\left(\tau\big(\alpha(\mathrm{t})\big)\right)\right)\frac{d}{dt}\left(\mathcal{Z}\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right)\right) \tag{13}$$

From (11) and (12), we get:

$$¥(\mathrm{t}) \le (1+2\gamma)\frac{\mathcal{Z}\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right)}{\pi\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right)}$$

Or

$$\frac{\pi\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right)}{(1+2\gamma)}¥(t) \le \mathcal{Z}\left(\alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big))\right) \tag{14}$$

$$\begin{gathered}\delta(\mathrm{t}) = \alpha^{-1}(\tau\big(\alpha(\mathrm{t})\big)) < \mathrm{t}\\ \delta(\alpha^{-1}(\mathrm{t})) = \alpha^{-1}(\tau(\mathrm{t}))\\ \delta(\alpha^{-1}\big(\tau^{-1}(\mathrm{t})\big)) = \alpha^{-1}(\mathrm{t})\\ \delta(\alpha^{-1}\left(\tau^{-1}\left(\alpha\big(\alpha(\mathrm{t})\big)\right)\right)) = \alpha(\mathrm{t})\end{gathered}$$

From (10), we get:

$$\frac{d}{dt}\big(¥(\mathrm{t})\big) + \phi_1\Gamma_1(\mathrm{t})\mathcal{Z}\left(\delta(\alpha^{-1}\left(\tau^{-1}\left(\alpha\big(\alpha(\mathrm{t})\big)\right)\right))\right) \le 0 \quad \text{for } t \ge \mathrm{t}_2 \ge \mathrm{t}_1 \tag{15}$$

We combine (13) and (14), reducing:

$$\frac{d}{dt}\big(¥(\mathrm{t})\big) + \phi_1\Gamma_1(\mathrm{t})\mathcal{Z}\left(\delta(\alpha^{-1}\left(\tau^{-1}\left(\alpha\big(\alpha(\mathrm{t})\big)\right)\right))\right) \le 0 \quad \text{for } \mathrm{t} \ge \mathrm{t}_2 \ge \mathrm{t}_1 \tag{16}$$

From (13) and (15), we get:

$$\frac{d}{dt}\big(¥(\mathrm{t})\big) + \phi_1\Gamma_1(\mathrm{t})\frac{\pi\left(\delta(\alpha^{-1}\left(\tau^{-1}\left(\alpha\big(\alpha(\mathrm{t})\big)\right)\right))\right)}{(1+2\gamma)}¥\left(\delta(\alpha^{-1}\left(\tau^{-1}\big(\alpha(\mathrm{t})\big)\right)\right) \le 0,\ \forall \mathrm{t} \ge \mathrm{t}_2 \ge \mathrm{t}_1 \tag{17}$$

By theorem 2.2.9 in (30) the inequality (17) does not have eventually positive solution, a contradiction !

4.1. Illustrative Examples

In this section, we present some examples to support the obtained results

4.1.1. Example 1

Let us consider nonlinear neutral differential equation as follows:

$$\frac{d}{dt}\left[\mathrm{t}\frac{d}{dt}\left(\mathfrak{x}(\mathrm{t})+\frac{1}{2\mathrm{t}}\mathfrak{x}(\mathrm{t}-1)\right)\right]+\frac{1}{2\mathrm{t}}\mathfrak{x}^2(\mathrm{t}-2)+\frac{1}{2\mathrm{t}}\,\mathfrak{x}^2(\mathrm{t}-3)=0,$$

$$\mathrm{t}\geq \mathrm{t}_0=2 \tag{18}$$

To satisfy all conditions of theorem (1):

$$\int_{\mathrm{t}_0}^{\infty}\frac{1}{\rho(\mathrm{t})}d\mathrm{t}=\int_{2}^{\infty}\frac{1}{\mathrm{t}}d\mathrm{t}=\infty$$

$\Gamma_1(\mathrm{t})=\min\left\{\gamma_2(\mathrm{t}),\gamma_2\left(\alpha^{-1}(\tau(\alpha(\mathrm{t}))\right),\gamma_2\left(\sigma^{-1}(\tau(\sigma(\mathrm{t}))\right)\right\}$,

$\Gamma_2(\mathrm{t})=\min\left\{\gamma_3(\mathrm{t}),\gamma_3\left(\alpha^{-1}(\tau(\alpha(\mathrm{t}))\right),\gamma_3\left(\sigma^{-1}(\tau(\sigma(\mathrm{t}))\right)\right\}$.

$$=Min\left\{\frac{1}{2\mathrm{t}},\frac{1}{2(\mathrm{t}-1)},\right\}=\frac{1}{2(\mathrm{t}-2)},\mathrm{t}\geq 2$$

$$\int_{\mathrm{t}_0}^{\infty}\Gamma_1(\mathrm{t})\,d\mathrm{t}=\int_{2}^{\infty}\frac{1}{2(\mathrm{t}-2)}d\mathrm{t}$$

$$=\frac{1}{2}ln(\mathrm{t}-2)\Big]_2^{\infty}=\frac{1}{2}(ln\,\infty-ln\,0)=\infty$$

Also, we get:

$$,\int_{\mathrm{t}_0}^{\infty}\Gamma_2(\mathrm{t})d\mathrm{t}=\int_{2}^{\infty}\frac{1}{2(\mathrm{t}-2)}d\mathrm{t}=\infty$$

We satisfied all conditions for theorem (1) are satisfied, so all solutions of eq. (1) oscillate.

4.1.2. Example 2

Let us consider nonlinear neutral differential equation as follows:

$$\frac{d}{dt}\left[(\mathrm{t}+1)\,\frac{d}{dt}\left(\mathfrak{x}(\mathrm{t})+\frac{1}{\mathrm{t}}\mathfrak{x}(\mathrm{t}-6)\right)\right]+\frac{5}{ln\,\mathrm{t}}\mathfrak{x}^2(\mathrm{t}-6)+12\,\mathfrak{x}^2(\mathrm{t}-7)=0,\ \ \mathrm{t}\geq \mathrm{t}_0$$

$$\mathrm{t}\geq \mathrm{t}_0=1 \tag{19}$$

To satisfy all conditions of theorem (2):

$$\int_{\mathrm{t}_0}^{\infty}\frac{1}{\rho(\mathrm{t})}d t=\int_{1}^{\infty}\frac{1}{\mathrm{t}+1}d\mathrm{t}=\infty$$

$$\pi(\mathrm{t})=\int_{\mathrm{t}_0}^{\mathrm{t}}\frac{1}{\rho(\xi)}d\xi=\int_{1}^{\mathrm{t}}\frac{1}{\xi+1}d\xi=ln(\mathrm{t}+1)$$

$$\delta(\mathrm{t})=\mathrm{t}-1$$

$$\pi\left(\delta(\alpha^{-1}\left(\tau^{-1}\left(\alpha(\alpha(\mathrm{t})\right)\right)\right)\right)=\pi(\mathrm{t}-1)=ln\,\mathrm{t},$$

$\Gamma_2(\mathrm{t})=\min\left\{\gamma_3(\mathrm{t}),\gamma_3\left(\alpha^{-1}(\tau(\alpha(\mathrm{t}))\right),\gamma_3\left(\sigma^{-1}(\tau(\sigma(\mathrm{t}))\right)\right\}$.

$$=Min\left\{\frac{1}{ln\,\mathrm{t}},\frac{1}{ln(\mathrm{t}-6)},\frac{1}{ln(\mathrm{t}-6)}\right\}=\frac{1}{ln\,\mathrm{t}},\mathrm{t}\geq 1$$

To satisfy the condition (9) with $\phi=6$ and $\gamma=1$

$$\lim_{\mathrm{t}\to\infty}inf\int_{\mathrm{t}-1}^{\mathrm{t}}\phi\,\Gamma_1(\xi)\frac{\pi(\xi-1)}{(1+2\gamma)}d\xi$$

$$\liminf_{t\to\infty} \int_{t-1}^{t} 6 \frac{1}{\ln \xi} \frac{\ln \xi}{3} d\xi = 2 > \frac{1}{e}$$

All the conditions of theorem (2) are satisfied, so all solutions of eq. (1) oscillate.

5. Discussion

Through the paper, qualitative properties are studied as oscillatory properties. It can be noted that the influence of changing delays in main results takes a huge role to control in sufficient conditions to obtain the oscillation property; it is explained in application examples.

6. Conclusion

In this search, oscillatory behavior of second-order NDE is considered in the nonlinear case. Specific conditions have been established for known functions in the delay differential eq. (1). A new relationship between the deviating arguments and known functions to get oscillatory property has been built. Application examples are presented to show the importance of our results compared with other previous studies. In theorem 2, a new condition is submitted to get oscillatory behavior of equation (1) with an application example. We improved some previous theorems in the literature by taking the delays as functions. As future works, it would be interesting to extend the results of this article to higher-order nonlinear Des.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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