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New Lifetime Alpha Power Rayleigh Weibull Distribution: Structure and Properties

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Abstract

Alpha power transformation family distributions (APT) are considered one of the modern transformations that have received the attention of researchers in the past decade and are concerned with adding a shape parameter. Based on APT, this paper proposes a new distribution by adding a new parameter to the two-parameter Rayleigh Weibull distribution, a new continuous distribution called alpha power Rayleigh Weibull distribution (APRWD) including three parameters $(\alpha, \rho, and\ \delta)$ where ρ is known as shape parameter and α, δ as scale parameters. This paper also includes a presentation of the mathematical construction of the basic statistical functions: cumulative, probability density, survival, and hazard functions. We also include an expansion of the cumulative and probability density functions to use them in investigating and finding several mathematical statistical properties of (APRWD) such as moments, skewness, Kurtosis, moment generating, characteristics, mode, quantile, and factorial moments generating functions.

Keywords alpha-power family, Rayleigh Weibull distribution, survival function**,** moments about the origin, moment generating function.

1. Introduction

The statistical analysis and fitting of lifetime data have earned plenty of attention in all applied fields over the years. However, because the world is constantly changing, new kinds of data and statistical problems develop as new problems are discovered. In this paper, we propose and investigate a new model called APRWD according to APT. The shape parameter α in the APRWD introduces a flexibility-enhancing transformation that empowers the distribution to model a wider variety of empirical phenomena, especially when data deviate from the assumptions of classical Rayleigh or Weibull models. This makes APRWD a valuable tool in reliability analysis, environmental modeling, and other statistical applications requiring versatile distributional assumptions. APT was pointed out (1) that after the eighties, most of the proposed and developed methods depended on composition or addition. However, more recently, modern methods depended on creating distributions by adding parameters through families that generate distributions. It wasproposed the incorporation of a scaling parameter to generate new distributions, leading to the development of the APT family (2). The cumulative distribution function (CDF) and probability density function (PDF) of the APT family are defined as follows

$$G(x)=\begin{cases}\frac{\alpha^{F(x)}-1}{\alpha-1}, x, \alpha>0, and\ \alpha\neq 1\\ F(x) \qquad \alpha=1\end{cases} \tag{1}$$

$$g(x)=\begin{cases}\frac{log(\alpha)}{\alpha-1}\alpha^{F(x)}f(x) \qquad \alpha\neq 1\\ f(x) \qquad \alpha=1\end{cases} \tag{2}$$

This new approach of (APT) is applied by many researchers to produce new models such as alpha power Weibull (3, 4), alpha power Lindly (5), alpha power Pareto (6), alpha power Teissier (7), alpha power inverse Weibull (8), alpha power exponential Weibull (9), alpha power exponentiated inverse rayleigh (10), and discrete of alpha power transformation (11). While other researchers took another approach in using generalization, weighting, and expansion methods for (ATP) as in (12), (13), (14), (15), (16), (17), (18), and (19).

The basic idea of this paper is to produce a sub-model resulting from the use of a distribution that is essentially the result of a combination of two distributions. The idea of the combination was originally introduced by (20), This method has been exploited to provide Rayleigh Weibull by (21), also provide Shanker Weibull (22) and exponential Rayleigh (23), The two parameters Rayleigh Weibull distribution introduced by (24) with Cdf and Pdf take forms as follows:

$$F(x)=1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}, x, \rho, \delta>0 \tag{3}$$

$$f(x)=\ (\rho x+\delta x^{\delta-1})\, e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)} \tag{4}$$

The Alpha Power Transformations family is used to increase the elasticity of basic distributions by introducing an additional parameter that allows the cumulative distribution function (CDF) to be modified nonlinearly. When combined with the Rayleigh-Weibull distribution, which is already relatively elastic, the result is a three-parameter distribution that can more accurately represent life or reliability data.

The next moves are summarized in the mathematical construction of (APRWD), the expansion of the pdf, and the investigation of the mathematical and statistical properties of (APRWD).

2. Structure of New APRWD.

The random variable x > 0 and α, δ>0 where alpha and beta are classified as scale parameters and ρ > 0the gamma parameter is classified as a shape parameter.

$$G(x)=\begin{cases}\frac{\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}-1}{\alpha-1} \qquad \alpha\neq 1\\ 1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)} \qquad \alpha=1\end{cases} \tag{5}$$

$$g(x)=\begin{cases}\frac{\alpha\, log(\alpha)}{\alpha-1}\ \alpha^{-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}\ e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}\left(\rho x+\delta x^{\delta-1}\right) \quad \alpha\neq 1\\ \left(\rho x+\delta x^{\delta-1}\right)\ e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)} \qquad \alpha=1\end{cases} \tag{6}$$

Because $x>0;\ \alpha,\delta,\rho>0$, and $\alpha\neq 1$, all imply $g(x)>0$.

Let's now demonstrate that $\int_0^\infty g(x)\,dx=1$.

$$\int_0^\infty g(x)\,dx = \int_0^\infty \frac{\alpha\, log(\alpha)}{\alpha-1}\,\alpha^{-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}\;e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}\left(\rho x+\delta x^{\delta-1}\right)dx$$

$$= \left[\frac{\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}-1}{\alpha-1}\right]_0^\infty = \left(\frac{\alpha-1}{\alpha-1}-\frac{1-1}{\alpha-1}\right) = 1$$

The following is a definition of survival and hazard functions.

$$S(x) = \begin{cases} 1-\left(\dfrac{\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}-1}{\alpha-1}\right) & \alpha \neq 1 \\ e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)} & \alpha = 1 \end{cases} \tag{7}$$

$$h(x) = \begin{cases} \dfrac{\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}\,log(\alpha)\;e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}\left(\rho x+\delta x^{\delta-1}\right)}{\alpha-\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}} & \alpha \neq 1 \\ \left(\rho x+\delta x^{\delta-1}\right) & \alpha = 1 \end{cases} \tag{8}$$

2.1. The shapes of (APRWD)

$$\lim_{x\to\infty} g(x) = \frac{\alpha\, log(\alpha)}{\alpha-1}\lim_{x\to\infty}\left(\frac{\left(\rho x+\delta x^{\delta-1}\right)}{\alpha^{e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}\,e^{\left(\frac{\rho}{2}x^2+x^\delta\right)}}\right) = 0$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(\frac{\left(\rho x+\delta x^{\delta-1}\right)log(\alpha)}{\alpha-1}\,\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}}\,e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}\right) = 0$$

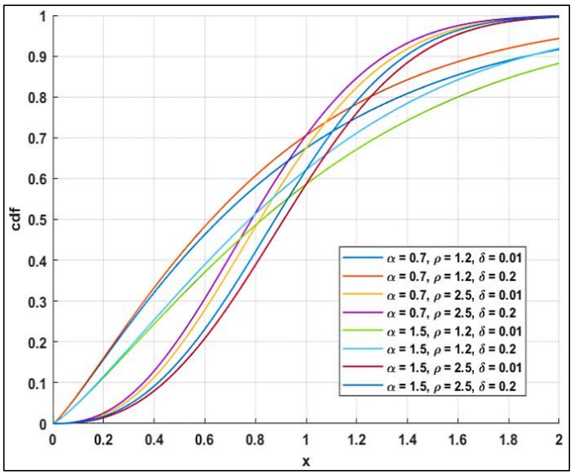

Figure 1. Plots of cdf

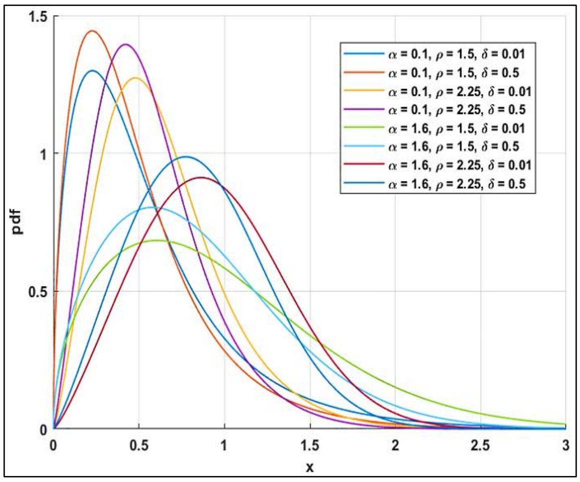

Figure 2. Plots of pdf

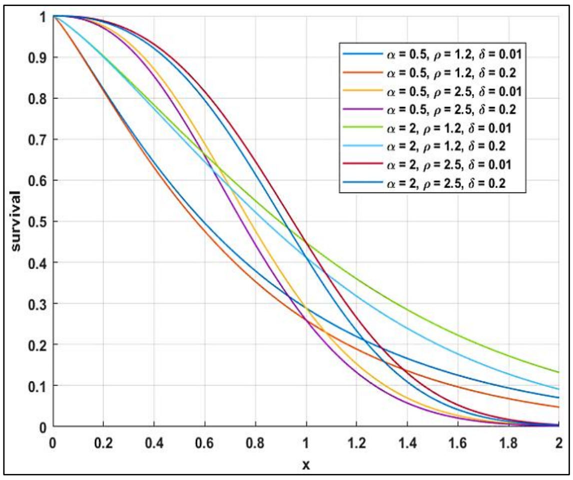

Figure 3. Plots of S(x)

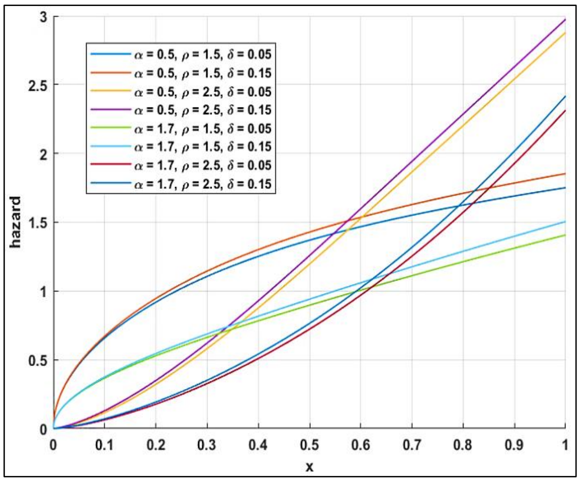

Figure 4. Plots of h(x)

The effect of the distribution parameters is represented by the following:
(α) controls the shape of the distribution, especially the tail, (δ) the scaling parameter, and (ρ): the basic shape parameter (related to the Weibull). Each of these parameters affects distribution properties such as kurtosis and skewness**.**

2.2. Expanding the pdf

To make the process of determining the statistical characteristics of the APRWD distribution simpler, we employ a few mathematical methods to expand the cumulative and probability density functions (25)and (26).

$$\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}} = \alpha\cdot\alpha^{-e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}} = \alpha\cdot\left(e^{log(\alpha)}\ \right)^{-e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}} = \alpha\cdot e^{-log(\alpha)\cdot e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}}$$

$$e^{-log(\alpha)\cdot e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}} = \sum_{i=0}^{\infty}\frac{(-1)^i\left(log(\alpha)\right)^i}{i!}e^{-i\left(\frac{\rho}{2}x^2+x^\delta\right)}$$

$$g(x) = \frac{log(\alpha)}{\alpha-1}\alpha^{1-e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}}e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}\left(\rho x+\delta x^{\delta-1}\right)$$

$$= \frac{\alpha\, log(\alpha)}{\alpha-1}\left(\rho x+\delta x^{\delta-1}\right)e^{-\,log(\alpha)\,e^{-\left(\frac{\rho}{2}x^2+X^\delta\right)}}\ e^{-\left(\frac{\rho}{2}x^2+x^\delta\right)}$$

$$= \frac{\alpha}{\alpha-1}\sum_{i=0}^{\infty}\frac{(-1)^i\left(log(\alpha)\right)^{i+1}}{i!}\left(\rho x+\delta x^{\delta-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}e^{-(i+1)x^\delta}$$

$$e^{-(i+1)X^\delta} = \sum_{j=0}^{\infty}\frac{(-1)^j(i+1)^j}{j!}x^{j\delta}$$

$$g(x) = \frac{\alpha}{\alpha-1}\sum_{i=j=0}^{\infty}\frac{(-1)^{i+j}\,(i+1)^j\left(log(\alpha)\right)^{i+1}}{i!\,j!}\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}$$

$$let\ \xi_{i,j} = \frac{\alpha}{\alpha-1}\sum_{i=j=0}^{\infty}\frac{(-1)^{i+j}\,(i+1)^j\left(log(\alpha)\right)^{i+1}}{i!\,j!}$$

$$g(x) = \xi_{i,j}\cdot\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2} \quad (9)$$

3. Mathematical and statistical properties of (APRWD)

3.1. Moments

Moments are among the most significant distribution properties since they affect a wide range of other distribution characteristics, including mean, variance, skewness, and kurtosis (27) and (28), the moments of (APRWD) could be obtained as follows:

$$M_r' = E(x^r) = \int_0^{\infty} x^r\cdot g(x)\,dx = \xi_{i,j}\int_0^{\infty}\left(\rho x^{j\delta+r+1}+\delta x^{\delta(j+1)+r-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}dx$$

$$let\ y = (i+1)\frac{\rho}{2}x^2 \Rightarrow x = \sqrt{\frac{2y}{\rho(i+1)}} \Rightarrow dx = \frac{dy}{\sqrt{y}\sqrt{2\rho(i+1)}}$$

$$M'_r = \rho\xi_{i,j}\int_0^\infty \left(\frac{2y}{\rho(i+1)}\right)^{\frac{1}{2}(j\delta+r+1)} e^{-y}\frac{dy}{\sqrt{y}\sqrt{2\rho(i+1)}} + \delta\xi_{i,j}\int_0^\infty \left(\frac{2y}{\rho(i+1)}\right)^{\frac{1}{2}(\delta(j+1)+r-1)} e^{-y}\frac{dy}{\sqrt{y}\sqrt{2\rho(i+1)}}y$$

$$= \rho\xi_{i,j}\frac{2^{\frac{1}{2}(j\delta+r+1)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(j\delta+r+1)+\frac{1}{2}}}\int_0^\infty y^{\frac{1}{2}(j\delta+r+1)-\frac{1}{2}}e^{-y}dy + \delta\xi_{i,j}\frac{2^{\frac{1}{2}(\delta(j+1)+r-1)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(\delta(j+1)+r-1)+\frac{1}{2}}}\int_0^\infty y^{\frac{1}{2}(\delta(j+1)+r-1)-\frac{1}{2}}e^{-y}dy$$

$$M'_r = \rho\xi_{i,j}\frac{2^{\frac{1}{2}(j\delta+r+1)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(j\delta+r+1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+r+1)+\frac{1}{2}\right)} + \delta\xi_{i,j}\frac{2^{\frac{1}{2}(\delta(j+1)+r-1)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(\delta(j+1)+r-1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+r-1)+\frac{1}{2}\right)} \quad (10)$$

$$\mu_x = M'_1 = \rho\xi_{i,j}\frac{2^{\frac{1}{2}(j\delta+2)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(j\delta+2)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+2)+\frac{1}{2}\right)} + \delta\xi_{i,j}\frac{2^{\frac{1}{2}(\delta(j+1))-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(\delta(j+1))+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1))+\frac{1}{2}\right)}$$

$$M'_2 = \rho\xi_{i,j}\frac{2^{\frac{1}{2}(j\delta+3)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(j\delta+3)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+3)+\frac{1}{2}\right)} + \delta\xi_{i,j}\frac{2^{\frac{1}{2}(\delta(j+1)+1)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(\delta(j+1)+1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+1)+\frac{1}{2}\right)}$$

$$Var(x) = M'_2 + (M'_1)^2$$

3.2. The Mode

Finding the point at which is made the probability density function in a way that reaches the highest value.

$$g(x) = \frac{\alpha}{\alpha-1}\sum_{i=j=0}^{\infty}\frac{(-1)^{i+j}\,(i+1)^j\,\left(log(\alpha)\right)^{i+1}}{i!\,j!}\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}$$

$$e^{-(i+1)\frac{\rho}{2}x^2} = \sum_{u=0}^{\infty}\frac{(-1)^u(i+1)^u\rho^u}{u!\,2^u}x^{2u}$$

$$g(x) = \frac{\alpha}{\alpha-1}\sum_{i=j=u=0}^{\infty}\frac{(-1)^{i+j+u}\,\rho^u(i+1)^{j+u}\,\left(log(\alpha)\right)^{i+1}}{2^s i!\,j!\,u!}\left(\rho x^{j\delta+2u+1} + \delta x^{\delta(j+1)+2u-1}\right)$$

$$\Psi_{i,j,u} = \frac{\alpha}{\alpha-1}\sum_{i=j=u=0}^{\infty}\frac{(-1)^{i+j+u}\,\rho^u(i+1)^{j+u}\,\left(log(\alpha)\right)^{i+1}}{2^s i!\,j!\,u!}$$

$$g(x) = \Psi_{i,j,u}\left(\rho x^{j\delta+2u+1} + \delta x^{\delta(j+1)+2u-1}\right)$$

$$\frac{\partial g(x)}{\partial x} = \Psi_{i,j,u}\left(\rho(j\delta + 2u + 1)x^{j\delta+2u} + \delta(\delta(j + 1) + 2u - 1)x^{\delta(j+1)+2u-2}\right)$$

$$\frac{\partial g(x)}{\partial x} = 0$$

$$\rho(j\delta + 2u + 1)x^{j\delta+2u} + \delta(\delta(j + 1) + 2u - 1)x^{\delta(j+1)+2u-2} = 0$$

Divided both sides of the above equation by $\left(x^{j\delta+2u}\right)$

$$\rho(j\delta + 2u + 1) + \delta(\delta(j + 1) + 2u - 1)x^{\delta-2} = 0$$

$$x^{\delta-2} = \frac{-\rho(j\delta + 2u + 1)}{\delta(\delta(j + 1) + 2u - 1)}$$

$$x_{mode} = \left(\frac{-\rho(j\delta + 2s + 1)}{\delta(\delta(j + 1) + 2s - 1)}\right)^{\frac{1}{\delta-2}} \tag{11}$$

3.3. Quantile function

Quantile holds significant importance both in theory and in practice. From a theoretical standpoint, it facilitates the determination of statistical measures such as skewness and kurtosis (29) and (30). Practically, it is instrumental in generating data for simulations, making it a valuable tool in various applications.

$$G(x) = \frac{a^{1-e^{-\left(\frac{\rho}{2}x^2+X^{\delta}\right)}} - 1}{a - 1}$$

$$G(x) = u \;\Rightarrow x = G^{-1}(u)$$

$$u = \frac{a^{1-e^{-\left(\frac{\rho}{2}x^2+x^{\delta}\right)}} - 1}{a - 1}$$

$$u(a - 1) + 1 = a^{1-e^{-\left(\frac{\rho}{2}x^2+x^{\delta}\right)}}$$

$$log(u(a - 1) + 1) = \left(1 - e^{-\left(\frac{\rho}{2}x^2+X^{\delta}\right)}\right) \cdot log(a)$$

$$e^{-\left(\frac{\rho}{2}x^2+x^{\delta}\right)} = 1 - \left(\frac{log(u(a - 1) + 1)}{log(a)}\right)$$

$$\frac{\rho}{2}x^2 + x^{\delta} = -\,log\left(1 - \left(\frac{log(u(a - 1) + 1)}{log(a)}\right)\right)$$

$$x^{\delta} + \frac{\rho}{2}x^2 + log\left(1 - \left(\frac{log(u(a - 1) + 1)}{log(a)}\right)\right) = 0 \tag{12}$$

Finding the roots of equation (12) represented by values of x requires using methods for solving nonlinear equations. To clarify, values can be given for the parameter $\delta = 1$ & 2 as follows.

$$\frac{\rho}{2}x^2 + x + log\left(1 - \left(\frac{log(u(a - 1) + 1)}{log(a)}\right)\right) = 0$$

$$x_{u,\delta=1} = \frac{-1 \pm \sqrt{1 - 2\rho\, log\left(1 - \left(\frac{log(u(a - 1) + 1)}{log(a)}\right)\right)}}{\rho}$$

$$\left(\frac{\rho}{2} + 1\right)x^2 + log\left(1 - \left(\frac{log(u(a - 1) + 1)}{log(a)}\right)\right) = 0$$

$$x_{u,\delta=2} = \sqrt{\frac{-\,log\left(1-\left(\frac{log(u(\mathrm{a}-1)+1)}{log(\mathrm{a})}\right)\right)}{\left(\frac{\rho}{2}+1\right)}}$$

Median point at which the cumulative distribution function equal to 0.5 $(\mathrm{u}=0.5)$.

$$x^{\delta}+\frac{\rho}{2}x^{2}+log\left(1-\left(\frac{log(0.5\mathrm{a}+0.5)}{log(\mathrm{a})}\right)\right)=0$$

$$x_{0.5,\delta=1} = \frac{-1 \pm \sqrt{1-2\rho\, log\left(1-\left(\frac{log(0.5\mathrm{a}+0.5)}{log(\mathrm{a})}\right)\right)}}{\rho}$$

$$x_{0.5,\delta=2} = \sqrt{\frac{-\,log\left(1-\left(\frac{log(0.5\mathrm{a}+0.5)}{log(\mathrm{a})}\right)\right)}{\left(\rho/_{2}+1\right)}}$$

3.4. Characteristic Function

$$\phi_x(it)=E\left(e^{itx}\right)=\int_0^\infty e^{itx}\; g(x)\,dx=\int_0^\infty e^{itx}\xi_{i,j}\cdot\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}\,dx$$

$$e^{itx}=\sum_{\mathfrak{b}=0}^{\infty}\frac{(it)^{\mathfrak{b}}}{\mathfrak{b}!}x^{\mathfrak{b}}$$

$$\phi_x(it)=\xi_{v,j}\sum_{\mathfrak{b}=0}^{\infty}\frac{(it)^{\mathfrak{b}}}{\mathfrak{b}!}x^{\mathfrak{b}}\int_0^\infty\left(\rho x^{j\delta+\mathfrak{b}+1}+\delta x^{\delta(j+1)+\mathfrak{b}-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}\,dx$$

$$\begin{aligned}&=\rho\xi_{i,j}\sum_{\mathfrak{b}=0}^{\infty}\frac{(it)^{\mathfrak{b}}2^{\frac{1}{2}(j\delta+\mathfrak{b}+1)-\frac{1}{2}}}{\mathfrak{b}!\left(\rho(i+1)\right)^{\frac{1}{2}(j\delta+\mathfrak{b}+1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+\mathfrak{b}+1)+\frac{1}{2}\right)}\\&+\delta\xi_{i,j}\sum_{b=0}^{\infty}\frac{(it)^{\mathfrak{b}}2^{\frac{1}{2}(\delta(j+1)+\mathfrak{b}-1)-\frac{1}{2}}}{\mathfrak{b}!\left(\rho(i+1)\right)^{\frac{1}{2}(\delta(j+1)+\mathfrak{b}-1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+\mathfrak{b}-1)+\frac{1}{2}\right)}\end{aligned}\qquad(13)$$

3.5. Factorial Moments Generating Function

$$\mathcal{M}_x(t)=E(t^x)=\int_0^\infty t^x\; g(x)\,dx=\int_0^\infty e^{xlnt}\,\xi_{i,j}\cdot\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}\,dx$$

$$e^{xlnt}=\sum_{q=0}^{\infty}\frac{(lnt)^q}{q!}x^q$$

$$\mathcal{M}_x(t)=\xi_{i,j}\sum_{q=0}^{\infty}\frac{(lnt)^q}{q!}\int_0^{\infty}\left(\rho x^{j\delta+q+1}+\delta x^{\delta(j+1)+q-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}\,dx$$

$$=\rho\xi_{i,j}\sum_{q=0}^{\infty}\frac{(lnt)^q 2^{\frac{1}{2}(j\delta+q+1)-\frac{1}{2}}}{q!\left(\rho(i+1)\right)^{\frac{1}{2}(j\delta+q+1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+q+1)+\frac{1}{2}\right)} \tag{14}$$

$$+\delta\xi_{i,j}\sum_{q=0}^{\infty}\frac{(lnt)^q 2^{\frac{1}{2}(\delta(j+1)+q-1)-\frac{1}{2}}}{q!\left(\rho(i+1)\right)^{\frac{1}{2}(\delta(j+1)+q-1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+q-1)+\frac{1}{2}\right)}$$

3.6. Moment Generating Function

$$\mathrm{M}_{,x}(t)=E(e^{tx})=\int_0^{\infty}e^{tx}\;g(x)\,dx=\int_0^{\infty}e^{tx}\cdot\xi_{i,j}\cdot\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}\,dx$$

$$e^{tx}=\sum_{k=0}^{\infty}\frac{t^k}{k!}x^k$$

$$\mathrm{M}_{,x}(t)=\xi_{i,j}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}\left(\rho x^{j\delta+1}+\delta x^{\delta(j+1)-1}\right)e^{-(i+1)\frac{\rho}{2}x^2}\,dx$$

$$=\rho\xi_{i,j}\sum_{k=0}^{\infty}\frac{t^k 2^{\frac{1}{2}(j\delta+k+1)-\frac{1}{2}}}{k!\left(\rho(i+1)\right)^{\frac{1}{2}(j\delta+k+1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+k+1)+\frac{1}{2}\right)} \tag{15}$$

$$+\delta\xi_{i,j}\sum_{k=0}^{\infty}\frac{t^k 2^{\frac{1}{2}(\delta(j+1)+k-1)-\frac{1}{2}}}{k!\left(\rho(i+1)\right)^{\frac{1}{2}(\delta(j+1)+k-1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+k-1)+\frac{1}{2}\right)}$$

3.7. Coefficients of Skewness(C.S) and Kurtosis(C.K)

$$M_r'=\rho\xi_{i,j}\frac{2^{\frac{1}{2}(j\delta+r+1)-\frac{1}{2}}}{\left(\rho(i+1)\right)^{\frac{1}{2}(j\delta+r+1)+\frac{1}{2}}}\cdot P_{\left(\frac{1}{2}(j\delta+r+1)+\frac{1}{2}\right)}$$

$$+\delta\xi_{i,j}\frac{2^{\frac{1}{2}(\delta(j+1)+r-1)-\frac{1}{2}}}{\left(\rho(i+1)\right)^{\frac{1}{2}(\delta(j+1)+r-1)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+r-1)+\frac{1}{2}\right)}$$

$$M_3'=\rho\xi_{i,j}\frac{2^{\frac{1}{2}(j\delta+4)-\frac{1}{2}}}{\left(\rho(i+1)\right)^{\frac{1}{2}(j\delta+4)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(j\delta+4)+\frac{1}{2}\right)}$$

$$+\delta\xi_{i,j}\frac{2^{\frac{1}{2}(\delta(j+1)+2)-\frac{1}{2}}}{\left(\rho(i+1)\right)^{\frac{1}{2}(\delta(j+1)+2)+\frac{1}{2}}}\cdot\Gamma_{\left(\frac{1}{2}(\delta(j+1)+2)+\frac{1}{2}\right)}$$

$$C.S=\frac{M_3'}{(M_2')^{\frac{3}{2}}} \tag{16}$$

$$M'_4 = \rho \xi_{i,j} \frac{2^{\frac{1}{2}(j\delta+5)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(j\delta+5)+\frac{1}{2}}} \cdot \Gamma_{\left(\frac{1}{2}(j\delta+5)+\frac{1}{2}\right)} + \delta \xi_{i,j} \frac{2^{\frac{1}{2}(\delta(j+1)+3)-\frac{1}{2}}}{(\rho(i+1))^{\frac{1}{2}(\delta(j+1)+3)+\frac{1}{2}}} \cdot \Gamma_{\left(\frac{1}{2}(\delta(j+1)+3)+\frac{1}{2}\right)}$$

$$C.K = \frac{M'_4}{(M'_2)^2} - 3 \tag{17}$$

Table 1. 1st - 4th moments, skewness, kurtosis, variance

α	δ	γ	μ'_1	μ'_2	μ'_3	μ'_4	K	S	var
1.5	0.5	0.7	0.7315	1.0761	2.0581	4.6415	1.0080	1.8436	0.5411
	0.8	1.2	0.6584	0.7097	0.9762	1.5820	0.1410	1.6329	0.2762
2	0.8	0.5	0.8785	1.3159	2.5673	5.9767	0.4517	1.7008	0.5441
	1.5	1.2	0.7519	0.7478	0.8902	1.2123	-0.8319	1.3767	0.1824
3.5	2.5	1.5	0.8046	0.7545	0.7870	0.8916	-1.4340	1.2007	0.1071
	2	2.5	0.6993	0.5901	0.5666	0.6015	-1.2726	1.2500	0.1011

4. Conclusion

This study introduced the Alpha Power Rayleigh Weibull Distribution (APRWD), derived from the alpha-power family. Core functions including the pdf, cdf, survival, and hazard rate were rigorously formulated. Key statistical properties such as moments, moment-generating function, skewness, and kurtosis were also derived using precise mathematical expressions. The results demonstrate the flexibility and applicability of APRWD in reliability and survival analysis.. This highlights the importance of the new distribution in modeling failure and life data and opens the door to its use in diverse applications such as engineering, biological systems, and financial data analysis. It is recommended that additional properties of the distribution, such as its moments, in-depth hazard functions, and the development of versions based on time-varying data or dual distributions, be studied in the future. Other estimation methods such as Bayesian estimation can also be studied to improve the efficiency of parameters in small samples. AIC, BIC, K-S test, Anderson-Darling, Cramér–von Mises can be used to compare the performance of the APRWD distribution with distributions such as Weibull, Rayleigh, Gamma, Lognormal.

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Conflict of Interest

There are no conflicts of interest.

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