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New Exponentiated Exponential- Rayleigh Distribution: Structure and Properties

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Abstract

In this research, we will introduce a new distribution, which is called the new exponentiated exponential-Rayleigh distribution, which is built by adding the shape parameter to the cumulative function of the exponential-Rayleigh distribution resulting from merging the cdf function of three parameters continuous distribution. Two of these parameters are scale parameters, and another is the shape parameter and this is to make this new continuous statistical distribution more flexible than other continuous statistical distributions. Besides, discuss the mathematical and statistical properties of this new distribution, including (the mode, the median, the moments about the origin, the coefficients of skewness and kurtosis, the characteristic function, the moment-generating function, the factorial moments-generating function, and the mean time to failure). The shape property of each of the pdf, cdf, and hazard functions was also studied. Multiple shapes were drawn with different parameter values for each of the PDF, CDF, survival, and hazard functions. An organisational table was also created for some of these properties.

Keywords: Exponential distribution, Rayleigh distribution, Exponential Rayleigh distribution, Moments about the origin, Quantile function.

1. Introduction

In order for statistical distributions to be important tools in various disciplines, these distributions had to be flexible distributions that help in understanding the data (1-3). Because the current distributions are restricted distributions and these restrictions make them insufficient in modelling diverse data, many scientists have resorted to integrating or adding a new parameter to these distributions in order to be more flexible (4-6). There are several methods to add a new parameter to statistical distributions, among these methods is adding a new parameter to the cumulative distribution function (CDF) (7, 8). There have been many scientists who have followed this method to add a new parameter to the distribution (9,10). In 2008, it was presented the exponentiated exponential distribution, a novel distribution that serves as a failure time distribution or the generalised exponential distribution (11). In 2010, it was presented a study on a mixture of the Pareto distribution and the Exponentiated exponential distribution (12). Also in 2012, it was introduced the extended Exponentiated Weibull distribution with Gamma. (13), Also in 2012, it was presented the inverted Exponentiated exponential Weibull distribution (14). In 2013, it was introduced the class of generalised exponentiated exponential distributions, which is a new class of distributions that extends the exponentiated exponential type distributions. (15), Also in 2013, it was presented the extension of the family of Exponentiated exponential T-X distributions with some applications where it was noted Hence, the T-X distribution can be generated with an upper limit of -log (1-F(x)), and it is obvious that alternative T-X distribution types can be generated with different higher limits. (16). Also in this year, it was presented the complementary exponentiated exponential geometric distribution of life span, where they proposed expanding the exponential distribution by fusing it with a geometric distribution to create a new distribution family (17), It was presented the modified exponential gamma distribution in 2014: Generalisation and extension of the Gamma distribution probabilities (18). In 2015, it was presented the exponentiated power Lindley distribution, which is a new distribution with three parameters called the exponential Lindley distribution [EPLD] (19). In 2017, it was presented the Exponentiated-Exponential Weibull distribution with applications, where they presented a novel four-parameter model that uses the (EW-G) distribution as a competitive extension of the Weibull distribution, which offers some new distributions and can be used to access several contemporary distributions (20). In 2018, it was was able to present the Exponentiated BURR X exponential distribution, which is the Burr X exponential distribution (BrxEE) and is a novel version of the (EE) model from which some of its statistical properties, including moments, incomplete moments, moment generating function, average deviations, and probability-weighted moments, were derived, and others (21). In the same year, it was introduced the Exponentiated Exponential Gamma distribution, where he defined the exponential distribution of Gamma and also introduced the Shannon entropy of (EGED) (22). In 2019, it was presented about the Exponentiated Weibull Rayleigh distribution (EWRD) (23). In 2020, it was presented the Exponentiated Exponential Weibull Distribution with mathematical properties and application and came up with the proposal of the Exponential Weibull Distribution (EEWD) based on the idea of (24). Also in the same year, it was presented estimation techniques and applications to engineering data for the Odd Exponentiated half-logistic exponential distribution (25). In 2021, it was presented a new exponentiated generalised linear exponential distribution with properties and applications (26). In 2022, Dhungana G. and Kumar V. reported the Exponentiated Odd Lomax exponential distribution with application to COVID-19 death cases of Nepal (27). In 2023, it was presented an extension of the Exponentiated Rayleigh distribution with properties and applications, adding the shape parameter to improve and enhance the flexibility of this distribution (28). Finally, Lamyaa Khalid Hussein and Iden Hassan Hussein in 2023 represent a new distribution that is mixed between exponential and Rayleigh distributions (29). In this research, we will present a new mixture of continuous distributions, which we call the New Exponentiated Exponential-Rayleigh distribution properties and Applications by adding the shape parameter to the (CDF) function (30). The results of the document include the following: In Section 2, we introduced the fundamental statistical functions, including the probability density function, reciprocal density function, survival function, and hazard rate function, and explained how we extended the probability density function to obtain the Exponentiated Exponential-Rayleigh Distribution. The statistical and mathematical characteristics of the new distribution, including the mode, moments around the origin, variance, skewness coefficient, characteristic function, moment generating function, and factorial moment generating function, are finally represented in Section 3.

2.The Exponenatied Exponential- Rayleigh Distribution:

$$F_{EER}(x) = (1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda} \quad \delta, \alpha, \lambda > 0, x > 0 \tag{1}$$

$$f_{EER}(x; \delta, \alpha, \lambda) = \lambda(\delta + \alpha x)e^{-(\delta x + \frac{\alpha}{2}x^2)}(1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda - 1} \tag{2}$$

and survival (reliability) function defined as:

$$S_{EER}(x) = 1 - F(x) = 1 - (1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda} \tag{3}$$

So the hazard rate function is:

$$\mathrm{h(x)} = \frac{f(x)}{s(x)} = \frac{\lambda(\delta + \alpha x)e^{-(\delta x + \frac{\alpha}{2}x^2)}(1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda - 1}}{1 - (1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda}} \tag{4}$$

$f(x) \geq 0 \quad \forall \quad \delta, \alpha, \lambda > 0, x > 0$

To proof $\int_0^{\infty} f(x)dx = 1$

$$\int_0^{\infty} \lambda(\delta + \alpha \mathrm{x})\mathrm{e}^{-(\delta \mathrm{x} + \frac{\alpha}{2}\mathrm{x}^2)}(1 - \mathrm{e}^{-(\delta \mathrm{x} + \frac{\alpha}{2}\mathrm{x}^2)})^{\lambda - 1}\mathrm{dx} = \left[(1 - \mathrm{e}^{-(\delta \mathrm{x} + \frac{\alpha}{2}\mathrm{x}^2)})^{\lambda}\right]_0^{\infty}$$

$$= \left[(1 - \mathrm{e}^{-(\infty)})^{\lambda} - (1 - \mathrm{e}^0)^{\lambda}\right] = \left[(1 - 0)^{\lambda} - (1 - 1)^{\lambda}\right] = 1 - 0 = 1$$

2.1. Expand the Probability Density function:

This step has been explained in the previous steps.

$$f(x) = \lambda(\delta + \alpha x)e^{-(\delta x + \frac{\alpha}{2}x^2)}(1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda}(1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{-1}$$

$$f(x) = \lambda(\delta + \alpha x)e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)} \sum_{k=0}^{\infty}(-1)^k \binom{\lambda}{k} e^{-k(\delta x + \frac{\alpha}{2}x^2)} \sum_{j=0}^{\infty} \frac{\Gamma(1 + j)}{j!\ \Gamma(1)} e^{-j(\delta x + \frac{\alpha}{2}x^2)}$$

$$f(x) = \lambda(\delta + \alpha x) \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^k \binom{\lambda}{k} \frac{\Gamma(1 + j)}{j!} e^{-(k + j + 1)(\delta x + \frac{\alpha}{2}x^2)}$$

$$e^{-(k + j + 1)(\frac{\alpha}{2}x^2)} = \sum_{b=0}^{\infty} \frac{(-1)^b (k + j + 1)^b \alpha^b}{2^b\ b!} x^{2b}$$

Let $\Psi_{k,j,b} = \lambda \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{b=0}^{\infty} \frac{(-1)^{k+b} (k+j+1)^b \alpha^b \Gamma(1+j)}{2^b\ b!\ j!}$

$$f(x) = \Psi_{k,j,b}\ x^{2b}\ (\delta + \alpha x)\ e^{-(k+j+1)\delta x}$$

2.1.1 The shapes of the New Exponentiated Exponential- Rayleigh Distribution

Knowing the shape of the (SHWD) helps us understand the behavior and approach of distribution functions in dealing with data, to understand this mathematically, especially through the limit values of the probability density and hazard functions when$(x \to 0\ \& x \to \infty)$.

$$lim_{x \to 0} f(x) = lim_{x \to 0}\left[\lambda(1 - e^{-(\delta x + \frac{\alpha}{2}x^2)})^{\lambda - 1} e^{-(\delta x + \frac{\alpha}{2}x^2)} (\delta + \alpha x)\right] = \lim_{x \to 0}\left[\lambda(1 - e^0)^{\lambda - 1}e^0(\delta + 0)\right] = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\lambda(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda-1}\, e^{-(\delta x+\frac{\alpha}{2}x^2)}\,(\delta+\alpha x)\right]$$

$$= \lim_{x\to\infty}\frac{\lambda(\delta+\alpha x)(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda}}{e^{(\delta x+\frac{\alpha}{2}x^2)}(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})}$$

$$= \lim_{x\to\infty}\frac{\lambda(\delta+\alpha x)}{(e^{(\delta x+\frac{\alpha}{2}x^2)}-1)(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{-\lambda}}$$

2.2. Applied L'Hospital's Rule:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\lambda\alpha}{\infty} = 0$$

As we continue to derive for x the result of the numerator will be equal constant as considered an integer value for β and as contained to derive the denominator it always contains the exponential part which is equal to $\infty\ as\ (x\to\infty)$. The final result of the limit is a constant divide by ∞ which equal to zero $\mathrm{Lim}_{x\to\infty} f(x) = 0$.

$$lim_{x\to 0} F(x) = \lim_{x\to 0}(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda} = 0$$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda} = 1$$

The shape of hazard function can be defined as follows :-

$$\mathrm{h(x)} = \frac{\lambda(\delta+\alpha x)e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-1}}{1-(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda}}$$

$$\lim_{x\to 0} h(x) = \frac{\lambda\delta e^{0}(1-e^{0})^{\lambda-1}}{1-(1-e^{0})^{\lambda}} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\frac{\lambda(\delta+\alpha x)\, e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-1}}{\left[1-(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda}\right]}$$

$$\lim_{x\to\infty}\frac{\lambda\,(\delta+\alpha x)}{e^{\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{-\lambda}\left[1-(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda}\right]}$$

$$\lim_{x\to\infty}\frac{\lambda\,(\delta+\alpha x)}{(e^{\left(\delta x+\frac{\alpha}{2}x^2\right)}-1)(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{-\lambda}\left[1-(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda}\right]}$$

$$\lim_{x\to\infty}\frac{\lambda\,(\delta+\alpha x)}{(e^{\left(\delta x+\frac{\alpha}{2}x^2\right)}-1)\left[(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{-\lambda}-1\right]} = \frac{\infty}{0}$$

By using L'Hopital's Rule:

$$= \lim_{x\to\infty}\frac{\lambda\,\alpha}{(e^{\left(\delta x+\frac{\alpha}{2}x^2\right)}-1)\left[-\lambda(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{-(\lambda+1)}e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}(-(\delta+\alpha x))\right]+\left[(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{-\lambda}-1\right]e^{\left(\delta x+\frac{\alpha}{2}x^2\right)}(\delta+\alpha x)}$$

$$= \frac{\lambda\,\alpha}{(e^{\infty}-1)[-\lambda(1-e^{-\infty})^{-(\lambda+1)}e^{-\infty}(-(\delta+\alpha.\infty))]+[(1-e^{-\infty})^{-\lambda}-1]e^{\infty}(\delta+\alpha.\infty)}$$

$$= \frac{\lambda\,\alpha}{0+\infty} = 0$$

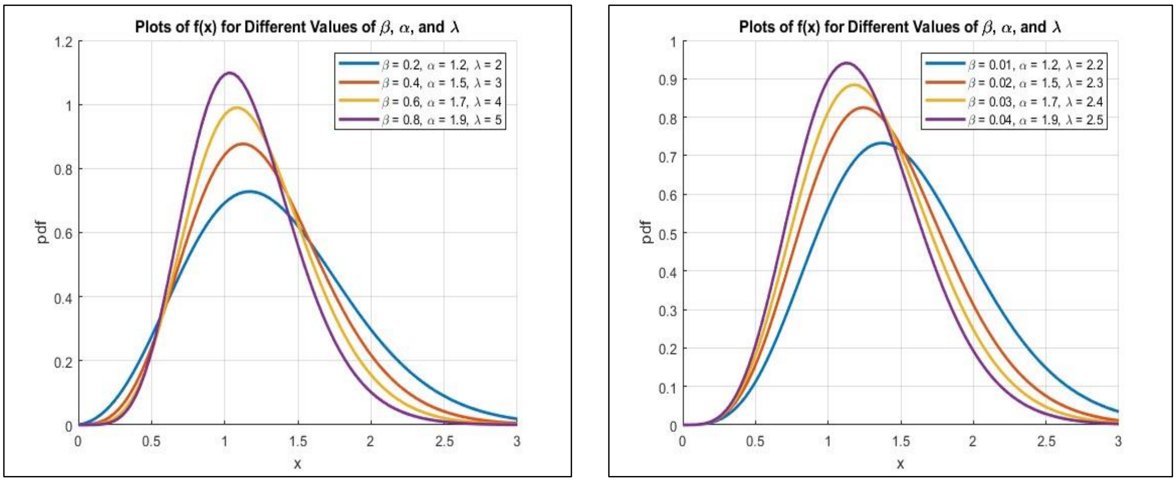

Figure 1. The shape of pdf for new Exponentiated Exponential- Rayleigh Distribution with different values of δ, α, λ.

This figure illustrates the graph of this function when $x \to \infty$ the PDF value is zero, and when $x \to 0$ the PDF value is also zero.

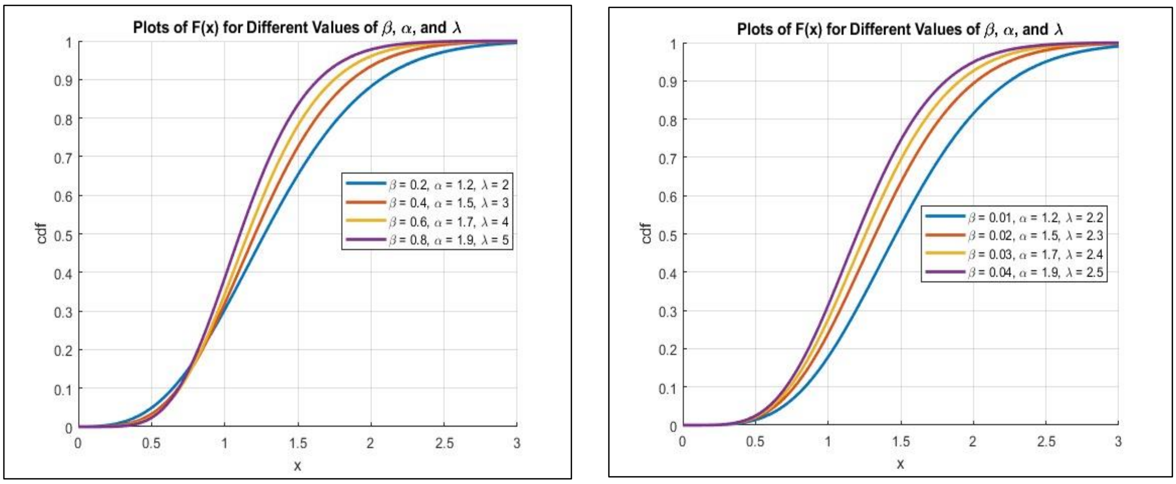

Figure 2. The shape of cdf for new Exponentiated Exponential- Rayleigh Distribution with different values of δ, α, λ.

This graph shows that the CDF function equals zero when $x \to 0$ and equals one when $x \to \infty$, which indicates that the CDF function is an increasing function.

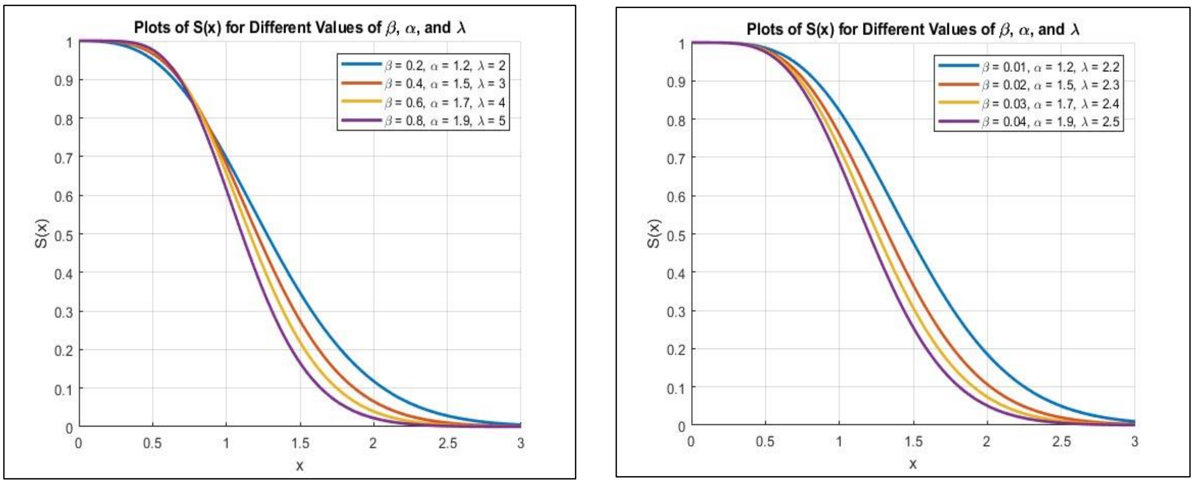

Figure 3. The shape of S(x) for new Exponentiated Exponential- Rayleigh Distribution with different values of δ, α, λ.

This graph shows that the function S(x) is a decreasing function.

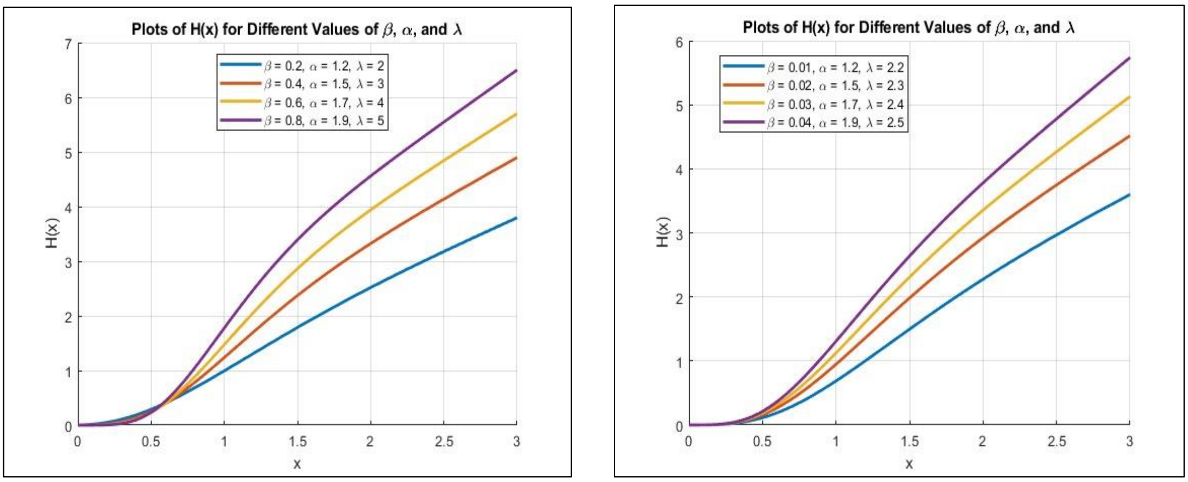

Figure 4. The shape of h(x) for new Exponentiated Exponential- Rayleigh Distribution with different values of $\delta, \alpha, and\, \lambda$.

In the special case, if $\lambda = 1$; $\delta > 0$, then the distribution becomes the exponential Rayleigh distribution, also if $\lambda = 1$; $\delta = 0$, then the distribution becomes the Rayleigh distribution. Finally, if $\lambda = 1$; $\alpha = 0$, then the distribution becomes exponential distribution.

2.3. Quantile Function

This function defined by : $F(x) = u \quad ; u \sim u(0\,,1)\,; x = F^{-1}\,(u)$

$$(1 - e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)})^{\lambda} = u$$

$$1 - e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)} = u^{\frac{1}{\lambda}}$$

$$1 - u^{\frac{1}{\lambda}} = e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)}$$

$$\mathrm{Ln}\left(1 - u^{\frac{1}{\lambda}}\right) = -\left(\delta x + \frac{\alpha}{2}x^2\right)$$

$$\frac{\alpha}{2}x^2 + \delta x + \mathrm{Ln}\left(1 - u^{\frac{1}{\lambda}}\right) = 0$$

$$\mathrm{x} = \frac{-\delta \pm \sqrt{\delta^2 - 2\alpha\ \mathrm{Ln}\left(1 - \mathrm{u}^{\frac{1}{\lambda}}\right)}}{\alpha}$$

In spaical case if $\mathrm{u} = 0.5$ then the Medeain is :

$$\mathrm{x} = \frac{-\,\delta \pm \sqrt{\delta^2 - 2\alpha\ \mathrm{Ln}\left(1 - (0.5)^{\frac{1}{\lambda}}\right)}}{\alpha}$$

3. Mathematical and statistical properties of new (EERD)

3.1. The Mode

The mode is defined as the point at which the probability density function achieves its maximum value, the mode of (EER) is obtained as follows:

From **Equation 2** the mode of (EER) is defined by: $\frac{\partial f(x)}{\partial x} = 0$

$$\frac{\partial f(x)}{\partial x} = \lambda(\delta + \alpha x)\, e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)} \left((\lambda - 1)(1 - e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)})^{\lambda - 2}\ e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)}\,(\delta + \alpha x) \right.$$
$$\left. + (1 - e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)})^{\lambda - 1} \left(-\lambda(\delta + \alpha x)^2\, e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)} + \lambda\alpha\, e^{-\left(\delta x + \frac{\alpha}{2}x^2\right)} \right) \right)$$

$$\lambda(\lambda-1)(\delta+\alpha x)^2\, e^{-2\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-2} + \lambda\, e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-1}(-(\delta+\alpha x)^2+\ \alpha)=0$$

$$\lambda\, e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-2}\left((\lambda-1)(\delta+\alpha x)^2\, e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)} + (\ \alpha-(\delta+\alpha x)^2)(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})\right)=0$$

since $\lambda>0$, $\delta>0$, $\alpha>0$ and $x>0$ then $\lambda\, e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-2}\neq 0$

$$(\lambda-1)(\delta+\alpha x)^2\, e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)}+(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})(\alpha-(\delta+\alpha x)^2)=0$$

This equation is difficult to solve; it is solved using MATLAB programs.

3.2. Moments about the origin (31)

The rth moment about the origin can be obtained by:

$$M_r'(x)=\int_0^\infty x^r\, f(x)dx=\int_0^\infty x^r\left(\lambda\,(\delta+\alpha x)e^{-(\delta x+\frac{\alpha}{2}x^2)}(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda-1}\right)dx$$

$$(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda-1}=(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda}\ (1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{-1}$$

$$(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda}\ =\sum_{k=0}^{\infty}(-1)^k\binom{\lambda}{k}\, e^{-k(\delta x+\frac{\alpha}{2}x^2)}$$

$$(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{-1}=\sum_{j=0}^{\infty}\frac{\Gamma\,(1+j)}{j!}\, e^{-j(\delta x+\frac{\alpha}{2}x^2)}$$

$$(1-e^{-\left(\delta x+\frac{\alpha}{2}x^2\right)})^{\lambda-1}=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k\,\Gamma\,(1+j)}{j!}\binom{\lambda}{k}\, e^{-(k+j)\left(\delta x+\frac{\alpha}{2}x^2\right)}$$

$$M_r'(x)=\ \lambda\ \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k\,\Gamma\,(1+j)}{j!}\binom{\lambda}{k}\int_0^\infty(\delta x^r+\ \alpha x^{r+1})\, e^{-(k+j+1)\left(\delta x+\frac{\alpha}{2}x^2\right)}\,dx$$

Since $e^{-(k+j+1)\left(\delta x+\frac{\alpha}{2}x^2\right)}=\ e^{-(k+j+1)\delta x}\ e^{-(k+j+1)\frac{\alpha}{2}x^2}$

$$e^{-(k+j+1)\frac{\alpha}{2}x^2}\ =\ \sum_{b=0}^{\infty}\frac{(-1)^b\ (k+j+1)^b\alpha^b}{2^b\, b!}x^{2b}$$

$$M_r'(x)=\lambda\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{j+b}\ (k+j+1)^b\alpha^b\Gamma\,(j+1)}{2^b\, j!\ b!}\int_0^\infty(\delta x^{r+2b} +\ \alpha x^{r+2b+1})\, e^{-(k+j+1)\delta x}dx$$

Suppose that

$$\lambda\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{j+b}\ (k+j+1)^b\alpha^b\Gamma\,(j+1)}{2^b\, j!\ b!}\ =\Psi_{k,j,b}$$

$$M_r'(x)=\ \Psi_{k,j,b}\left(\int_0^\infty\ \delta x^{r+2b}e^{-(k+j+1)\delta x}dx+\int_0^\infty\ \alpha x^{r+2b+1}e^{-(k+j+1)\delta x}dx\right)$$

$Let\ y=(k+j+1)\delta x$, $x=\frac{y}{\delta(k+j+1)}$, $dx=\frac{dy}{\delta(k+j+1)}$

$$M_r'(x) = \Psi_{k,j,b}\left(\int_0^{\infty} \delta \frac{y^{r+2b}}{(\delta(k+j+1))^{r+2b}} e^{-y} \frac{dy}{\delta(k+j+1)} + \int_0^{\infty} \alpha \frac{y^{r+2b+1}}{\big(\delta(k+j+1)\big)^{r+2b+1}} e^{-y} \frac{dy}{\delta(k+j+1)}\right)$$

$$M_r'(x) = \Psi_{k,j,b} \frac{1}{\delta^{r+2b}\,(k+j+1)^{r+2b+1}}\,\Gamma(r+2b+1) + \Psi_{k,j,b} \frac{\alpha}{\big(\delta(k+j+1)\big)^{r+2b+2}}\,\Gamma(r+2b+2)$$

As a direct result it Can be found

$\mathrm{E}(\mathrm{x}) = M_1'(x)$, $E(x^2) = M_2'(x)$ *and* $Var(x)$ *as follows* :

$$\mathrm{E(x)} = \mathrm{M}_1'(\mathrm{x}) = \Psi_{\mathrm{k,j,b}} \frac{1}{\delta^{2\mathrm{b}+1}\,(\mathrm{k+j+1})^{2\mathrm{b}+2}}\,\Gamma(2\mathrm{b}+2) + \Psi_{\mathrm{k,j,b}} \frac{\alpha}{\big(\delta(\mathrm{k+j+1})\big)^{2\mathrm{b}+3}}\,\Gamma(2\mathrm{b}+3) \quad (5)$$

3.3. Mean Time to failure

We can find this property by find the expected : $\mathrm{E(x)} = M_1'(x)$ as in **Equation 5**

$$\mathrm{E}(\mathrm{x}^2) = \mathrm{M}_2'(\mathrm{x}) = \Psi_{\mathrm{k,j,b}} \frac{1}{\delta^{2\mathrm{b}+2}\,(\mathrm{k+j+1})^{2\mathrm{b}+3}}\,\Gamma(2\mathrm{b}+3) + \Psi_{\mathrm{k,j,b}} \frac{\alpha}{\big(\delta(\mathrm{k+j+1})\big)^{2\mathrm{b}+4}}\,\Gamma(2\mathrm{b}+4)$$

$$\mathrm{var(x)} = \mathrm{M}_2'(\mathrm{x}) - \big(\mathrm{M}_1'(\mathrm{x})\big)^2$$

3.4. Coefficients of Skewness and Kurtosis (32)

Depending on the moment, the coefficients skewness $(C.S)$ and kurtosis$(C.K)$ can be found through the following formulas (30).:

$$M_3'(x) = \Psi_{k,j,b} \frac{1}{\delta^{2b+3}\,(k+j+1)^{2b+4}}\,\Gamma(2b+4) + \Psi_{k,j,b} \frac{\alpha}{\big(\delta(k+j+1)\big)^{2b+5}}\,\Gamma(2b+5)$$

$$\mathrm{C.S} = \frac{\mathrm{M}_3'\,(\mathrm{x})}{\mathrm{M}_2'\,(\mathrm{x})^{\frac{3}{2}}}$$

$$M_4'(x) = \Psi_{k,j,b} \frac{1}{\delta^{2b+4}\,(k+j+1)^{2b+5}}\,\Gamma(2b+5) + \Psi_{k,j,b} \frac{\alpha}{\big(\delta(k+j+1)\big)^{2b+6}}\,\Gamma(2b+6)$$

$$\mathrm{C.K} = \frac{\mathrm{M}_4'(\mathrm{x})}{(\mathrm{M}_2'(\mathrm{x}))^2} - 3$$

Table 1. The first - fourth moments, variance, skewness, and kurtosis for the distribution

δ	α	λ	μ_1'	μ_2'	μ_3'	μ_4'	K	S	var
2.5	2	1.5	0.4106	0.2538	0.2026	0.1946	0.0213	1.5849	0.0852
	1.5	1	0.3388	0.2039	0.1679	0.1708	1.1071	1.8228	0.0891
2	0.1	0.5	0.3005	0.2543	0.3407	0.6124	6.4725	2.6571	0.1640
	0.6	0.9	0.4168	0.3327	0.3747	0.5312	1.7988	1.9524	0.1590
1.5	2	0.7	0.3606	0.2448	0.2236	0.2474	1.1273	1.8457	0.1148
	2.8	0.5	0.2658	0.1536	0.1217	0.1173	1.9733	2.0217	0.0830

The distribution is skewed to the right based on the provided data values because the skewness values are positive. The values of the specified features determine the flatness. For numbers more than three, it is flattened; for values less than three, it is pointed.

3.5. Characteristic function (33)

$$\phi_x(it) = E\left(e^{itx}\right) = \int_0^\infty e^{itx}\ f(x)\,dx =$$

$$\int_0^\infty e^{itx}\left(\lambda\,(\delta+\alpha x)e^{-(\delta x+\frac{\alpha}{2}x^2)}(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{\lambda}(1-e^{-(\delta x+\frac{\alpha}{2}x^2)})^{-1}\right)\,dx$$

$$\phi_x(it) = \int_0^\infty e^{itx}\,\Psi_{k,j,b}\ x^{2b}\ \ (\delta+\alpha x)\ \ e^{-(k+j+b)\delta x}dx$$

$$= \Psi_{k,j,b}\left[\delta\int_0^\infty e^{itx}x^{2b}\quad e^{-(k+j+b)\delta x}\,dx + \alpha\int_0^\infty e^{itx}x^{2b+1}\quad e^{-(k+j+1)\delta x}\ dx\right]$$

$$= \Psi_{k,j,b}\,\delta\int_0^\infty x^{2b}\quad e^{-((k+j+b)\delta-it)x}\,dx + \Psi_{k,j,b}\,\alpha\int_0^\infty x^{2b+1}\quad e^{-((k+j+1)\delta-it)x}dx$$

$$let\ \left((k+j+b)\delta-it\right)x = y \qquad x = \frac{y}{(k+j+b)\delta-it} \qquad dx = \frac{dy}{(k+j+b)\delta-it}$$

$$= \Psi_{k,j,b}\,\delta\int_0^\infty\left(\frac{y}{(k+j+b)\delta-it}\right)^{2b}\quad e^{-y}\,\frac{dy}{(k+j+b)\delta-it}$$

$$+\Psi_{k,j,b}\,\alpha\int_0^\infty\left(\frac{y}{(k+j+b)\delta-it}\right)^{2b+1}\quad e^{-y}\,\frac{dy}{(k+j+b)\delta-it}$$

$$= \Psi_{k,j,b}\ \frac{\delta}{\left((k+j+b)\delta-it\right)^{2b+1}}\int_0^\infty y^{2b}\quad e^{-y}dy$$

$$+\Psi_{k,j,b}\ \frac{\alpha}{\left((k+j+b)\delta-it\right)^{2b+2}}\int_0^\infty y^{2b+1}\quad e^{-y}dy$$

$$\phi_x(it) = \Psi_{k,j,b}\ \frac{\delta}{\left((k+j+b)\delta-it\right)^{2b+1}}\Gamma\,(2b+1)$$

$$+\Psi_{k,j,b}\ \frac{\alpha}{\left((k+j+b)\delta-it\right)^{2b+2}}\Gamma\,(2b+2)$$

3.6. Moment Generating Function (34)

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}\ f(x)\,dx$$

$$M_x(t) = \lambda\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{i+b}(i+j+1)^b(\alpha)^b\Gamma\,(j+1)}{2^b\ j!\ b!}\left[\int_0^\infty\delta x^{2b}e^{-\left((k+j+1)\delta-t\right)x}dx\right.$$

$$\left.+\ \Psi_{k,j,b}\int_0^\infty\alpha e^{2b+1}\ e^{-\left((k+j+1)\delta-t\right)x}dx\right]$$

$$Let\ y = (k+j+1)\delta-\mathrm{t})\mathrm{x} \qquad x = \frac{y}{(k+j+1)\delta-t} \qquad dx = \frac{dy}{(k+j+1)\delta-t}$$

$$M_x(t) = \Psi_{k,j,b} \frac{\delta}{\left((k+j+1)\delta - t\right)^{2b+1}} \int_0^\infty y^{2b} e^{-y} d\,y$$

$$+ \Psi_{k,j,b} \frac{\alpha}{\left((k+j+1)\delta - t\right)^{2b+2}} \int_0^\infty y^{2b+1} e^{-y} dy$$

$$M_x(t) = \Psi_{k,j,b} \frac{\delta}{\left((k+j+1)\delta - t\right)^{2b+1}} \Gamma\,(2b+1) + \Psi_{k,j,b} \frac{\alpha}{\left((k+j+1)\delta - t\right)^{2b+2}} \Gamma\,(2b+2)$$

3.7. Factorial Moments Generating Function (35)

$$\mathcal{M}_x(t) = E(e^{tx}) = \int_0^\infty t^x\ f(x)\, dx = \int_0^\infty e^{x\ln t} f(x)\, dx$$

$$\mathcal{M}_x(t) = \int_0^\infty e^{x\ln t}\, \Psi_{k,j,b}\, x^{2b}(\delta + \alpha x)e^{-(k+j+1)\delta x} dx$$

$$\mathcal{M}_x(t) = \Psi_{k,j,b} \int_0^\infty e^{x\ln t}\, x^{2b}(\delta + \alpha x)e^{-(k+j+1)\delta x} dx$$

$$\mathcal{M}_x(t) = \Psi_{k,j,b}\left[\int_0^\infty x^{2b}\delta e^{-\left((k+j+1)\delta - \ln t\right)x} dx + \int_0^\infty \alpha x^{2b+1} e^{-\left((k+j+1)\delta - \ln t\right)x} dx\right]$$

let $y = [(k+j+1)\delta - \ln t]x \qquad x = \frac{y}{(k+j+1)\delta - \ln t} \qquad dx = \frac{dy}{(k+j+1)\delta - \ln t}$

$$\mathcal{M}_x(t) = \Psi_{k,j,b}\left[\int_0^\infty \delta \frac{y^{2b}}{[(k+j+1)\delta - \ln t]^{2b}} e^{-y} \frac{dy}{(k+j+1)\delta - \ln t}\right.$$

$$\left. + \int_0^\infty \alpha \frac{y^{2b+1}}{[(k+j+1)\delta - \ln t]^{2b+1}} e^{-y} \frac{dy}{(k+j+1)\delta - \ln t}\right]$$

$$\mathcal{M}_x(t) = \Psi_{k,j,b} \frac{\delta}{[(k+j+1)\delta - \ln t]^{2b+1}} \int_0^\infty y^{2b} e^{-y} dy$$

$$+ \frac{\alpha}{[(k+j+1)\delta - \ln t]^{2b+2}} \int_0^\infty y^{2b+1} e^{-y} dy$$

$$\mathcal{M}_x(t) = \Psi_{k,j,b} \frac{\delta}{[(k+j+1)\delta - \ln t]^{2b+1}} \Gamma\,(2b+1) + \Psi_{k,j,b} \frac{\alpha}{\left((k+j+1)\delta - \ln t\right)^{2b+2}} \Gamma\,(2b+2)$$

4. Conclusion

A new statistical lifetime distribution is represented by adding the shape parameter to the (cdf) function of the Exponenatied -Rayleigh Distribution called "Exponentied Exponential-Rayleigh Distribution.". In addition, the shape of the probability density function, the cumulative function, and the hazard function are discussed. Also, the basic statistical and mathematical properties of the new distribution, such as mode, median, r-moments around the origin, and the moment-generating function, are varied. Finally, introduce the table of applications with different values of parameters.

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Conflict of Interest

No conflicts of interest.

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