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A Numerical Study of Initial and Boundary Conditions Problem for Wave Equation Using the Operational Matrices

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Abstract

In this paper, orthogonal polynomials and their operational matrices will be utilized to address the initial and boundary value problems of the one-dimensional wave problem, where the domain of the space variable is bounded, which covers a variety of scientific and engineering operations. Six types of orthogonal polynomials, Include instead of such as the Genocchi, Bernoulli, Legendre, Boubaker, Chebyshev and Standard polynomials. The linear problem with its initial and boundary conditions are transformed to a linear algebraic equations, which can then be solved by utilizing Mathematica®12 to get an approximate solution for this problem. Some test problems related to the one-dimensional wave equation with different conditions are discussed and solved to show how reliable and efficient the proposed methods. The error norm L_∞ and the mean square error L_{rms}, were computed; these are presented through analytical tables and graphics showing the rapid convergence for these methods.

Keywords: Wave problem; Operational matrices,Orthogonal Polynomials,Approximate solutions.

1. Introduction

The mathematical foundation for many problems in mathematics, physics, engineering, and chemistry can be explained mathematically using partial differential equations (PDEs). In physics, for example, partial differential equations clear describe wave propagation and heat transfer. Furthermore, partial differential equations were utilized to describe the majority of physical processes in domains such as electricity, quantum mechanics, wave propagation in shallow water, fluid dynamics, plasma physics, and others (1).

There is a need to find effective and reliable approximate or analytical techniques that can handle PDEs because of these massive applications, there is a need to develop effective and trustworthy approximate or analytical procedures for dealing with PDEs. Many mathematicians and engineers have solved a wide range of functional equations using the VIM by He (2–4).

Many tapes of scientific, physical, and engineering problems can be classified as initial boundary value problems (IBVP). With a few exceptions, we are unable to find accurate analytical answers to the majority of these difficulties. There have been several attempts to develop approximate and analytical methods for solving the non-linear diffusion and wave problems, see (5–8).

Wave equations can only be solved analytically in highly particular situations, hence many realistic cases cannot be solved using commonly used analytical methods. In addition to discretization methods such as finite difference, finite volume, and finite element approaches, various more ways have been proposed to solve the wave problem. For example, the analytical approach to solving the wave equation (9), the numerical solution of the wave issue (10,11), and the optimal homotopy strategy for solving the nonlinear wave equation (12). Furthermore, there are numerous techniques that offer an approximate resolve for the differing kinds of the differential equations, see, (13–15).

Corrington suggested in 1973 that a system of algebraic linear equations could be created from linear differential and integral equations by using a least squares approximation and repeatedly integrating Walsh functions (16). In addition, Sparis and Mouroutsos in 1986 using the orthogonal polynomial series operational matrices to solve differential equations (17), many researchers have used orthogonal polynomials to find approximate solutions for many applications. See; (18–28). The authors were highly interested in since they were practical methods for resolving an extensive number of approximation theory and numerical analysis problems. On the other hand, the orthogonal polynomials and operational matrices stand out above other types due to their effective reduction of the required solution, which is achieved by applying the operational matrices technique to transform the linear differential equations into linear algebraic systems of equations, where any computer program can be used to solve them.

Farther more, Turkyilmazoglu in 2013 proposed an approximate analytical technique for resolving differential equations based on standard polynomials, which is used on various types of problems (29–33).

In 2023, Othman, et al. introduced several orthogonal polynomials, for instance, Hermite, Laguerre, Chebyshev, and others polynomials with inner product, to develop the computational method (CM) (34). Myasar, et al. (35,36), further introduced the Genocchi, Bernoulli, and Boubaker polynomials, which contribute to the effective computational method (ECM).

The outline for this paper is as follows: Section two shows the second-order linear wave equation's mathematical formulation. Section three: Preliminary of the orthogonal polynomials. Section four: Main results and applications of orthogonal polynomials to resolve some examples for the wave problem. Section five: gives the conclusions.

2. The Mathematical Model of the Wave Equation

Let us consider the one-dimensional wave problem (7):

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = h(x,t), \quad t \geq 0\,, x \in \Omega. \tag{1}$$

with initial conditions:

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad x \in \partial_D \Omega, \tag{2}$$

and boundary conditions:

$$u(x,t) \;= \hat{u}(x,t), \quad t \geq 0\,,\; x \in \partial_D \Omega, \tag{3}$$

$$Fu(x,t) = \;\hat{f}(x,t), \quad t \geq 0\,,\; x \in \partial_N \Omega. \tag{4}$$

where $h(x,t)$ is a particular function, Ω is the bounded domain, c is a constant, $\hat{f}(x,t)$ and $\hat{u}(x,t)$ are known functions, $[Fu](x,t) = \frac{\partial u}{\partial n}(x,t)$, and $n(x,t)$ the external normal vector to the boundary $\partial\Omega$, when $f = 0$ **Equation 1** will be homogeneous wave equation, and if $f \neq 0$ **Equation 1** will be non-homogeneous wave problem.

3. Preliminary of the orthogonal polynomials and their operational matrices

The orthogonal polynomials and their operational matrices play an important part applied and pure mathematics as well as numerical calculation. Six types of these orthogonal polynomials will be used: standard polynomials (x^i), Legendre, Chebyshev, Boubaker, Genocchi, and Bernoulli polynomials to resolve the wave problem with initial and boundary conditions.

3.1. The operational matrices for standard polynomials

Assume that the wave problem (1), with the initial and boundary conditions (2-4), has a unique solution, this can be implemented using a suitable linear transformation. Taking the basic functions

$\xi = \{\xi_0(x), \xi_1(x), \ldots, \xi_n(x), \ldots\}$, and $\eta = \{\eta_0(t), \eta_1(t), \ldots, \eta_n(t), \ldots\}$,

Let the solution of **Equation 1**, may be presented by using the double series extension in terms of a base functions

$$u(x,t) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} b_{hk}\, \xi_h(x)\, \eta_k(t), \tag{5}$$

The **Equation 5** may be written using the dot product, as follows:

$$u(x,t) = \sum_{h=0}^{n} \sum_{k=0}^{n} b_{hk}\, \xi_h(x)\, \eta_k(t) = \psi(x) \,.\, B \,.\, \phi(t)\ , \tag{6}$$

where b_{hk} the coefficients will be evaluated later, by use the definition:

$$\psi(x) = [1 \quad x \quad \ldots \quad x^n], \quad \phi(t) = [1 \quad t \quad \ldots \quad t^n]^T, \text{ and } B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix}^T,$$

Furthermore, the n^{th}-order partial derivatives for ψ and ϕ with respect x and t are obtained by:

$$\frac{\partial^n \psi(x)}{\partial x^n} = \psi(x).A_S\ , \quad \frac{\partial^n \phi(t)}{\partial t^n} = (A_S)^T \,.\, \phi(t),$$

where A_S is the derivative matrix that is dimensional $(n+1) \times (n+1)$ is given by:

$$A_S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & n \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Thus, the derivatives of $u(x,t)$ will be obtained using **Equation 6** have the following forms:

$$\frac{\partial^n u(x,t)}{\partial x^n} = \psi(x).A_S.B.\phi(t), \qquad \frac{\partial^n u(x,t)}{\partial t^n} = \psi(x).B.(A_S)^T.\phi(t), \qquad n \geq 1. \tag{7}$$

3.2. The operational matrices for Boubaker polynomials

The n^{th}-degree of Boubaker polynomials $P_n(x)$ are defined as follows (28,37):

$$P_n(x) = \sum_{k=0}^{(n/2)} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k} x^{n-2k},$$

also Boubaker polynomials can be computed by used the iterative procedure as follows:

$$P_{m+1}(x) = x\, P_m(x) - P_{m-1}(x), \qquad m \geq 2.$$

Furthermore, suppose that the unknown function $u(x,t)$, may be approximated by applying the double series based on the Boubaker polynomial:

$$u(x,t) = \sum_{h=0}^{n} \sum_{k=0}^{n} b_{hk}\, P_h(x)\, P_k(t) = \psi(x).\ B\ .\ \phi(t), \tag{8}$$

where $\psi(x) = [P_0(x), \ldots, P_n(x)]$, $\phi(t) = [P_0(t), \ldots, P_n(t)]^T$, and

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix}^T,$$

where b_{hk} are the Boubaker polynomials coefficients will be evaluated latter.

The derivatives of $u(x,t)$ may be transformed into matrices by utilizing the following formulation:

$$\frac{\partial^n u(x,t)}{\partial x^n} = \psi(x). A_{Bk}. B. \phi(t), \qquad \frac{\partial^n u(x,t)}{\partial t^n} = \psi(x). B. (A_{Bk})^T. \phi(t), \qquad n \geq 1. \tag{9}$$

The derivatives matrix of Boubaker polynomials denoted by A_{Bk}, and if N is even or $N$is odd, Defined A_{Bk}as follows, respectively:

$$A_{Bk\text{ even}} = \begin{bmatrix} a_{0,0} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{1,0} & 0 & \cdots & 0 & 0 & 0 \\ a_{2,1} & 0 & a_{2,0} & \cdots & 0 & 0 & 0 \\ 0 & a_{3,1} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{N-1,\frac{N-1}{2}} & 0 & a_{N-i,\frac{N-3}{2}} & \cdots & 0 & a_{N-1,0} & 0 \\ 0 & a_{N,\frac{N-1}{2}} & 0 & \cdots & a_{N,1} & 0 & a_{N,0} \end{bmatrix},$$

$$A_{Bk\text{ odd}} = \begin{bmatrix} a_{0,0} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{1,0} & 0 & \cdots & 0 & 0 & 0 \\ a_{2,1} & 0 & a_{2,0} & \cdots & 0 & 0 & 0 \\ 0 & a_{3,1} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & a_{N-1,\frac{N-2}{2}} & 0 & \cdots & 0 & a_{N-1,0} & 0 \\ a_{N,\frac{N}{2}} & 0 & a_{N,\frac{N-2}{2}} & \cdots & a_{N,1} & 0 & a_{N,0} \end{bmatrix}.$$

The following relations can be used to calculate the components $\{a_{n,k}\}_{n=0,k=0}^{N,\left[\frac{n}{2}\right]}$:

$$a_{n,0} = 1, \qquad a_{n,1} = -(n-4), \qquad a_{n,k} = (-1)^k \frac{n-4k}{k!} \prod_{j=k+1}^{2k-1}(n-j), \quad k = 2, 3, \dots, \left[\frac{n}{2}\right].$$

3.3. The operational matrices for Bernoulli polynomials

Bernoulli polynomials $B_m(x)$ of $m^{th} -$ degree is given by (27,38,39):

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_j\, x^{m-j},$$

where B_j is the Bernoulli number, such that $B_j = -\sum_{j=0}^{n+1} \frac{(-1)^j}{j} \binom{n+1}{j} \sum_{h=1}^{j} h^n$, for $n \geq 0$, $n \neq 1$,

The first Bernoulli numbers define as follows:

$B_0 = 1, B_1 = -\frac{1}{2}\ ,\ B_2 = \frac{1}{6}\,.$

Additionally, the unknown function $u(x,t)$ can be approximated by using the following dot product based on the Bernoulli polynomials:

$$u(x,t) = \sum_{h=0}^{n} \sum_{k=0}^{n} b_{hk} P_h(x) P_k(t) = \psi(x). B. \phi(t), \tag{10}$$

where $\psi(x) = [B_0(x), \dots, B_n(x)]$, $\phi(t) = [B_0(t), \dots, B_n(t)]^T$, and

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix}^T,$$

where b_{hk} is the unknown coefficients of Bernoulli polynomials determined latter.

Moreover, the derivatives of $u(x,t)$ can be written in matrices form by the following formula:

$$\frac{\partial^n u(x,t)}{\partial x^n} = \psi(x). A_S. B. \phi(t), \qquad \frac{\partial^n u(x,t)}{\partial t^n} = \psi(x). B. (A_S)^T. \phi(t), \qquad n \geq 1. \tag{11}$$

where A_{Br} is the derivative matrix of Bernoulli polynomials, is given as below:

$$A_{Br} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & k & 0 \end{bmatrix},$$

3.4. The operational matrices for Legendre polynomials

Legendre polynomials $L_m(x)$ of m^{th} −degree is given as follows (24,40):

$L_0(x) = 1, \quad L_1(x) = x, \ \ldots., \quad L_{n+1}(x) = \frac{x\,(2n+1)\ L_n(x) - nL_{n-1}(x)}{n+1}, \ \ n = 2, 3, \ldots.$

Furthermore, the Legendre polynomials will be get by using the formula as follows:

$L_k(x) = \sum_{i=0}^{n} (-1)^{n+i} \frac{(n+i)!}{2^i\,(n-i)!\,(i!)^2} (x+1)^i.$

By using, the linear combination as follow can be approximated the unknown function $u(x,t)$.

$$u(x,t) = \sum_{h=0}^{n} \sum_{k=0}^{n} b_{hk} L_h(x) L_k(t) = \psi(x).B.\phi(t), \tag{12}$$

where $\psi(x) = [L_0(x)\,, \ldots, L_n(x)]$, $\phi(t) = [L_0(t), \ldots, L_n(t)]^T$, and

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix}^T,$$

Moreover, b_{hk} is the coefficients of Legendre polynomials evaluated latter.

The derivative of $u(x,t)$ with a respect to x and t may be written in matrices as follows:

$$\frac{\partial^n u(x,t)}{\partial x^n} = \psi(x).A_L.B.\phi(t), \qquad \frac{\partial^n u(x,t)}{\partial t^n} = \psi(x).B.(A_L)^T.\phi(t), \qquad n \geq 1. \tag{13}$$

The derivatives matrix of the Legendre polynomials denoted by A_L, and we obtained by (34):

$$A_L = \begin{cases} (2j+1), & j = i - n, \\ 0 & \text{otherwise.} \end{cases}$$

If k is even then $n = 1, 3, \ldots, k-1$, and if k is odd then $n = 1, 3, \ldots, k$.

3.5. The operational matrices for Genocchi polynomials

The Genocchi polynomials $G_n(x)$ of n^{th} −degree is given by (20):

$G(x) = \sum_{j=0}^{n} \binom{n}{j} G_n\, x^{n-j},$

By using, the linear combination as follow can be approximated the function $u(x,t)$.

$$u(x,t) = \sum_{h=0}^{n} \sum_{k=0}^{n} b_{hk} G_h(x) G_k(t) = \psi(x).B.\phi(t), \tag{14}$$

where $\phi(t) = [G_0(t), \ \ldots, G_n(t)]^T$, $\psi(x) = (G_0(x), \ldots, G_n(x))$ and

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix}^T$$

Moreover, b_{hk} is the unknown coefficient of Genocchi polynomials evaluated latter.

The derivatives of $u(x,t)$ can be written in matrices form as follows:

$$\frac{\partial^n u(x,t)}{\partial x^n} = \psi(x).G.B.\phi(t), \qquad \frac{\partial^n u(x,t)}{\partial t^n} = \psi(x).B.(G)^T.\phi(t), \qquad n \geq 1. \tag{15}$$

The derivative matrix of Genocchi polynomials is denoted by G and given by:

$$G = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & n-1 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & n & 0 \end{bmatrix},$$

3.6. The operational matrices for Chebyshev polynomials

Chebyshev polynomials $H_m(x)$ of $m^{th}-$ degree are defined as follows (25,41):

$H_m(x)=\sum_{i=0}^{m}(-1)^{m-i}2^i\frac{(m+i-1)!}{(m-i)!(2i)!}(x+1)^i,$

In addition, the function $u(x,t)$ may be approximated by using the linear combination as following:

$$u(x,t)=\sum_{h=0}^{n}\sum_{k=0}^{n}b_{hk}H_h(x)H_k(t)=\psi(x).B.\phi(t), \quad (16)$$

where $\phi(t)=[H_0(t),\dots,H_n(t)]^T$, $\psi(x)=[H_0(x),\dots,H_n(x)]$, and

$$B=\begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix}^T,$$

Furthermore, b_{hk} represents the unknown coefficients of Chebyshev polynomials calculated latter.

The derivatives of $u(x,t)$ may be rewritten in matrices using the following formula:

$$\frac{\partial^n u(x,t)}{\partial x^n}=\psi(x).A_{Ch}.B.\phi(t), \qquad \frac{\partial^n u(x,t)}{\partial t^n}=\psi(x).B.(A_{Ch})^T.\phi(t), \qquad n\geq 1. \quad (17)$$

The derivative matrix of the Chebyshev polynomials is denoted by A_G, and obtained by:

$$A_{Ch}=\begin{cases}\frac{2!}{\xi_i} & i=k-m, \\ 0 & \text{otherwise,}\end{cases}$$

If j is odd then $m=1,3,\dots,j$ and if j is even then $m=1,3,\dots,j-1$, and $\xi_0=2$, and $\xi_n=1$ for $n\geq 1$.

3.7. Algorithm

1. Input (integer) n.

Input (double series) tools.

Input (array) a_{old}=a_{00}, (initial approximation, a_{00} with $n+1$ dimension, are chosen so that the boundary conditions are satisfied).

2. $\widehat{\mathrm{G}}=(a_{\text{old}}): a_{\text{new}}=\widehat{\mathrm{F}}$ is a system of linear algebraic equations that has been solved and a_{new} is obtained.

Go to (2).

2.1 If $|a_{\text{old}}-a_{\text{new}}|$ <tol then $a_{\text{new}}=a$, break (the program is finished).

2.2 Else then $a_{\text{old}}\leftarrow a_{\text{new}}$.

3. Go to (2).

4. Main Results and Applications of Orthogonal Polynomials in Applied Science

In this section, will be applied the proposed methods to resolve some test problems of the non-homogeneous wave equation to get the approximate solution. We use two distinct error criteria to assess the accuracy of suggested methods, which were calculated by using Mathematica®12. The norm L_{rms} is defined by:

$$L_{rms}=\left(\frac{\sum_{j=1}^{N}\left(u_{approx}(x_j,t_j)-u_{exact}(x_j,t_j)\right)^2}{\sum_{j=1}^{N}\left(u_{exact}(x_j,t_j)\right)^2}\right)^{1/2},$$

and the norm L_∞ which is defined by:

$$L_\infty=\mathbf{max}_{j=1,2,\dots\dots,N}\left|u_{approx}(x_j,t_j)-u_{exact}(x_j,t_j)\right|.$$

where u_{approx} and u_{exact} represent the numerical and analytical solutions, respectively.

Example 1. Consider the following inhomogeneous wave problem:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+1, \qquad t\geq 0,\ 0\leq x\leq 1. \quad (18)$$

with initial conditions:

$$u(x,0) = x, \qquad \frac{\partial u}{\partial t}(x,0) = x, \tag{19}$$

and Dirichlet boundary conditions:

$$u(0,t) = \frac{t^2}{2}\ , \qquad u(1,t) = 1 + t + \frac{t^2}{2}. \tag{20}$$

and the analytical solution is: $u(x,t) = x + xt + \frac{t^2}{2}$.

By applying the proposed method to resolve this problem with initial and boundary condition gives in **Equations 19**, and **20**, respectively. More specifically, by converting the unknown function $u(x,t)$ with partial derivatives in to linear equations system. Furthermore, assume that $n = 2$.

First: Applying the standard polynomials.

By inserting the **Equations 6**, and **7** into **Equations 18, 19** and **20** the 2^{th}-order partial derivatives and conditions are converted into matrices, such that:

$$\psi(x).B.\left(A_S{}^2\right)^T.\phi(t) - \psi(x).A_S{}^2.B.\phi(t) = 1,$$

$$[1 \quad x \quad x^2]\begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}[1 \quad t \quad t^2]^T -$$

$$[1 \quad x \quad x^2]\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}[1 \quad t \quad t^2]^T = 1,$$

$$2b_{02} - 2b_{20} + 2b_{12}x - 2b_{21}t + 2b_{22}x^2 - 2b_{22}t^2 = 1, \tag{21}$$

For $n = 2$ the **Equation 6**, becomes as following:

$$u(x,t) = [1 \quad x \quad x^2]\begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}[1 \quad t \quad t^2\ ]^T,$$

$$u(x,t) = b_{00} + b_{01}t + b_{02}t^2 + b_{10}x + b_{11}xt + b_{12}xt^2 + b_{20}x^2 + b_{21}x^2t + b_{22}x^2t^2, \tag{22}$$

From **Equation 7**, the first derivatives of $u(x,t)$ with a respective t its:

$$\frac{\partial\, u(x,t)}{\partial\, t} = \psi(x).B.(A_S)^T.\phi(t) = [1 \quad x \quad x^2]\begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}[1 \quad t \quad t^2]^T,$$

$$= b_{01} + 2b_{02}t + b_{11}x + 2b_{12}tx + b_{21}x^2 + 2b_{22}tx^2\ , \tag{23}$$

By substituting the IC from **Equation 19**, when $t = 0$ in to **Equations 22,** and **23**, will be obtained:

$$u(x,0) = \psi(x).A.\phi(0) = b_{00} + b_{10}x + b_{20}x^2 = x, \tag{24}$$

$$\frac{\partial\, u(x,0)}{\partial\, t} = \psi(x).B.(A_S)^T.\phi(0) = b_{21}x^2 + b_{11}x + b_{01} = x, \tag{25}$$

Also, by inserting the BC from **Equation 20** when $x = 0, 1$ in to **Equation 22**, will be get:

$$u(0,t) = \psi(0).B.\phi(t) = b_{00} + b_{01}t + b_{02}t^2 = \frac{t^2}{2}, \tag{26}$$

$$u(1,t) = \psi(1).B.\phi(t),$$

$$u(1,t) = (b_{02} + b_{12} + b_{22})t^2 + (b_{01} + b_{11} + b_{21})t + (b_{00} + b_{10} + b_{20}) = 1 + t + \frac{t^2}{2}, \tag{27}$$

By resolve the linear algebraic system consisting of the **Equations 21, 24, 25, 26,** and **27**, to find the values of b_{00}, b_{01}, b_{02}, b_{10}, b_{11}, b_{12}, b_{20}, b_{21} and b_{22}, using the Mathematica®12, we get:

$$B = \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} -1.1102 \times 10^{-15} & 0.2143 & 0.4643 \\ 1 & 0.3571 & 0.1071 \\ 3.3307 \times 10^{-16} & 0.4286 & -0.0714 \end{bmatrix}.$$

By replacing these values in **Equations 6** or **22**, therefore, the approximate solution using standard polynomials (x^i) as follows:

$$u(x,t) \approx -1.1102\times10^{-15} + x + 3.3307\times10^{-16}x^2 + t^2(0.4643 + 0.1071x - 0.0714x^2) + t(0.2143 + 0.3571x + 0.4286x^2).$$

Finally, by using the other proposed methods such as: Boubaker, Bernoulli, Chebyshev, Genocchi, and Legendre, the approximate solution for test example 1 as follows:

$$u(x,t) \approx x + t(-0.0455 + 1.13636x - 0.0909x^2) + t^2(0.5410 - 0.1410x + 0.1000x^2).$$

$$u(x,t) \approx x - 1.9660\times10^{-17}x^2 + t^2(0.5313 - 0.0938x + 0.0625x^2) + t(0.0625 + 0.8125x + 0.125x^2).$$

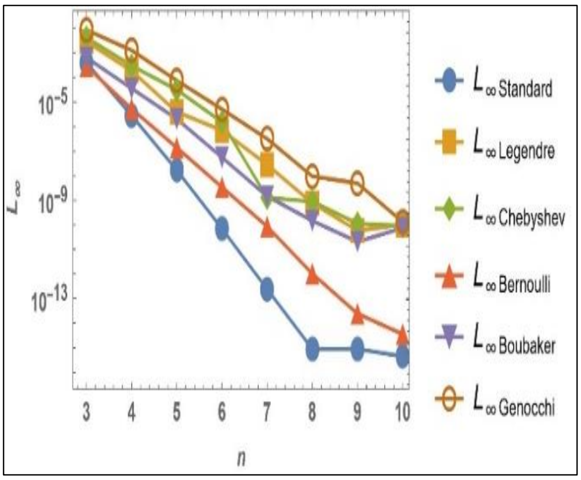
$$u(x,t) \approx 1.7764\times10^{-15} + x - 1.1102\times10^{-15}x^2 + t^2(0.6429 - 0.42857x + 0.2857x^2) + t(0.2143 + 0.3571x + 0.4286x^2).$$

$$u(x,t) \approx -3.5527\times10^{-15} + x + 2.8866\times10^{-15}x^2 + t(-0.7059 + 3.1176x - 1.4118x^2) + t^2(2.3529 - 5.5588x + 3.7059x^2).$$

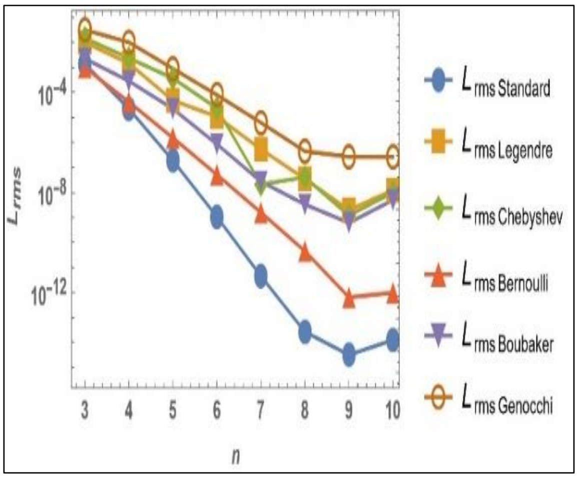
$$u(x,t) \approx 2.7756\times10^{-17} + x - 5.5511\times10^{-17}x^2 + t^2(0.5713 - 0.2139x + 0.1426x^2) + t(0.2213 + 0.3361x + 0.4426x^2).$$

Table 1. The values of L_∞ and L_{rms} for example 1, when $n = 3, 4, \dots, 10$.

n	Norm	Standard Poly.	Boubaker Poly.	Bernoulli Poly.	Chebyshev Poly.	Genocchi Poly.	Legendre Poly.
3	L_∞	3.9389×10^{-4}	6.3650×10^{-4}	2.8073×10^{-4}	4.1435×10^{-3}	8.4519×10^{-3}	2.8457×10^{-3}
	L_{rms}	1.3868×10^{-3}	2.2434×10^{-3}	9.8839×10^{-4}	1.4605×10^{-2}	3.1633×10^{-2}	1.0040×10^{-2}
4	L_∞	2.8110×10^{-6}	3.5392×10^{-5}	5.2614×10^{-6}	2.9529×10^{-4}	1.2557×10^{-3}	1.8394×10^{-4}
	L_{rms}	2.2211×10^{-5}	2.8541×10^{-4}	4.1481×10^{-5}	2.3525×10^{-3}	1.0075×10^{-2}	1.4611×10^{-3}
5	L_∞	1.6801×10^{-8}	2.0271×10^{-6}	1.3806×10^{-7}	3.1093×10^{-5}	7.3779×10^{-5}	3.9023×10^{-6}
	L_{rms}	1.9194×10^{-7}	2.2173×10^{-5}	1.5364×10^{-6}	3.6629×10^{-4}	8.9684×10^{-4}	4.7726×10^{-5}
6	L_∞	7.0099×10^{-11}	5.5441×10^{-8}	3.4199×10^{-9}	1.5229×10^{-6}	5.0341×10^{-6}	6.9510×10^{-7}
	L_{rms}	1.0776×10^{-9}	8.5531×10^{-7}	5.1635×10^{-8}	2.3571×10^{-5}	7.9422×10^{-5}	1.0778×10^{-5}
7	L_∞	2.4247×10^{-13}	1.4491×10^{-9}	8.5369×10^{-11}	1.2737×10^{-9}	3.1418×10^{-7}	2.7870×10^{-8}
	L_{rms}	4.5606×10^{-12}	2.7474×10^{-8}	1.5928×10^{-9}	2.1049×10^{-8}	6.1840×10^{-6}	5.3380×10^{-7}
8	L_∞	8.8818×10^{-16}	1.4573×10^{-10}	1.0250×10^{-12}	8.8493×10^{-10}	9.4655×10^{-9}	7.6410×10^{-10}
	L_{rms}	2.7022×10^{-14}	3.3249×10^{-9}	4.5441×10^{-11}	4.0375×10^{-8}	4.4433×10^{-7}	3.4891×10^{-8}
9	L_∞	8.8818×10^{-16}	2.0440×10^{-11}	2.4425×10^{-14}	1.1189×10^{-10}	4.8860×10^{-9}	5.7948×10^{-11}
	L_{rms}	3.2342×10^{-15}	6.1451×10^{-10}	6.4498×10^{-13}	1.2092×10^{-9}	2.6727×10^{-7}	1.9853×10^{-9}
10	L_∞	4.4409×10^{-16}	7.4571×10^{-11}	3.5527×10^{-15}	9.2631×10^{-11}	1.3047×10^{-10}	1.0119×10^{-10}
	L_{rms}	1.2699×10^{-14}	4.9801×10^{-9}	9.8615×10^{-13}	1.0156×10^{-8}	2.6093×10^{-7}	1.2359×10^{-8}



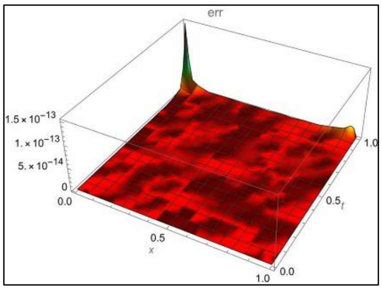
Norm $L_\infty, n = 3, 4, \ldots, 10$.



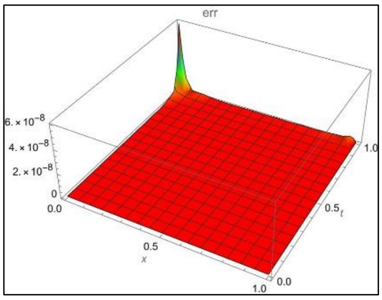
Norm $L_{rms}, n = 3, 4, \ldots, 10$.

Figure 1. The plots logarithmic of L_{rms} and L_∞ , for example 1, when $n = 3, 4, \ldots, 10$.

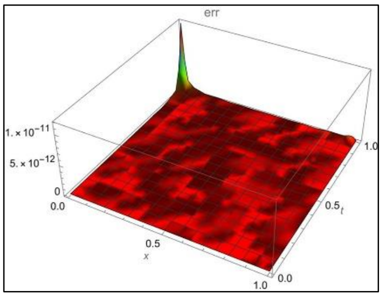
In **Table 1** and **Figure 1**, it can be seen the values of the norm L_{rms} and L_∞. They depend on the value of n, which represents the degree of the polynomial. The higher the value of n gives the lower error. The best results were achieved using the standard polynomials when, $n = 10$, the values of L_∞ and L_{rms} equals, 4.4409×10^{-16} ,1.2699×10^{-14}, respectively, followed by Bernoulli polynomials.



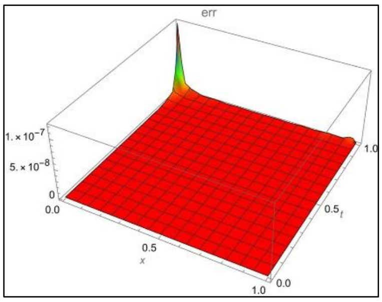
Standard polynomials



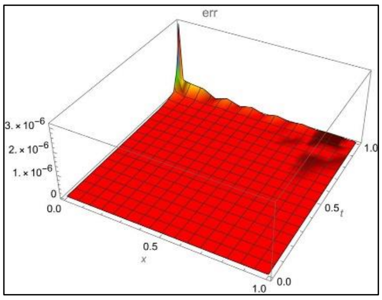
Boubaker Polynomials



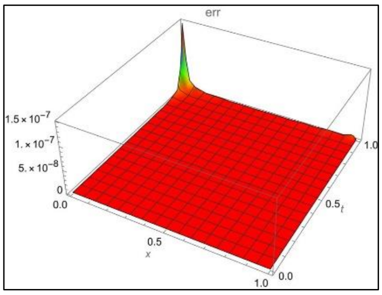
Bernolli Polynomials



Chebyshev Polynomials

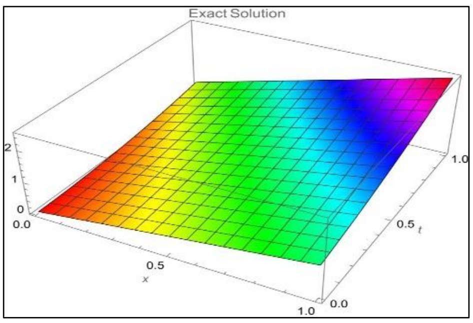


Genocchi Polynomials

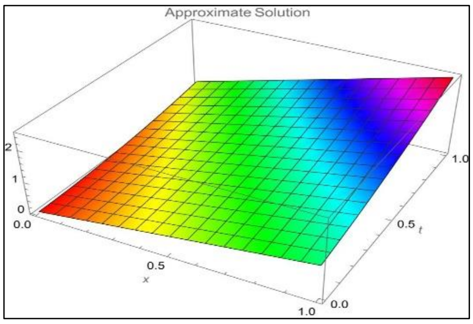


Legendre Polynomials

Figure 2. Plots the norm L_∞ for example 1, when $n = 10$.



Exact solution



Approximate solution

Figure 3. Comparison between analytical and numerical solution for example 1 by the standard poly.

Example 2. Consider the inhomogeneous wave problem as follows:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+(x^2-2)\cosh(t), \qquad 0\le x\le 1, t\ge 0, \tag{28}$$

with initial conditions:

$$u(x,0)=x^2, \qquad \frac{\partial u}{\partial t}(x,0)=0, \tag{29}$$

and Dirichlet boundary conditions:

$$u(0,t)=0, \qquad u(1,t)=\cosh(t). \tag{30}$$

and the analytical solution is: $u(x,t)=x^2\cosh(t)$.

$$\psi(x).B.\left({A_n^*}^2\right)^T.\phi(t)-\psi(x).{A_n^*}^2.B.\phi(t)=(x^2-2)\cosh(t),$$

$$u(x,0)=\psi(x).B.\phi(0)=x^2, \qquad \frac{\partial u}{\partial t}(x,0)=\psi(x).B.(A_n^*)^T.\phi(0)=0,$$

$$u(0,t)=\psi(0).B.\phi(t)=0, \qquad u(1,t)=\psi(1).B.\phi(t)=\cosh(t),$$

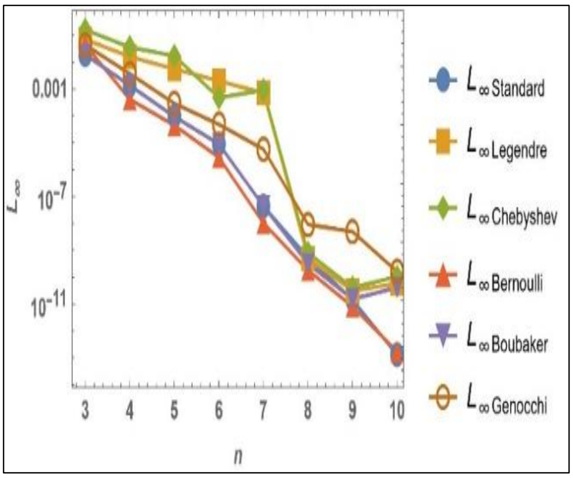
such that A_n^* represented the operational matrices of all proposed methods.

Let us use the Mathematica®12 to solve the given test problem and determine the L_∞ and L_{rms}:

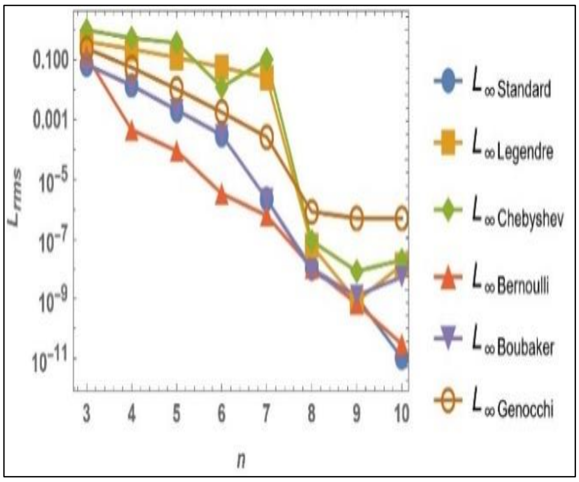
Table 2. The values of L_∞ and L_{rms} for example 2, when $n=3,4,\dots,10$.

n	Norm	Standard Poly.	Boubaker Poly.	Bernoulli Poly.	Chebyshev Poly.	Genocchi Poly.	Legendre Poly.
3	L_∞	0.0188309	0.0188309	0.0436566	0.1571468	0.0430718	0.0689526
	L_{rms}	0.0704147	0.0704147	0.1338318	0.9958798	0.2414356	0.4054831
4	L_∞	1.3482×10^{-3}	1.3482×10^{-3}	4.0844×10^{-4}	0.0355224	3.6705×10^{-3}	0.0157769
	L_{rms}	0.0140385	0.0140385	4.5347×10^{-4}	0.5368480	0.0560399	0.2322642
5	L_∞	9.3338×10^{-5}	9.3338×10^{-5}	4.4291×10^{-5}	0.0168202	2.8721×10^{-4}	5.5056×10^{-3}
	L_{rms}	2.1050×10^{-3}	2.1050×10^{-3}	9.0735×10^{-5}	0.3734015	0.0106238	0.1194339
6	L_∞	8.9516×10^{-6}	8.9516×10^{-6}	2.8350×10^{-6}	4.7210×10^{-4}	4.7884×10^{-5}	1.9671×10^{-3}
	L_{rms}	3.0845×10^{-4}	3.0845×10^{-4}	3.3746×10^{-6}	0.0121814	1.7861×10^{-3}	0.0570002
7	L_∞	4.3799×10^{-8}	4.3799×10^{-8}	1.0233×10^{-8}	9.011×10^{-4}	5.7300×10^{-6}	7.0847×10^{-4}
	L_{rms}	2.2095×10^{-6}	2.2095×10^{-6}	6.2162×10^{-7}	0.1040413	2.6369×10^{-4}	0.0253115
8	L_∞	5.3188×10^{-10}	3.1343×10^{-10}	1.8148×10^{-10}	8.0279×10^{-10}	9.0985×10^{-9}	4.9988×10^{-10}
	L_{rms}	1.0668×10^{-8}	8.4181×10^{-9}	1.1232×10^{-8}	8.7810×10^{-8}	8.3036×10^{-7}	6.2008×10^{-8}
9	L_∞	1.4315×10^{-11}	1.5375×10^{-11}	8.1493×10^{-12}	4.0713×10^{-11}	4.9242×10^{-9}	3.1324×10^{-11}
	L_{rms}	1.1569×10^{-9}	1.3045×10^{-9}	7.0331×10^{-10}	8.4630×10^{-9}	5.0793×10^{-7}	8.4263×10^{-10}
10	L_∞	1.2901×10^{-13}	4.1190×10^{-11}	1.7109×10^{-13}	1.0473×10^{-10}	1.6116×10^{-10}	5.8388×10^{-11}
	L_{rms}	1.0793×10^{-11}	5.4818×10^{-9}	3.0960×10^{-11}	2.0187×10^{-8}	4.9575×10^{-7}	1.3722×10^{-8}

The error norm and mean square error values are readily seen in **Table 2**, and **Figure 4**, which can be obtained by solving example 2 using Mathematica. Some methods gave close results, but the Bernoulli polynomials gave better results when $n = 9$, such that L_∞, L_{rms} equals $8.1493 \times 10^{-12}, 7.0331 \times 10^{-10}$, respectively, but when $n = 10$ the Bernoulli and standard polynomials give the same results.

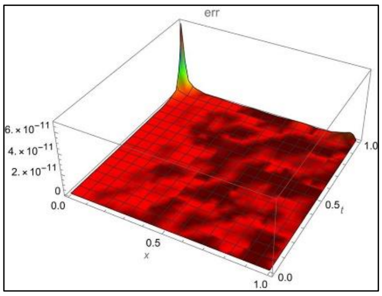


Norm $L_\infty, n = 3, 4, \ldots, 10$.

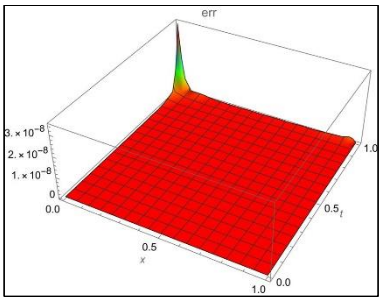


Norm $L_{rms}, n = 3, 4, \ldots, 10$.

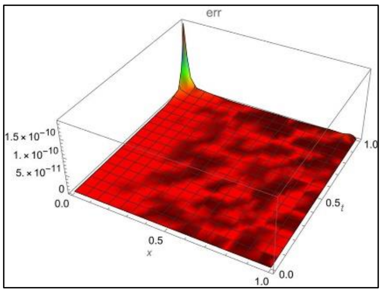
Figure 4. The plots logarithmic of L_{rms} and L_∞ , for example 2, when $n = 3, 4, \ldots, 10$.



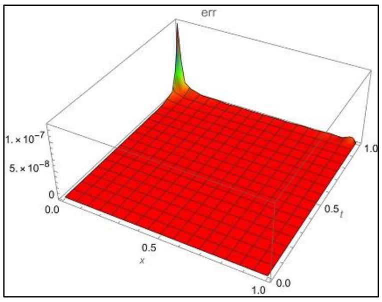
Standard Polynomials



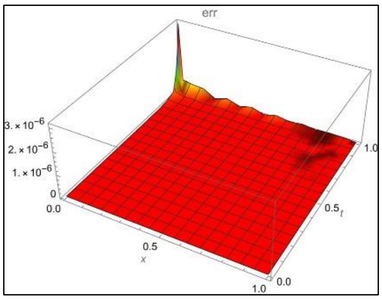
Boubaker Polynomials



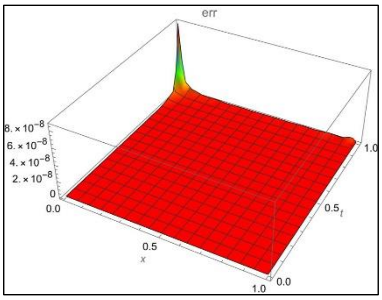
Bernoulli Polynomials



Chebyshev Polynomials

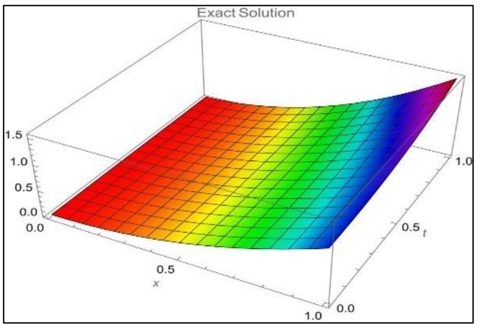


Genocchi Polynomials

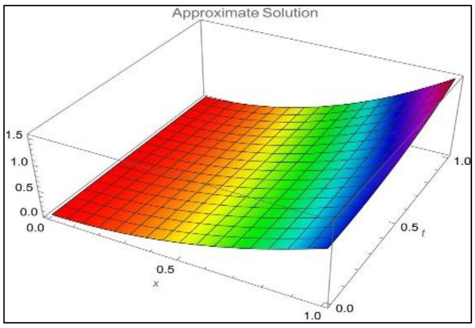


Legendre Polynomials

Figure 5. Plots the norm L_∞ for example 2, when $n = 10$.



Exact solution



Approximate solution

Figure 6. Comparison between analytical and numerical solution for example 2 by the Bernoulli poly.

Example 3. Consider the homogeneous wave problem as follows (7):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq x \leq 1, \ \ t \geq 0. \tag{31}$$

with initial conditions:

$$u(x,0) = x, \qquad \frac{\partial u}{\partial t}(x,0) = \cos(x), \tag{32}$$

and Neumann boundary conditions:

$$\frac{du}{dn}(0,t) = 1, \qquad \frac{du}{dn}(1,t) = 1 - \sin(1)\sin(t). \tag{33}$$

and the analytical solution is: $u(x,t) = x + \cos(x)\sin(t)$.

By implementing the proposed methods will be get the following linear equations system:

$\psi(x).B.\left({A_n^*}^2\right)^T.\phi(t) - \psi(x).{A_n^*}^2.B.\phi(t) = 0,$

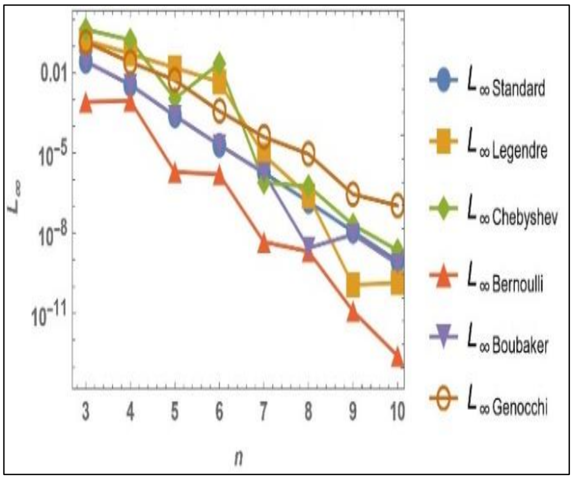
$u(x,0) = \psi(x).A.\phi(0) = x,$ $\qquad \frac{\partial u}{\partial t}(x,0) = \psi(x).B.(A_n^*)^T.\phi(0) = \cos(x),$

$u(0,t) = \psi(0).A_n^*.B.\phi(t) = 1,$ $\qquad \frac{du}{dn}(1,t) = \psi(1).A_n^*.B.\phi(t) = 1 - \sin(1)\sin(t).$

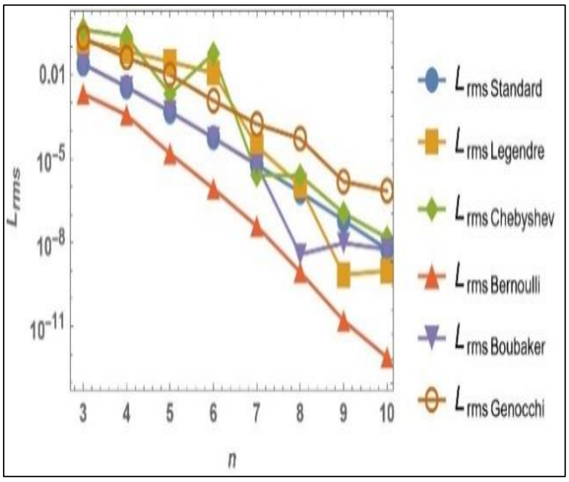
Table 3. The values of L_∞ and L_{rms} for example 3, when $n = 3, 4, \ldots, 10$.

n	Norm	Standard Poly.	Boubaker Poly.	Bernoulli Poly.	Chebyshev Poly.	Genocchi Poly.	Legendre Poly.
3	L_∞	0.026769	0.026769	8.1346×10^{-4}	0.422276	0.144075	0.165196
	L_{rms}	0.023984	0.023984	2.0740×10^{-3}	0.409013	0.204080	0.158717
4	L_∞	3.5980×10^{-3}	3.5980×10^{-3}	9.0742×10^{-4}	0.171641	0.025237	0.051985
	L_{rms}	3.6621×10^{-3}	3.6621×10^{-3}	3.6562×10^{-4}	0.231415	0.040730	0.069231
5	L_∞	2.4744×10^{-4}	2.4744×10^{-4}	1.9915×10^{-6}	1.0927×10^{-3}	5.3252×10^{-3}	0.015482
	L_{rms}	4.7965×10^{-4}	4.7965×10^{-4}	1.5268×10^{-5}	2.0580×10^{-3}	9.9614×10^{-3}	0.028927
6	L_∞	2.0079×10^{-5}	2.0079×10^{-5}	1.5870×10^{-6}	0.022490	3.7142×10^{-4}	4.5766×10^{-3}
	L_{rms}	5.5328×10^{-5}	5.5328×10^{-5}	8.5427×10^{-7}	0.056908	1.1897×10^{-3}	0.011584
7	L_∞	1.8479×10^{-6}	1.8268×10^{-6}	4.7259×10^{-9}	7.9157×10^{-7}	3.9991×10^{-5}	8.3372×10^{-6}
	L_{rms}	6.1196×10^{-6}	6.1167×10^{-6}	3.9008×10^{-8}	2.5910×10^{-6}	1.7296×10^{-4}	2.7798×10^{-5}
8	L_∞	1.5123×10^{-7}	2.8128×10^{-9}	2.1089×10^{-9}	5.7829×10^{-7}	9.2256×10^{-6}	2.5362×10^{-7}
	L_{rms}	6.3784×10^{-7}	3.7864×10^{-9}	8.9858×10^{-10}	2.4624×10^{-6}	5.1617×10^{-5}	1.0919×10^{-6}
9	L_∞	1.1721×10^{-8}	9.0889×10^{-9}	1.2325×10^{-11}	2.0718×10^{-8}	2.8711×10^{-7}	1.1747×10^{-10}
	L_{rms}	6.2317×10^{-8}	9.6519×10^{-9}	1.6392×10^{-11}	1.1274×10^{-7}	1.5240×10^{-6}	7.1236×10^{-10}
10	L_∞	8.9957×10^{-10}	7.1069×10^{-10}	2.6179×10^{-13}	2.3150×10^{-9}	1.0916×10^{-7}	1.4462×10^{-10}
	L_{rms}	5.8471×10^{-9}	5.8575×10^{-9}	8.0077×10^{-13}	1.4899×10^{-8}	7.0608×10^{-7}	9.8726×10^{-10}

From **Table 3**. and **Figure 7.**, the error norm and mean square error values are visible, as these values decrease as the value of n increases, which represents the degree of the polynomial. The best methods for approximation are the Bernoulli polynomials, such that when $n = 5$, the values of L_∞ and L_{rms} equals 8.1346×10^{-4}, 2.0740×10^{-3}, respectively, but when $n = 10$, it can be seen the values of L_∞ and L_{rms} gives 2.6179×10^{-13}, 8.0077×10^{-13}.

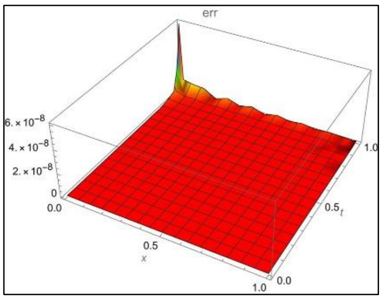


Norm $L_\infty, n = 3, 4, \ldots, 10$.

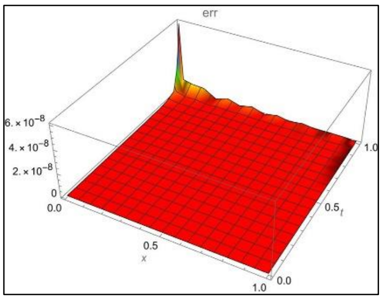


Norm $L_{rms}, n = 3, 4, \ldots, 10$.

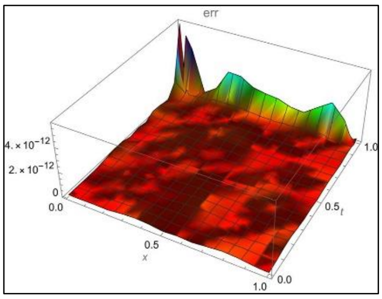
Figure 7. The plots logarithmic of L_{rms} and L_∞ , for example 3, when $n = 3, 4, \ldots, 10$.



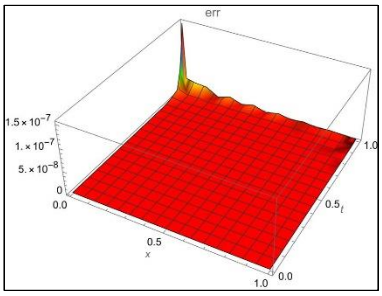
Standard Polynomials



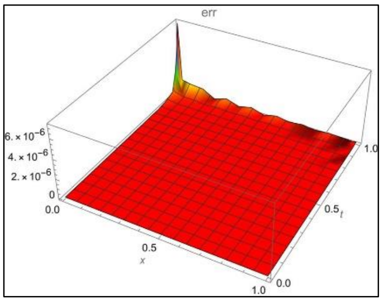
Boubaker Polynomials



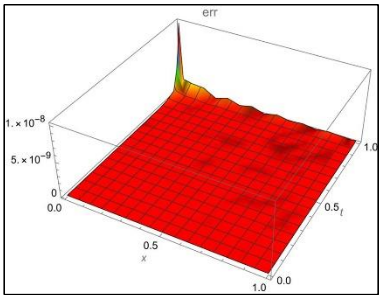
Bernoulli Polynomials



Chebyshev Polynomials

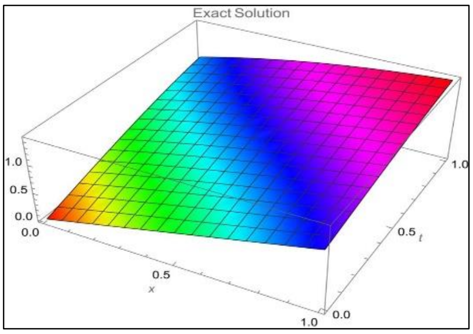


Genocchi Polynomials

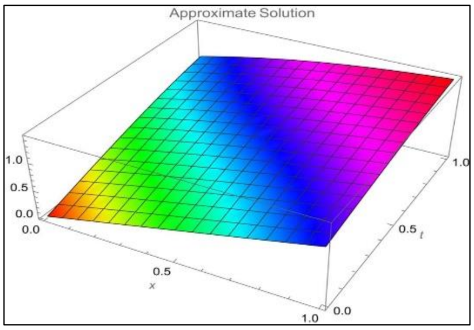


Legendre Polynomials

Figure 8. Plots the norm L_∞ for example 3, when $n = 10$.



Exact solution



Approximate solution

Figure 9. Comparison between analytical and numerical solution for example 3 by the Bernoulli poly.

Example 4: Consider the following inhomogeneous wave problem (7):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 6t + 2x, \qquad 0 \le x \le 1, t \ge 0, \tag{34}$$

with initial conditions:

$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \sin(x),$ (35)

and Neumann boundary conditions:

$\frac{du}{dn}(0,t) = t^2, \qquad \frac{du}{dn}(1,t) = t^2 + \cos(1)\sin(t).$ (36)

and the analytical solution is: $u(x,t) = t^3 + t^2 x + \sin(x)\sin(t)$.

$\psi(x).B.\left({A_n^*}^2\right)^T.\phi(t) - \psi(x).{A_n^*}^2.B.\phi(t) = 6t + 2x,$

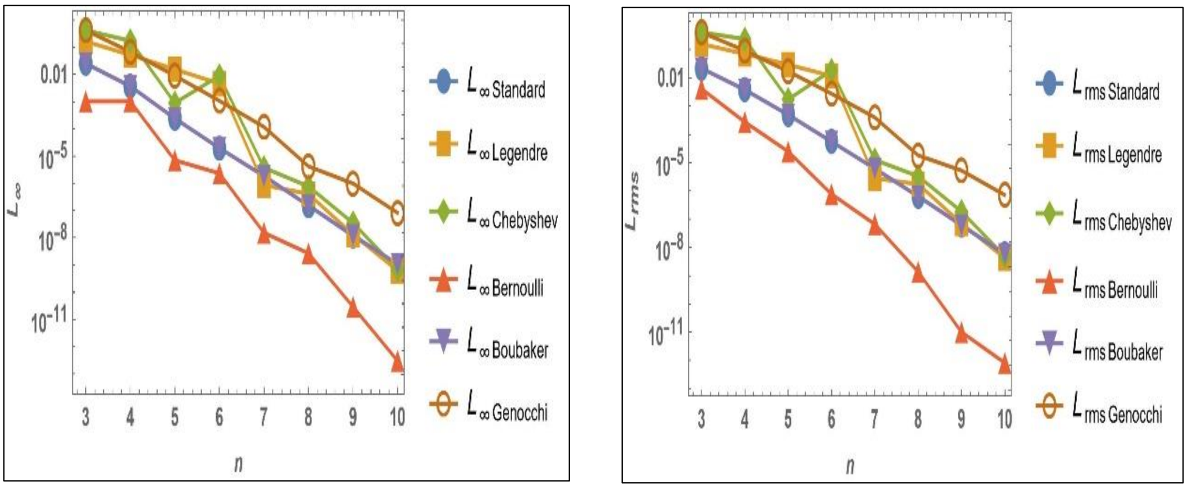
$u(x,0) = \psi(x).B.\phi(0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x).B.(A_n^*)^T.\phi(0) = \sin(x),$

$\frac{du}{dn}(0,t) = \psi(0).A_n^*.B.\phi(t) = t^2, \qquad \frac{du}{dn}(1,t) = \psi(1).A_n^*.B.\phi(t) = t^2 + \cos(1)\sin(t).$

Table 4. The values error norm L_∞ and L_{rms} for example 4, when $n = 3, 4, \ldots, 10$.

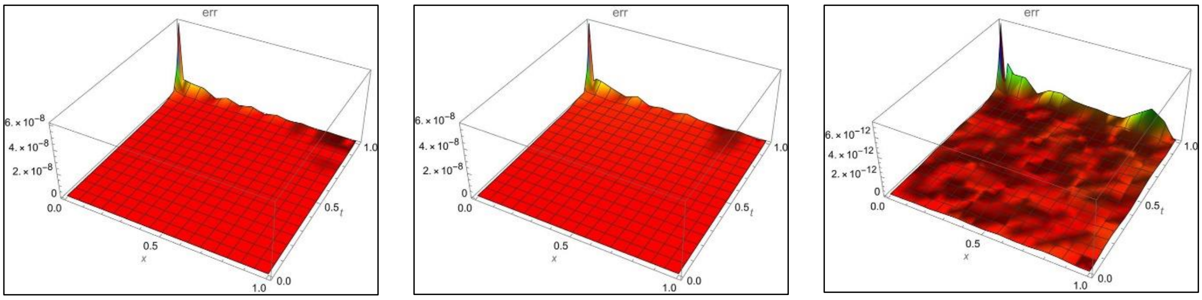
n	Norm	Standard Poly.	Boubaker Poly.	Bernoulli Poly.	Chebyshev Poly.	Genocchi Poly.	Legendre Poly.
3	L_∞	0.025288	0.025287	0.0010292	0.420795	0.420795	0.163715
	L_{rms}	0.021544	0.021544	0.0040845	0.409652	0.409652	0.157369
4	L_∞	3.6917×10^{-3}	3.6917×10^{-3}	1.0348×10^{-3}	0.171735	6.7708×10^{-2}	5.2079×10^{-2}
	L_{rms}	3.7901×10^{-3}	3.7901×10^{-3}	2.7305×10^{-4}	0.233360	9.1241×10^{-2}	6.9889×10^{-2}
5	L_∞	2.4244×10^{-4}	2.4244×10^{-4}	6.9924×10^{-6}	8.9554×10^{-4}	8.9811×10^{-3}	1.5477×10^{-2}
	L_{rms}	4.9382×10^{-4}	4.9382×10^{-4}	2.4738×10^{-5}	1.7230×10^{-3}	1.6941×10^{-2}	2.9166×10^{-2}
6	L_∞	1.9630×10^{-5}	1.9630×10^{-5}	2.3295×10^{-6}	8.9054×10^{-3}	1.0740×10^{-3}	4.5761×10^{-3}
	L_{rms}	5.5389×10^{-5}	5.5389×10^{-5}	7.7029×10^{-7}	1.8313×10^{-2}	2.7441×10^{-3}	1.1675×10^{-2}
7	L_∞	1.8584×10^{-6}	1.8584×10^{-6}	1.5232×10^{-8}	3.7324×10^{-6}	1.1798×10^{-4}	8.5907×10^{-7}
	L_{rms}	6.1401×10^{-6}	6.1401×10^{-6}	6.6218×10^{-8}	1.2450×10^{-5}	3.9714×10^{-4}	2.7754×10^{-6}
8	L_∞	1.5181×10^{-7}	1.5182×10^{-7}	2.6662×10^{-9}	7.7151×10^{-7}	4.1914×10^{-6}	4.0559×10^{-7}
	L_{rms}	6.4355×10^{-7}	6.4355×10^{-7}	1.3797×10^{-9}	3.3122×10^{-6}	1.8038×10^{-5}	1.7364×10^{-6}
9	L_∞	1.1711×10^{-8}	1.2419×10^{-8}	2.9055×10^{-11}	3.4713×10^{-8}	9.8285×10^{-7}	1.3215×10^{-8}
	L_{rms}	6.2850×10^{-8}	5.9882×10^{-8}	1.0129×10^{-11}	1.8559×10^{-7}	5.2592×10^{-6}	7.0957×10^{-8}
10	L_∞	8.9913×10^{-10}	1.0768×10^{-9}	2.9887×10^{-13}	6.7981×10^{-10}	8.1974×10^{-8}	6.0933×10^{-10}
	L_{rms}	5.8928×10^{-9}	5.7708×10^{-9}	8.2474×10^{-13}	4.8477×10^{-9}	7.1460×10^{-7}	4.1418×10^{-9}

From **Table 4**, and **Figure 10**, it can be seen the values of the L_{rms} and L_∞. The best approximation method was the Bernoulli polynomial, where the value of the error norm was equal to 0.0010292 and the value of the mean square error was equal to 0.0040845 when $n = 3$, also when $n = 10$, the values of L_∞, and L_{rms} are 2.9887×10^{-13}, 8.2474×10^{-13}, respectively.

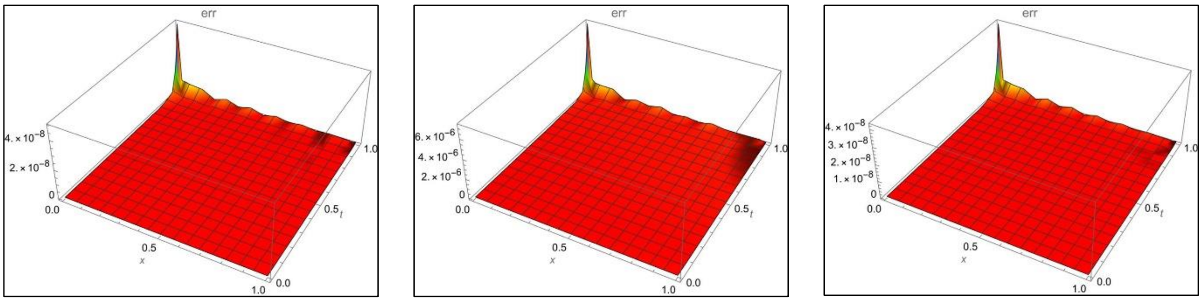


Norm $L_\infty, n = 3, 4, \dots, 10$. Norm $L_{rms}, n = 3, 4, \dots, 10$.

Figure 10. The plots logarithmic of L_{rms} and L_∞ , for example 4, when $n = 3, 4, \dots, 10$.

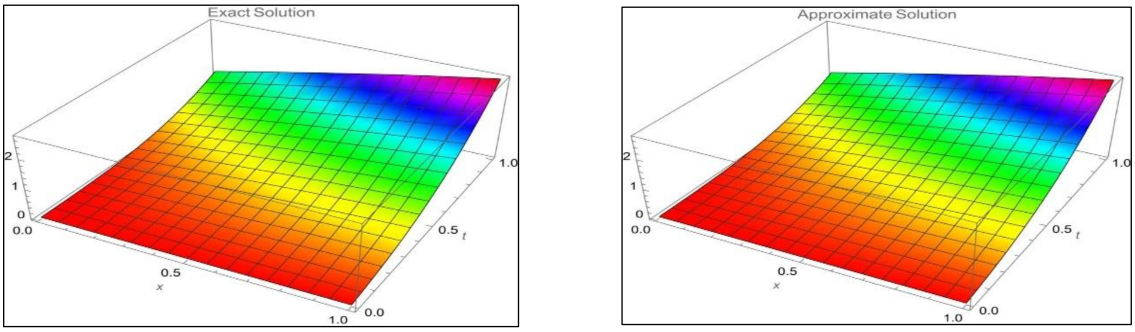


Standard Polynomials Boubaker Polynomials Bernoulli Polynomials



Chebyshev Polynomials Genocchi Polynomials Legendre Polynomials

Figure 11. Plots the norm L_∞ for example 4, when $n = 10$.



Exact solution Approximate solution

Figure 12. Comparison between analytical and numerical solution for example 4 by the Bernoulli poly.

Example 5. Consider the following homogeneous wave problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \; 0 \leq x \leq 1, \tag{37}$$

with initial conditions:

$$u(x,0) = x^2, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \tag{38}$$

and mixed boundary conditions:

$$u(0,t) = t^2 \quad , \qquad \frac{\partial u}{\partial n}(1,t) = 2. \tag{39}$$

and the analytical solution is: $u(x,t) = x^2 + t^2$.

$\psi(x).B.\left({A_n^*}^2\right)^T.\phi(t) - \psi(x).{A_n^*}^2.B.\phi(t) = 0,$

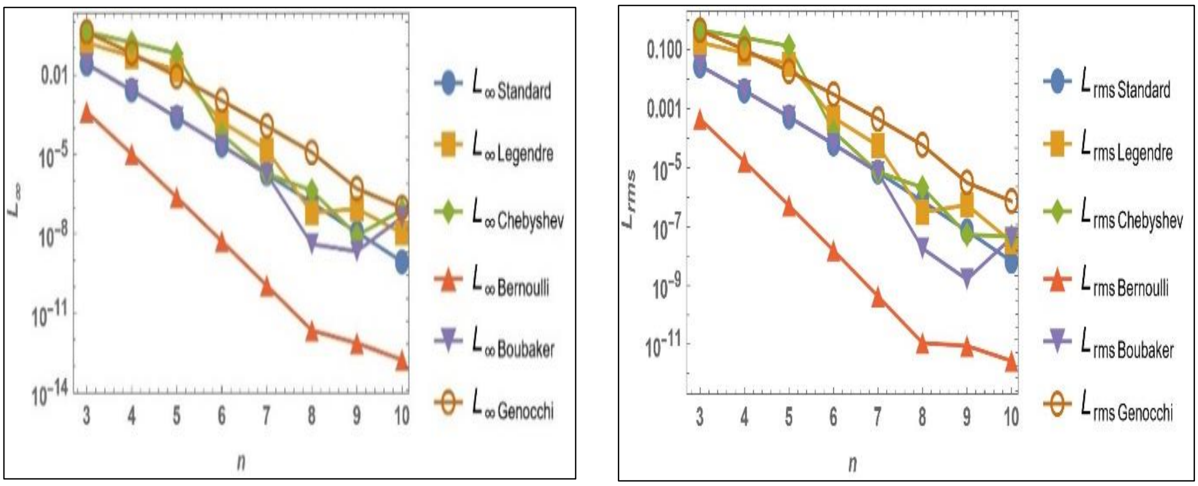
$u(x,0) = \psi(x).B.\phi(0) = x^2,$ $\qquad \frac{\partial u}{\partial t}(x,0) = \psi(x).B.(A_n^*)^T.\phi(0) = 0,$

$u(0,t) = \psi(0).B.\phi(t) = t^2,$ $\qquad \frac{\partial u}{\partial n}(1,t) = \psi(1).A_n^*.B.\phi(t) = 2.$

Table 5. The values L_∞ and L_{rms} for example 5, when $n = 3, 4, \ldots, 10$.

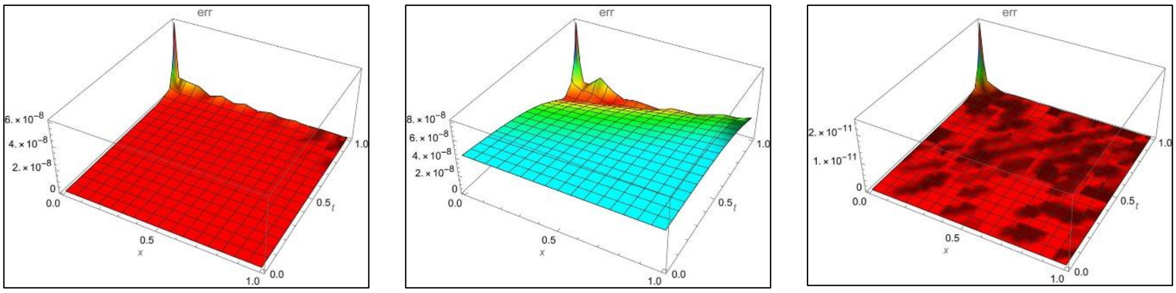
n	Norm	Standard Poly.	Boubaker Poly.	Bernoulli Poly.	Chebyshev Poly.	Genocchi Poly.	Legendre Poly.
3	L_∞	0.026367	0.02637	4.1199×10^{-4}	0.42187	0.421875	0.16479
	L_{rms}	0.028829	0.028829	4.5045×10^{-4}	0.46126	0.461264	0.18018
4	L_∞	2.6674×10^{-3}	2.6674×10^{-3}	1.0419×10^{-5}	0.17071	6.6684×10^{-2}	5.1055×10^{-2}
	L_{rms}	4.0600×10^{-3}	4.0600×10^{-3}	1.5859×10^{-5}	0.25984	0.10150	7.7711×10^{-2}
5	L_∞	2.4968×10^{-4}	2.4968×10^{-4}	2.4382×10^{-7}	6.3917×10^{-2}	8.9883×10^{-3}	1.5484×10^{-2}
	L_{rms}	5.2356×10^{-4}	5.2356×10^{-4}	5.1129×10^{-7}	0.13403	1.8848×10^{-2}	3.2469×10^{-2}
6	L_∞	2.1965×10^{-5}	2.1965×10^{-5}	5.3626×10^{-9}	5.9419×10^{-5}	1.0763×10^{-3}	1.7436×10^{-4}
	L_{rms}	6.2408×10^{-5}	6.2408×10^{-5}	1.5236×10^{-8}	1.6882×10^{-4}	3.0580×10^{-3}	4.9539×10^{-4}
7	L_∞	1.8433×10^{-6}	1.8433×10^{-6}	1.1251×10^{-10}	1.8693×10^{-6}	1.1797×10^{-4}	1.4866×10^{-5}
	L_{rms}	6.9147×10^{-6}	6.9147×10^{-6}	4.2211×10^{-10}	7.0124×10^{-6}	4.4254×10^{-4}	5.5767×10^{-5}
8	L_∞	1.4915×10^{-7}	4.0861×10^{-9}	2.2808×10^{-12}	4.3597×10^{-7}	1.2081×10^{-5}	6.3073×10^{-8}
	L_{rms}	7.1560×10^{-7}	1.7778×10^{-8}	1.0964×10^{-11}	2.0917×10^{-6}	5.7964×10^{-5}	3.0262×10^{-7}
9	L_∞	1.1727×10^{-8}	2.2072×10^{-9}	7.5362×10^{-13}	8.7881×10^{-9}	5.1274×10^{-7}	8.9391×10^{-8}
	L_{rms}	6.9919×10^{-8}	1.5722×10^{-9}	8.6578×10^{-12}	5.2527×10^{-8}	3.0578×10^{-6}	5.3297×10^{-7}
10	L_∞	9.0098×10^{-10}	4.2117×10^{-8}	1.7586×10^{-13}	7.9146×10^{-8}	9.7206×10^{-8}	1.0858×10^{-8}
	L_{rms}	6.5670×10^{-9}	4.0507×10^{-8}	2.6847×10^{-12}	4.4758×10^{-8}	7.1679×10^{-7}	2.9672×10^{-8}

The mean square error and the error norm values can be easily observed in **Table 5**, and **Figure 13**, which we obtained by solving example 5 using Mathematica. Some methods gave close results; except for the Bernoulli polynomial, which gave better results in general. The values of L_∞, and L_{rms} when $n = 3$ can be observed to be equals to 4.1199×10^{-4}, 4.5045×10^{-4} and equals 1.7586×10^{-13},2.6847×10^{-12}, respectively, when $n = 10$.

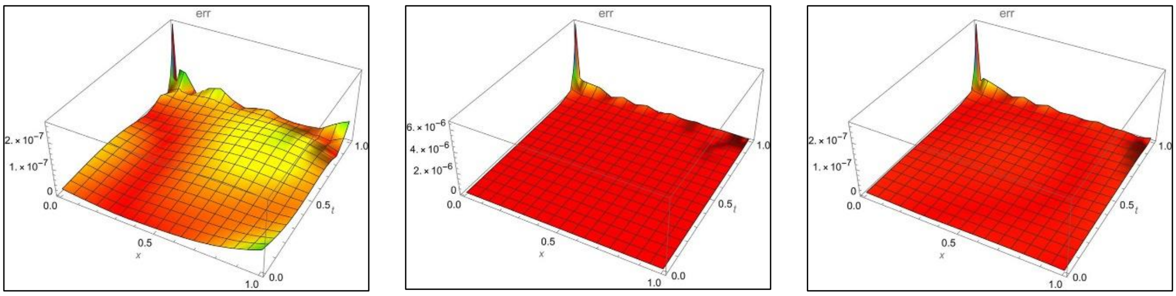


Norm $L_\infty, n = 3, 4, \dots, 10$. Norm $L_{rms}, n = 3, 4, \dots, 10$.

Figure 13. The plots logarithmic of L_{rms} and L_∞ , for example 5, when $n = 3, 4, \dots, 10$.

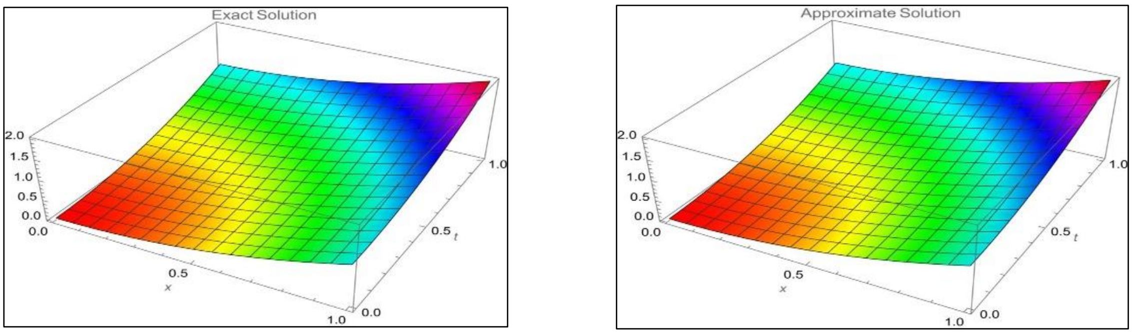


Standard Polynomials Boubaker Polynomials Bernoulli Polynomials



Chebyshev Polynomials Genocchi Polynomials Legendre Polynomials

Figure 14. Plots the norm L_∞ for example 5, when $n = 10$.



Exact solution Approximate solution

Figure 15. Comparison between analytical and numerical solution for example 5 by the Bernoulli poly.

Example 6. Consider the following homogeneous wave problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0,\ 0 \leq x \leq 1, \tag{40}$$

with initial conditions:

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \cosh(x), \tag{41}$$

and mixed boundary conditions:

$$u(0,t) = \sinh(t), \qquad \frac{\partial u}{\partial n}(1,t) = \sinh(1)\sinh(t), \tag{42}$$

and the analytical solution is: $u(x,t) = \cosh(x)\sinh(t)$.

$\psi(x).B.\left(A_n^{*2}\right)^T.\phi(t) - \psi(x).A_n^{*2}.B.\phi(t) = 0,$

$u(x,0) = \psi(x).B.\phi(0) = 0,$ $\quad \frac{\partial u}{\partial t}(x,0) = \psi(x).B.(A_n^*)^T.\phi(0) = \cosh(x),$

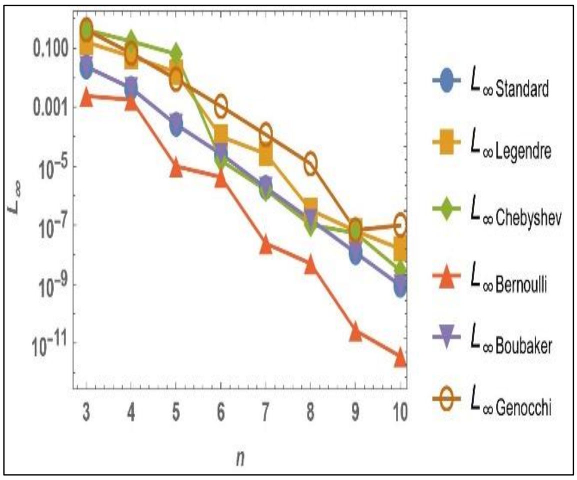
$u(0,t) = \psi(0).B.\phi(t) = \sinh(t),$

$\frac{\partial u}{\partial n}(1,t) = \psi(1).A_n^*.B.\phi(t) = \sinh(1)\sinh(t).$

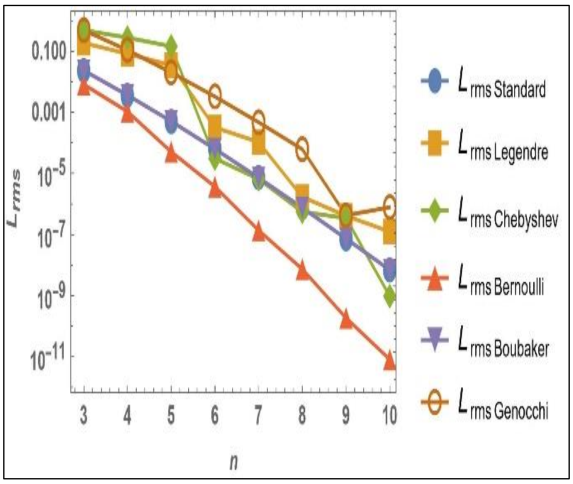
Table 6. The values of L_∞ and L_{rms} for example 6, when $n = 3, 4, \dots, 10$.

n	Norm	Standard Poly.	Boubaker Poly.	Bernoulli Poly.	Chebyshev Poly.	Genocchi Poly.	Legendre Poly.
3	L_∞	0.02371	0.02371	0.00239	0.41921	0.41921	0.16213
	L_{rms}	2.3180×10^{-2}	2.3180×10^{-2}	8.0865×10^{-3}	0.48249	0.48249	0.18379
4	L_∞	4.4663×10^{-3}	4.4663×10^{-3}	1.8093×10^{-3}	0.17251	6.8483×10^{-2}	5.2854×10^{-2}
	L_{rms}	3.6227×10^{-3}	3.6227×10^{-3}	1.0568×10^{-3}	0.27538	0.10710	8.1823×10^{-2}
5	L_∞	2.5933×10^{-4}	2.5933×10^{-4}	9.8953×10^{-6}	6.3926×10^{-2}	8.9980×10^{-3}	1.5493×10^{-2}
	L_{rms}	5.0699×10^{-4}	5.0698×10^{-4}	5.1090×10^{-5}	0.14239	1.9981×10^{-2}	3.4457×10^{-2}
6	L_∞	2.6267×10^{-5}	2.6267×10^{-5}	4.3072×10^{-6}	1.5617×10^{-5}	1.0806×10^{-3}	1.0011×10^{-4}
	L_{rms}	6.3034×10^{-5}	6.3034×10^{-5}	3.7037×10^{-6}	3.0898×10^{-5}	3.2466×10^{-3}	3.1860×10^{-4}
7	L_∞	1.8672×10^{-6}	1.8672×10^{-6}	2.4094×10^{-8}	1.5815×10^{-6}	1.1799×10^{-4}	2.6682×10^{-5}
	L_{rms}	7.2126×10^{-6}	7.2126×10^{-6}	1.3786×10^{-7}	6.5372×10^{-6}	4.7018×10^{-4}	1.0614×10^{-4}
8	L_∞	1.5414×10^{-7}	1.5414×10^{-7}	4.9978×10^{-9}	1.0245×10^{-7}	1.2086×10^{-5}	3.2711×10^{-7}
	L_{rms}	7.5332×10^{-7}	7.5332×10^{-7}	7.4583×10^{-9}	5.5511×10^{-7}	6.1594×10^{-5}	1.7006×10^{-6}
9	L_∞	1.1754×10^{-8}	1.1754×10^{-8}	2.6757×10^{-11}	5.4474×10^{-8}	6.7884×10^{-8}	6.3814×10^{-8}
	L_{rms}	7.4120×10^{-8}	7.4123×10^{-8}	1.8809×10^{-10}	3.4480×10^{-7}	4.2685×10^{-7}	4.1004×10^{-7}
10	L_∞	9.0452×10^{-10}	9.0442×10^{-10}	3.5627×10^{-12}	2.8841×10^{-9}	9.8860×10^{-8}	1.5768×10^{-8}
	L_{rms}	6.9709×10^{-9}	6.9705×10^{-9}	8.3315×10^{-12}	9.7122×10^{-10}	7.8001×10^{-7}	1.2507×10^{-7}

From **Table 6**, and **Figure 16**, the error norm and mean square error values are presented. The best approximation method was the Bernoulli polynomial, where the value of the error norm was equal to 3.5627×10^{-12} and the value of the mean square error was equal to 8.3315×10^{-12} when $n = 10$.

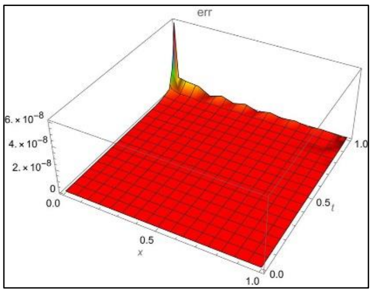


Norm $L_\infty, n = 3, 4, \dots, 10.$

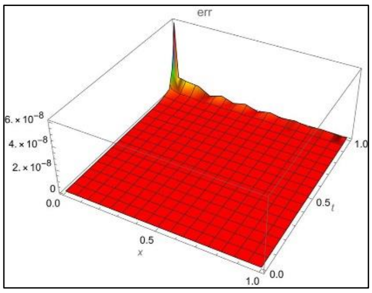


Norm $L_{rms}, n = 3, 4, \dots, 10.$

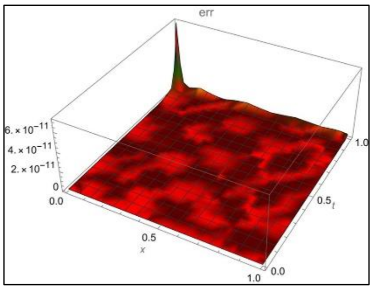
Figure 16. The plots logarithmic of L_{rms} and L_∞ , for example 6, when $n = 3, 4, \dots, 10.$



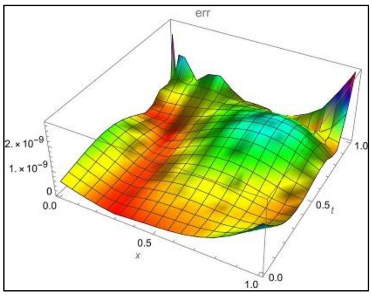
Standard Polynomials



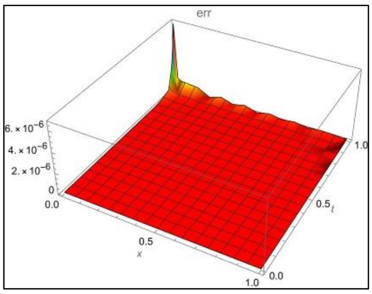
Boubaker Polynomials



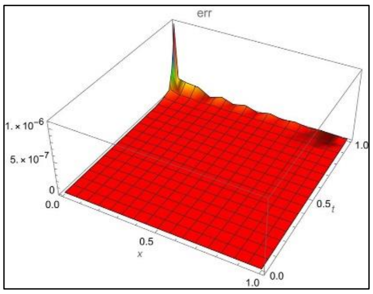
Bernoulli Polynomials



Chebyshev Polynomials

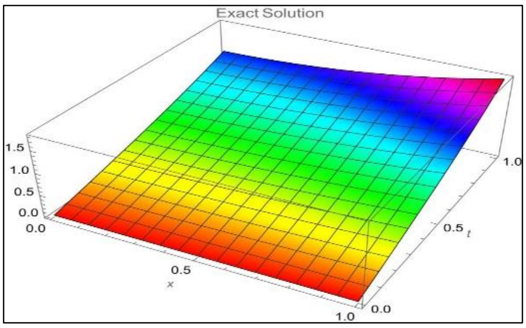


Genocchi Polynomials

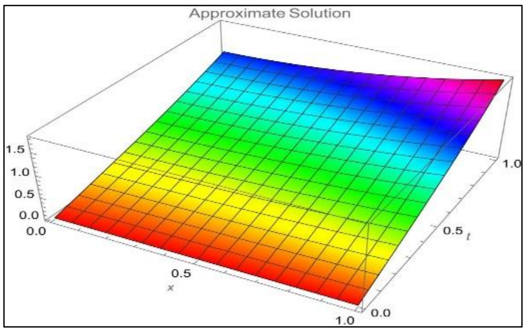


Legendre Polynomials

Figure 17. Plots the norm L_∞ for example 6, when $n = 10$.



Exact solution



Approximate solution

Figure 18. Comparison between analytical and numerical solution for example 6 by the Bernoulli poly.

5. Conclusion

In this work, the orthogonal polynomials and operational matrices based on standard polynomials were proposed and applied to resolve non-homogeneous wave problems. The approximate solutions were achieved and showed to be reliable and effective, even for low-order polynomials. Additionally, the mean square error L_{rms} and the norm L_∞ were calculated to assess the accuracy, dependability, and validity of the approaches. The results show that the recommended solutions have high accuracy and lower error rates. In all situations, the mean square error and error norm values drop as n grows. Furthermore, it is showed that the results of the L_∞, and L_{rms} by the recommended methods Bernoulli and standard polynomials fell greatly in comparison to the other orthogonal polynomials. The standard polynomial was the best in the first example when $n = 5$ to 10, and the values of the L_∞, and L_{rms} were equal to 4.4409×10^{-16} ,1.2699×10^{-14}, respectively, when $n = 10$, while the Bernoulli Polynomial was the best in the remaining examples as it gave the lowest error when $n = 3$ to 10.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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