Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2025, 38(4)

Nonlinear Dynamics and Stability of a Stage-Structured Predator–Prey System with Fear Factors

Bushra E. Kashem [1,2], Hassan F. Al-Husseiny[2*], Anwar Zeb[3]
[1,2]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
[2]College of Applied Science, University of Technology, Baghdad, Iraq
[3]Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan
*Corresponding Author.

Received: 23 April 2025 Accepted: 24 July 2025 Published: 20 October 2025
doi.org/10.30526/38.4.4097

Abstract

This paper presents the formulation and investigation for a stage-structured prey-predator system. We consider the stage- structure in both prey and predator populations, specifically dividing the population of prey into two distinct groups: immature prey and mature prey. We also divide the predator population into immature and mature groups. We assume that only immature predators are capable of attack, so they consume each immature and mature prey. Additionally, the rate of growth for immature prey based on the amount of mature prey, as immature prey does not have reproductive capability. We applied Holling Type I and Holling Type IV response functions to describe the consumption of immature and mature prey by immature predators, respectively. We conducted a mathematical analysis: boundedness of the solution, the presence of equilibrium points, and both local and global stability of the proposed system with respect to these equilibrium points. We also performed numerical simulations to verify the theoretical results.

Keywords: Prey - Predator System, Stage Structure, Holling type, Harvesting, Fear Effect.

1. Introduction

In recent years mathematical models have been become more important and effectiveness for understanding how populations change over time. Since the development of the Lotka-Volterra model, significant advancements in multi-species analysis and modeling have been made, often without considering stage structure. However, recent investigations have focused on stage-structured prey-predator systems for both species, analyzing their dynamic behavior under the effect of fear.

Many studies have focused on researching the dynamics of stage structure populations by modeling the two stages, whether they are prey, predator, or both , For example, Chauhan et al(1) studied the stage structure in fishery model. Bhattacharjee et al (2) predator species are analyzed in accordance with stag structure. Mortoja at el (3) examined stage structure in both species. Pang and Gao (4) explored the stage structure in prey species. Nasra at el (5) developed the stage structure prey predator model in four dimensions. Such that more research discussed prey- predator system in several group like immature and mature species, see (6–10).

Ecological models widely use the predator's response to fluctuations in prey relationships, known as the functional response, which can take various forms. Stage structure prey-predator relationships exhibit several well-known functional responses. For instance, Shi at el (11) examined at the prey-predator concept of stage structure with Holling Type II functional response.. Lu at el (12) proposed periodic solution for the prey predator system involving stage structure and Crowley-Martin response function. Chen and Chen (13) established stage structure prey predator model with Holling Type III functional response. Pandey at el (14) developed delay stage structure prey predator model with Holling Type I and type II functional response Didiharyono at el (15) studied the global stability to prey predator system with Crowley-Martin and stage structure in the predator. The response of predators to changes in prey availability is a critical aspect of ecological dynamics. Researchers have studied predator feeding behaviors and predation habits extensively, (16–20)

Harvesting is a great strategy for regulation predator and prey populations for ensuring healthy development while generating economic benefits (21–25)

Prey’s fear of predators is an essential component of the prey-predator system, describing the relationship between predators and their prey. This fear results from cooperative hunting behaviors among predators, which make prey more afraid of them. Recently, several researchers have begun investigating how fear alters the behavior of prey-predator system. For example, it was examined the impact of fear within the context of the prey predator Scavenger system by (26). It was studied the impact of fear within a prey-predator system including refuge of prey and gestation delay (27). It was analyzed a predator-prey model that includes fear in prey, cooperative hunting among predators, and harvesting (28). It was looked into how stable a fear-based predator-prey model is when there is interfering behaviour or common defence by (29). Additionally, it was examined stability and Hopf bifurcation in a delayed prey-predator system that includes fear, cooperative hunting, and the Allee effect by (30).

This work proposes and analyzes a prey-predator system with stage structured that incorporates the fear effect and several fundamental parameters. These parameters significantly influence interactions between immature prey, mature prey, immature predators, and mature predators.

This work is structured in this way: Section 2.1 provides the assumptions and formulation for the system. Sections 2.2–2.5 discuss the system's dynamic behavior, including the boundedness and existence of solutions, as well as the system's stability analysis. Additionally, Section 3 includes numerical simulations to support the theoretical findings. The conclusion is given in the last section.

2. System Formulation

This section presents a mathematical model for an ecological prey–predator system, incorporating biological aspects based on the following assumptions:

- The stage structure of both predator and prey divides the population of prey, M, into two classes: immature prey (x_1) and mature prey (x_2), so: $M = x_1 + x_2$. Similarly, the predator population, P, is categorized into immature predators (y_1) and mature predators (y_2), resulting in: $P = y_1 + y_2$.
- In the absence of predators and fear, the rate of birth for mature prey is directly proportional to the population of immature prey species. On other hand, a predatory fear of this spcies can have numerous implications. Specifically, the fear function represented by $\frac{1}{1+f\,y_2}$ affects the growth of mature prey, where f denotes the fear parameter.

- We recognize that only mature individuals are capable of reproducing. Therefore, the reproduction rate depends on the presence of the mature age class within a specific community. We assume that the density of prey (or predators) influences the reproduction rate. However, the rate at which immature prey (or predators) mature is independent of density. In our model, the transfer rate is solely dependent on density variables that influence predator growth.
- The immature predator is responsible only for attacking, utilizing Holling's Type I functional response for the intake of immature prey and Type IV functional response for matures prey.
- Assume that interspecies competition occurs only among prey species and that mature individuals are harvested solely by external forces.

The system is structured based on the previous assumptions as follows:

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{rx_2}{1+fy_2} - ax_1^2 - \alpha_1 x_1 - \gamma x_1 y_2 \ , \\ \frac{dx_2}{dt} &= \alpha_1 x_1 - bx_2^2 - \frac{c\,x_2 y_2}{1+\theta x_2^2} - h_1 x_2 \ , \\ \frac{dy_1}{dt} &= \beta_1 \gamma x_1 y_2 + \frac{\beta_2 c\,x_2 y_2}{1+\theta x_2^2} - \alpha_2 y_1 - d_1 y_1, \\ \frac{dy_2}{dt} &= \alpha_2 y_1 - d_2 y_2 - h_2 y_2. \end{aligned} \tag{1}$$

From a biological perspective, the initial condition $(X_1(0), X_2(0), Y_1(0), Y_2(0))$ must be in the first quadrant. **Table 1** below displays the system (1) parameters, which should have positive value:

Table 1. Parameters Description.

Parameter	Biological Description
r	The maximum growth rate per capita of mature prey.
f	The fear level of immature prey of mature predator.
a	The strength competition between immature prey.
b	The strength competition between mature prey.
α_1	The rate maturity from immature prey to mature prey.
α_2	The rate maturity from immature predator to mature predator.
β_1	The conversion from immature prey towards immature predator.
β_2	The conversion from mature prey towards immature predator.
θ	Level of defense.
γ	The rate attack of the mature predator to the immature prey.
c	The rate attack of the mature predator to the mature prey.
d_1	The rate natural death of immature predator.
d_2	The rate natural death of mature predator.
h_1	The rate harvest of mature prey.
h_2	The rate harvest of mature predator.

2.1. System Boundedness

The system domain is defined as $\mathbb{R}_+^4 = \{(x_1, x_2, y_1, y_2) \in R^4 \mid.\ \ x_1(0) \geq 0, x_2(0) \geq 0, y_1(0) \geq 0, y_2(0) \geq 0\}$ We assume that the functions $x_1(t), x_2(t), y_1(t)$, and $y_2(t)$, along with their derivatives, are continuous for all $t \geq 0$. This continuity implies that these functions are Lipschitz continuous in $\mathbb{R}_+^4$, which ensures the existence of a unique solution to system (1). The following theorem defines the boundaries for the solution of this system (1).

Theorem 1. The solutions to system (1), starting in $\mathbb{R}_+^4$, are uniformly bounded.

Proof. let $M = x_1 + x_2$, then:

$$\frac{dM}{dt}=\frac{dx_1}{dt}+\frac{dx_2}{dt}$$
$$=\frac{rx_2}{1+fy_2}-ax_1^2-\gamma x_1y_2-bx_2^2-\frac{c\,x_2y_2}{1+\theta x_2^2}-h_1x_2.$$

The next, we get:

$$\frac{dM}{dt}\leq(x_1-ax_1^2)+(rx_2-bx_2^2)-h_1x_2-x_1.$$

Therefore, we have concluded the following:

$$M(t)\leq\frac{B}{q}, \tag{2}$$

where $B=\frac{1}{a}+\frac{r}{b}$; and q = $\min\{1,h_1\}$.

Now, let $w(t)=x_1(t)+x_2(t)+y_1(t)+y_2(t)$,

then, differentiating it with respect to time yields:

$$\frac{dw}{dt}=\frac{rx_2}{1+fy_2}-(1-\beta_1)\gamma x_1y_2-(1-\beta_2)\frac{c\,x_2y_2}{1+\theta x_2^2}-h_1x_2-d_1y_1-(d_2+h_2)\,y_2,$$
$$\frac{dw}{dt}\leq rx_2-h_1x_2-d_1y_1-(d_2+h_2)\,y_2-rx_1+rx_1,$$
$$\frac{dw}{dt}\leq r(x_1+x_2)-rx_1-h_1x_2-d_1y_1-(d_2+h_2)\,y_2.$$

Thus, by applying the bound in **Equation 2**, we obtain:

$$\frac{dw}{dt}+Nw\leq r\frac{B}{q},$$

where $N=\min\{r,h_1,d_1,d_2+h_2\}$,

Therefore as t→ ∞, $w(t)\leq\frac{rB}{Nq}$, so the proof completed.

2.2. Existence of Equilibrium

For system stability, it is important to establish equilibrium points. We present the equilibrium point of system (1) below:

- The free of population equilibrium point (FPEP), $S_0=(0,0,0,0)$, exist in always.
- The predator free equilibrium point (FPDEP), $S_1=(\bar{x}_1,\bar{x}_2,0,0)$, where:

$$\bar{x}_2=\frac{\bar{x}_1(a\bar{x}_1+\alpha_1)}{r}, \tag{3.a}$$

Yet, $\bar{x}_1$ is a +ve root for 4th-order equation is as below:

$$W_1^{[1]}\bar{x}_1^4+W_2^{[1]}\bar{x}_1^3+W_3^{[1]}\bar{x}_1^2+W_4^{[1]}\bar{x}_1=0, \tag{3.b}$$

where

$$\left.\begin{aligned} W_1^{[1]}&=\frac{-a^2b}{r^2}<0\\ W_2^{[1]}&=\frac{-2ab\alpha_1}{r^2}<0\\ W_3^{[1]}&=-\left(\frac{ah_1}{r}+\frac{\alpha_1^2b}{r^2}\right)<0\\ W_4^{[1]}&=\alpha_1-\frac{\alpha_1h_1}{r}\end{aligned}\right\} \tag{3.c}$$

So, **Equation 3.b** gives only single positive root if the following condition is satisfied:

$$r>h_1\,. \tag{3.d}$$

3- The interior equilibrium point (IEP), $S_2=(x_1^*,y_2^*,y_1^*,y_2^*)$, where

$$\begin{aligned} x_1^*=&\frac{\alpha_2\left(1+\theta {x_2^*}^2\right)(d_2+h_2)+d_1\left(1+\theta {x_2^*}^2\right)(d_2+h_2)-\alpha_2\beta_2c\gamma x_2^*(d_2+h_2)}{\alpha_2\beta_1\gamma\left(1+\theta {x_2^*}^2\right)}-bx_2^*\\ &-\frac{\alpha_2c\,x_2^*}{\left(1+\theta {x_2^*}^2\right)(d_2+h_2)}y_1^*-h_1x_2^*,\end{aligned} \tag{4.a}$$

$$y_1^*=\frac{\alpha_1(\alpha_2+d_1)\left(1+\theta {x_2^*}^2\right)(d_2+h_2)-(h_1+bx_2^*)\left(1+\theta {x_2^*}^2\right)\alpha_2\beta_1\gamma x_2^*-\alpha_1\alpha_2\beta_2cx_2^*(d_2+h_2)}{\alpha_2\beta_1c\gamma x_2^*}, \tag{4.b}$$

$$y_2^* = \frac{\alpha_2 y_1^*}{(d_2+h_2)}, \tag{4.c}$$

while, $x_2{}^*$ is a +ve root of the following 12th-order equation:

$$\begin{gathered} W_1^{[2]} x_2^{*12} + W_2^{[2]} x_2^{*11} + W_3^{[2]} x_2^{*10} + W_4^{[2]} x_2^{*9} + W_5^{[2]} x_2^{*8} + W_6^{[2]} x_2^{*7} + W_7^{[2]} x_2^{*6} \\ + W_8^{[2]} x_2^{*5} + W_9^{[2]} x_2^{*4} + W_{10}^{[2]} x_2^{*3} + W_{11}^{[2]} x_2^{*2} + W_{12}^{[2]} x_2^{*} + W_{13}^{[2]} = 0, \end{gathered} \tag{4.d}$$

where $W_i^{[2]},\ \forall\, i = 0,1,\dots,13$, are determined using MATLAB. Their complexity and large size prevent us from presenting them here.

Then, the sign discarding rule ensures that **Equation 4.d** has a just single positive root when any of the following requirements are met:

$$\left.\begin{array}{c} W_1^{[2]} > 0 \ and\ W_i^{[2]} < 0,\ \ i = 2,\dots,13 \\ or \\ W_1^{[2]} < 0 \ and\ W_i^{[2]} > 0,\ \ i = 2,\dots,13 \end{array}\right\},$$

and if

$$x_1^* > 0$$

$$\begin{aligned} \alpha_1(\alpha_2+d_1)\left(1+\theta x_2^{*2}\right)(d_2+h_2)(d_2+h_2) > (h_1+bx_2^*)\left(1+\theta x_2^{*2}\right)\alpha_2\beta_1\gamma x_2^* \\ +\alpha_1\alpha_2\beta_2 c x_2^*(d_2+h_2). \end{aligned}$$

So, the (IPE) exists in the $\mathbb{R}_+^4$.

2.3. Local Stability Analysis:-

The section uses the linearization procedure for analyzing the local stability of the suggested system. System (1) located at (x_1, x_2, y_1, y_2) has the following Jacobian matrix (J.M):

$$J_i = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}, \tag{5}$$

where

$$a_{11} = -(2ax_1 + \alpha_1 + \gamma y_2), \qquad a_{12} = \frac{r}{1+fy_2},\ \ a_{14} = -\left(\frac{rfx_2}{(1+fy_2)^2} + \gamma x_1\right),$$

$$a_{21} = \alpha_1\,, \quad a_{22} = -\left(2bx_2 + \frac{cy_2 + 3c\theta\, x_2^2 y_2}{\left(1+\theta x_2^2\right)^2} + h_1\right), \quad a_{24} = -\frac{cx_2}{\left(1+\theta x_2^2\right)},$$

$$a_{31} = \beta_1\gamma y_2, \quad a_{32} = \frac{\beta_2 c y_2\left(1-\theta x_2^2\right)}{\left(1+\theta x_2^2\right)^2}, \quad a_{33} = -(\alpha_2 + d_1),$$

$$a_{34} = \beta_1\gamma x_1 + \frac{\beta_2 c x_2}{\left(1+\theta x_2^2\right)}, \quad a_{43} = \alpha_2.$$

If all the eigenvalues at an equilibrium point are negative, the point is referred to as locally asymptotically stable (LAS). Accordingly, the following theorem provides the conditions for local stability at each equilibrium point.

Theorem 2. The FPEP is LAS if the following conditions are met:

$$r < h_1. \tag{6}$$

Proof. The J.M at FPEP can write as:

$$J_0 = \begin{bmatrix} -\alpha_1 & r & 0 & 0 \\ \alpha_1 & -h_1 & 0 & 0 \\ 0 & 0 & -(\alpha_2+d_1) & 0 \\ 0 & 0 & \alpha_2 & -(d_2+h_2) \end{bmatrix}.$$

The characteristic equation of J_0 is:

$$[-(\alpha_2+d_1)-\lambda][-(d_2+h_2)-\lambda][\lambda^2 + tr_1\lambda + det_1] = 0,$$

where $tr_1 = -(\alpha_1 + h_1)$, $det_1 = \alpha_1(h_1 - r)$.

So, using the trace –determinate stability certain, FPEP is thought to be LAS if condition (6) is met. If not is called a saddle point.

Theorem 3. The FPDEP is LAS if the below conditions holds:

$$\alpha_2(\beta_1\gamma\bar{x}_1+\frac{c\beta_2\bar{x}_2}{(1+\theta\bar{x}_2^2)}) < (\alpha_2+d_1)(d_2+h_2) \tag{7}$$

Proof. The J.M at FPDEP is:

$$J_1=\begin{bmatrix} -(2a\bar{x}_1+\alpha_1) & 0 & 0 & -(rf\bar{x}_2+\gamma\bar{x}_1) \\ \alpha_1 & -(2b\bar{x}_2+h_1) & 0 & -\frac{c\bar{x}_2}{(1+\theta\bar{x}_2^2)} \\ 0 & 0 & -(\alpha_2+d_1) & \beta_1\gamma\bar{x}_1+\frac{\beta_2 c\bar{x}_2}{(1+\theta\bar{x}_2^2)} \\ 0 & 0 & \alpha_2 & -(d_2+h_2) \end{bmatrix}.$$

Then characteristic equation of J_1 is:

$[-(2b\bar{x}_2+h_1)-\lambda][-(2a\bar{x}_1+\alpha_1)-\lambda][\lambda^2+tr_2\lambda+det_2]=0,$

Such that, $tr_2=-(\alpha_2+d_1+d_2+h_1)$,

$$det_2=(\alpha_2+d_1)(d_2+h_2)-\alpha_2\left(\beta_1\gamma\bar{x}_1+\frac{\beta_2 c\bar{x}_2}{(1+\theta\bar{x}_2^2)}\right).$$

Then, according to the trace-determinant stability criterion, the FPDEP becomes (LAS) when condition (7) is satisfied. If not, it is a saddle point.

Theorem 4. The IEP is LAS in the $\Omega\in\mathbb{R}_+^4$ if the following condition satisfies:

$$\left.\begin{array}{c} (\beta_1-1)\,\gamma y_2^* < 2ax_1^*+\gamma y_2^*, \\ \frac{r}{1+fy_2^*}+\frac{(\beta_2-3\theta\, {x_2^*}^2-\theta\, {x_2^*}^2\beta_2-1)cy_2^*}{\left(1+\theta {x_2^*}^2\right)^2} < 2bx_2^*+h_1 \\ \left(\frac{rf}{(1+fy_2^*)^2}+\frac{(1+\beta_2)c}{\left(1+\theta {x_2^*}^2\right)}\right)x_2^*+(1+\beta_1)\gamma x_1^* < d_2+h_2 \end{array}\right\}. \tag{8}$$

Proof: At PEP, the J.M is written as:

$$J_2=\begin{bmatrix} a_{11}^* & a_{12}^* & 0 & a_{14}^* \\ a_{21}^* & a_{22}^* & 0 & a_{24}^* \\ a_{31}^* & a_{32}^* & a_{33}^* & a_{34}^* \\ 0 & 0 & a_{43}^* & a_{44}^* \end{bmatrix},$$

where

$a_{11}^*=-(2ax_1^*+\alpha_1+\gamma y_2^*)$, $a_{12}^*=\frac{r}{1+fy_2^*}$, $a_{14}^*=-(\frac{rfx_2^*}{(1+fy_2^*)^2}+\gamma x_1^*)$,

$a_{21}^*=\alpha_1$, $a_{22}^*=-(2bx_2^*+\frac{cy_2^*+3c\theta\, {x_2^*}^2y_2^*}{\left(1+\theta {x_2^*}^2\right)^2}+h_1)$, $a_{24}^*=-\frac{cx_2^*}{\left(1+\theta {x_2^*}^2\right)}$,

$a_{31}^*=\beta_1\gamma y_2^*$, $a_{32}^*=\frac{\beta_2 cy_2^*\left(1-\theta {x_2^*}^2\right)}{\left(1+\theta {x_2^*}^2\right)^2}$, $a_{33}^*=-(\alpha_2+d_1)$,

$a_{34}^*=\beta_1\gamma x_1^*+\frac{\beta_2 cx_2^*}{\left(1+\theta {x_2^*}^2\right)}$, $a_{43}^*=\alpha_2$, $a_{44}^*=-(d_2+h_2)$.

If the conditions are satisfied, we will apply the Gershgorin theorem (31).

$|a_{11}^*| > |a_{21}^*|+|a_{31}^*|+|a_{41}^*|$,

$|a_{22}^*| > |a_{12}^*|+|a_{32}^*|+|a_{42}^*|$,

$|a_{33}^*| > |a_{13}^*|+|a_{23}^*|+|a_{43}^*|$,

$|a_{44}^*| > |a_{14}^*|+|a_{24}^*|+|a_{34}^*|$.

Consequently, all the eigenvalues of J_2 at IPE are present in Ω, where

$$\Omega=\cup\left\{U^*\in C: \left|U^*-a_{ij}^*\right| < \sum_{\substack{i=1\\ i\neq j}}^4 \left|a_{ij}^*\right|\right\}.$$

Thus, each eigenvalue of J_2 lies within the disk centered at a^*_{ij}. Since the diagonal elements are negative and condition (8) is satisfied, every eigenvalue will be located in the left half-plane, contributing to the local asymptotic stability (LAS) of the IPE.

2.4 Global Stability Analysis

The theorem below analyzes the global stability (GAS) at each equilibrium point in system (1). If the Lyapunov function is satisfied, then the system is GAS.

Theorem 5. Suppose the FPEP is LAS in $\mathbb{R}^4_+$. If the condition (6) is satisfied, then it is GAS.

Proof. Assume this positive value is a definite function:

$U_0(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t)$.

Such that $U_0(t): \mathbb{R}^4_+ \to \mathbb{R}$ is continuously differentiable with $U_0(0,0,0,0) = 0$, and $U_0(x_1, x_2, y_1, y_2) > 0$, $\forall\ (x_1, x_2, y_1, y_2) \neq (0, 0, 0, 0)$.

Further,

$$\frac{dU_0}{dt} = \frac{rx_2}{1+fy_2} - ax_1^2 - \alpha_1 x_1 - \gamma x_1 y_2 + \alpha_1 x_1 - bx_2^2 - \frac{c\, x_2 y_2}{1+\theta x_2^2} - h_1 x_2$$
$$+\beta_1 \gamma x_1 y_2 + \frac{\beta_2 c\, x_2 y_2}{1+\theta x_2^2} - \alpha_2 y_1 - d_1 y_1 + \alpha_2 y_1 - d_2 y_2 - h_2 y_2.$$

After conducting additional calculations, we arrive at the following outcome:

$\frac{dU_0}{dt} \leq -(h_1 - r)x_2 - d_1 y_1 - (d_2 + h_2) y_2$.

So, U_0 is a Lyapunov function when we get $\frac{dU_0}{dt} < 0$ from condition (6), which implies that the FPEP is GAS.

Theorem 6. Consider FPDEP as LAS, thus, in the sub region it is GAS when:

$$\left.\begin{array}{r} \bar{x}_1 < x_1 \\ \bar{x}_2 < x_2 \\ \beta_1 \gamma x_1 < d_2 + h_2 \\ v_{12}^2 < 4 v_{22} v_{11} \end{array}\right\}, \qquad (9)$$

Where v_{12}, v_{11}, v_{22} are given in the proof.

Proof. Assume this positive value is a definite function:

$$U_1(t) = \frac{(x_1 - \bar{x}_1)^2}{2} + \frac{(x_2 - \bar{x}_2)^2}{2} + y_1 + y_2.$$

Where $U_1(t): \mathbb{R}^4_+ \to \mathbb{R}$ is continuously differentiable with $U_1(\bar{x}_1, \bar{x}_2, 0,0) = 0$, and $U_1(x_1, x_2, y_1, y_2) > 0$, $\forall\ (x_1, x_2, y_1, y_2) \neq (\bar{x}_1, \bar{x}_2, 0 ,0)$.

Also,

$$\frac{du_1}{dt} = (x_1 - \bar{x}_1)\frac{dx_1}{dt} + (x_2 - \bar{x}_2)\frac{dx_2}{dt} + \frac{dy_1}{dt} + \frac{dy_2}{dt},$$
$$\frac{dU_1}{dt} = (x_1 - \bar{x}_1)\left(\frac{rx_2}{1+fy_2} - ax_1^2 - \alpha_1 x_1 - \gamma x_1 y_2\right) + (x_2 - \bar{x}_2)\left(\alpha_1 x_1 - bx_2^2 - \frac{c\, x_2 y_2}{1+\theta x_2^2}\right.$$
$$\left. - h_1 x_2\right) + \beta_1 \gamma x_1 y_2 + \frac{\beta_2 c\, x_2 y_2}{1+\theta x_2^2} - \alpha_2 y_1 - d_1 y_1 + \frac{dy_2}{dt} + \alpha_2 y_1 - d_2 y_2 - h_2 y_2.$$

On completing far more calculations, we find the below result:

$$\frac{dU_1}{dt} \leq -[\sqrt{v_{11}}\,(x_1 - \bar{x}_1) - \sqrt{v_{22}}\,(x_2 - \bar{x}_2)]^2 - (rx_2 f y_2 + \gamma x_1 y_2 (1 + f y_2))(x_1 - \bar{x}_1)$$
$$-(1-\beta_2)\frac{c\, x_2 y_2}{1+\theta x_2^2}(x_2 - \bar{x}_2) - (d_2 + h_2 - \beta_1 \gamma x_1) y_2 - d_1 y_1\,,$$

where

$v_{11} = \alpha_1 a (x_1 - \bar{x}_1)$, $\ v_{12} = r + \alpha_1 (1 + f y_2)$, $\ v_{22} = b(x_2 + \bar{x}_2) + h_1$.

Consequently, when we estimated $\frac{dU_1}{dt} < 0$ from condition (9), that means U_1 is Lyapunov function, which implies that FPDEP is GAS.

Theorem 7. Under the assumption that the IEP is LAS, the basin of attraction meets these conditions:

$$\left.\begin{array}{l} \frac{4}{9} g_{11}\, g_{22} \geq g_{12}^2 \\ \frac{4}{9} g_{11}\, g_{33} \geq g_{13}^2 \\ \frac{4}{9} g_{11}\, g_{44} \geq g_{14}^2 \\ \frac{4}{9}\, g_{22}\, g_{33} \geq g_{23}^2 \\ \frac{4}{9}\, g_{22}\, g_{44} \geq g_{24}^2 \\ \frac{4}{9}\, g_{33} g_{44} \geq g_{34}^2 \\ g_{22} > 0 \\ g_{23} > 0 \\ g_{24} > 0 \end{array}\right\}, \qquad (10)$$

such that all symbols g_{ij} $, i, j = 1,2,3,4$ have been identified in proof.

Proof. let this positive value be a definite function:

$$U_2(t) == \frac{(x_1 - x_1^*)^2}{2} + \frac{(x_2 - x_2^*)^2}{2} + \frac{(y_1 - y_1^*)^2}{2} + \frac{(y_2 - y_2^*)^2}{2}.$$

Such that $U_2(t): \mathbb{R}_+^4 \to \mathbb{R}$ is continuously differentiable with $U_2(x_1^*, x_2^*, y_1^*, y_2^*) = 0$, and $U_2(x_1, x_2, y_1, y_2) > 0$, $\forall\ (x_1, x_2, y_1, y_2) \neq (x_1^*, x_2^*, y_1^*, y_2^*)$.

Then

$$\frac{du_1}{dt} = (x_1 - x_1^*)\frac{dx_1}{dt} + (x_2 - x_2^*)\frac{dx_2}{dt} + (y_1 - y_1^*)\frac{dy_1}{dt} + (y_2 - y_2^*)\frac{dy_2}{dt},$$

Therefore, by using system (1) and manipulating the algebra, we can describe $\frac{dU_2}{dt}$ as follows:

$$\begin{aligned} \frac{dU_2}{dt} \leq & -\left[\frac{g_{11}}{3}(x_1 - x_1^*)^2 - g_{12}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{g_{22}}{3}(x_2 - x_2^*)^2\right] \\ & -\left[\frac{g_{11}}{3}(x_1 - x_1^*)^2 - g_{13}(x_1 - x_1^*)(y_1 - y_1^*) + \frac{g_{33}}{3}(y_1 - y_1^*)^2\right] \\ & -\left[\frac{g_{11}}{3}(x_1 - x_1^*)^2 + g_{14}(x_1 - x_1^*)(y_2 - y_2^*) + \frac{g_{44}}{3}(y_2 - y_2^*)^2\right] \\ & -\left[\frac{g_{22}}{3}(x_2 - x_2^*)^2 + g_{23}(x_2 - x_2^*)(y_1 - y_1^*) + \frac{g_{33}}{3}(y_1 - y_1^*)^2\right] \\ & -\left[\frac{g_{22}}{3}(x_2 - x_2^*)^2 + g_{24}(x_2 - x_2^*)(y_2 - y_2^*) + \frac{g_{44}}{3}(y_2 - y_2^*)^2\right] \\ & -\left[\frac{g_{33}}{3}(y_1 - y_1^*)^2 - g_{34}(y_1 - y_1^*)(y_2 - y_2^*) + \frac{g_{44}}{3}(y_2 - y_2^*)^2\right]. \end{aligned}$$

So, applying condition (10) get the following:

$$\frac{dU_2}{dt} \le -\left[\sqrt{\frac{g_{11}}{3}}(x_1 - x_1^*) - \sqrt{\frac{g_{22}}{3}}(x_2 - x_2^*)\right]^2$$
$$-\left[\sqrt{\frac{g_{11}}{3}}(x_1 - x_1^*) - \sqrt{\frac{g_{33}}{3}}(y_1 - y_1^*)\right]^2$$
$$-\left[\sqrt{\frac{g_{11}}{3}}(x_1 - x_1^*) + \sqrt{\frac{g_{44}}{3}}(y_2 - y_2^*)\right]^2$$
$$-\left[\sqrt{\frac{g_{22}}{3}}(x_2 - x_2^*) + \sqrt{\frac{g_{33}}{3}}(y_1 - y_1^*)\right]^2$$
$$-\left[\sqrt{\frac{g_{22}}{3}}(x_2 - x_2^*) + \sqrt{\frac{g_{44}}{3}}(y_2 - y_2^*)\right]^2$$
$$-\left[\sqrt{\frac{g_{33}}{3}}(y_1 - y_1^*) - \sqrt{\frac{g_{44}}{3}}(y_2 - y_2^*)\right]^2.$$

Where

$g_{11} = a(x_1 + x_1^*) - \alpha_1 + \gamma y_2^*,$

$g_{12} = \frac{r}{1+fy_2^*} + \alpha_1,$

$g_{13} = \beta_1 \gamma y_2^*,$

$g_{14} = \frac{rfx_2^*}{(1+fy_2)(1+fy_2^*)} + \gamma x_1,$

$g_{22} = b(x_2 + x_2^*) + \frac{cy_2^* + c\theta\, x_2^* y_2^*(x_2 + x_2^* - 1)}{\left(1+\theta {x_2^*}^2\right)(1+\theta x_2^2)} - h_1,$

$g_{23} = \frac{\beta_2 c y_2^*(\theta x_2^*(x_2 + x_2^* - 1) - 1)}{\left(1+\theta {x_2^*}^2\right)(1+\theta x_2^2)},$

$g_{24} = \frac{cx_2(1-\theta x_2^*)}{\left(1+\theta {x_2^*}^2\right)(1+\theta x_2^2)},$

$g_{33} = (\alpha_2 + d_1),$

$g_{34} = \beta_1 \gamma x_1 + \alpha_2 + \frac{\beta_2 c x_2(1+\theta x_2^*)}{\left(1+\theta {x_2^*}^2\right)(1+\theta x_2^2)},$

$g_{44} = (d_2 + h_2).$

Consequently, in the region where condition (10) is satisfied, U_2 behaves as a Lyapunov function, and $\frac{dU_2}{dt} < 0$ implies that the IEP is globally asymptotically stable (GAS).

3. Numerical Simulation

This section presents a numerical analysis of the dynamic behavior of the proposed system (1), conducted using MATLAB R2009b software. Due to the absence of actual data for all system parameters, hypothetical values have been assigned to each parameter as follows:

$$\begin{array}{lllll} r = 0.9\,, & f = 0.05, & a = 0.01, & \alpha_1 = 0.3\,, & \beta_1 = 0.04\,, \\ \beta_2 = 0.02, & \theta = 0.3, & \gamma = 0.09, & b = 0.1\,, & \alpha_2 = 0.05, \\ h_1 = 0.01\,, & h_2 = 0.002, & c = 0.01\,, & d_1 = 0.004, & d_2 = 0.02. \end{array} \quad (11)$$

When analyzed with the specified parameter values and a variety of initial conditions, system (1) clearly meets the current requirements of IEP. The path of system (1) converge asymptotically to the point $S_2 = (6.4,\ 4.249, 0.877, 1.993\ )$ from a range of distinct initial points, are shown in **Figure 1**.

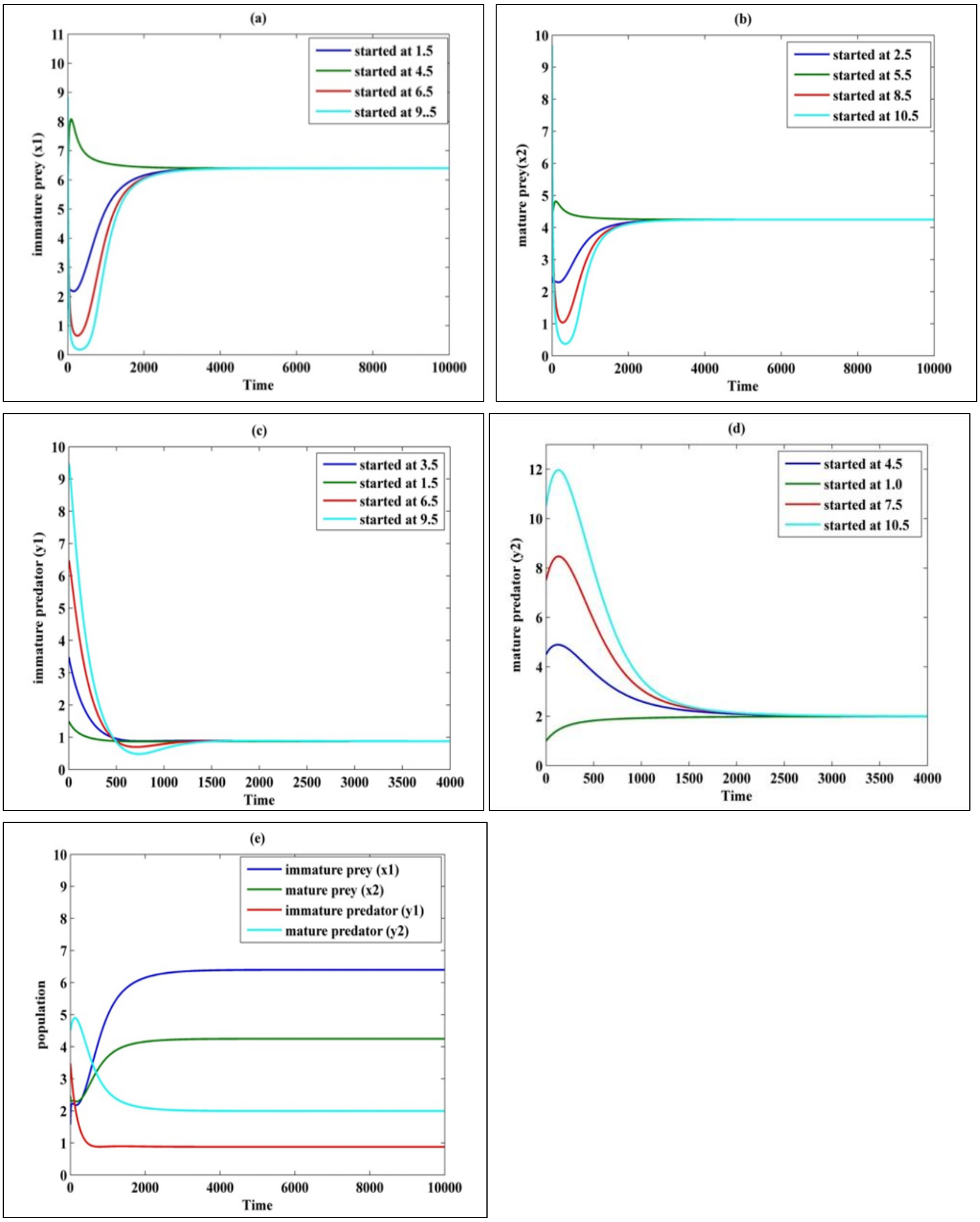

Figure 1. Asymptotic global stability to IEP for data (11) across a variety of initial conditions. (a) Immature prey paths. (b) Mature prey paths. (c) Immature predator paths. (d) Mature predator paths. (e) Every individual path.

Obviously, **Figure 1** confirms the theoretical results, showing that the IEP is GAS. However, if $h_1 = 1.0$, the path of system (1) asymptotically converges to FPEP, $S_0 = (0\,,0\,,0,0)$, as displayed in **Figure 2**.

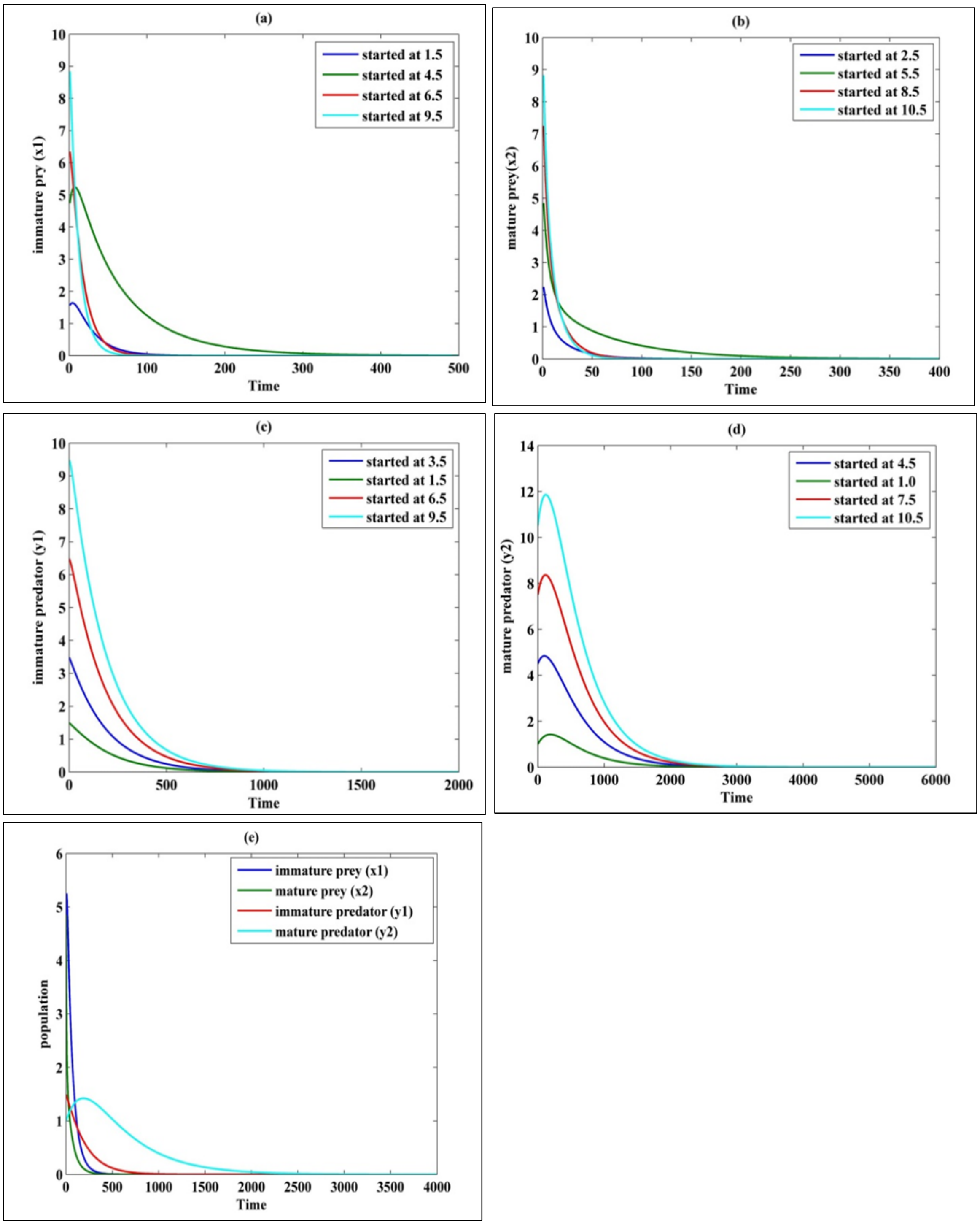

Figure 2. Asymptotic global stability to FPEP for data (11) across a variety of initial conditions. (a) Immature prey paths. (b) Mature prey paths. (c) Immature predator paths. (d) Mature predator paths. (e) Every individual path.

Furthermore, numerical simulations show that decreasing the parameter to $r = 0.4$, for the data in (11) causes the path of system (1) to converge to the globally stable fixed point FPDEP, $S_1 = (4.023, 3.424, 0, 0)$, as seen in **Figure 3**.

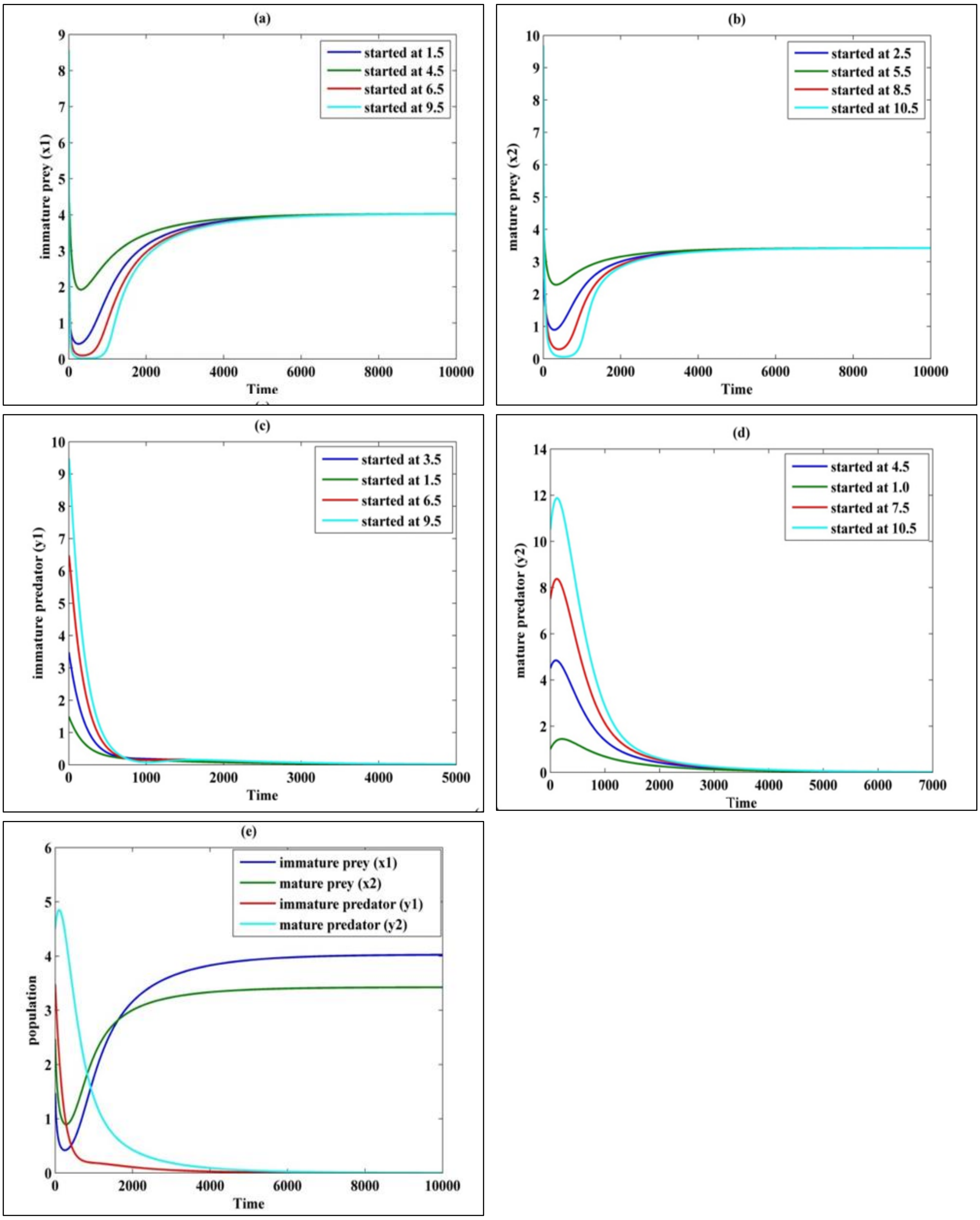

Figure 3. Asymptotic global stability to FPDEP for data (11) across a variety of initial conditions. (a) Immature prey paths. (b) Mature prey paths. (c) Immature predator paths. (d) Mature predator paths. (e) Every individual path.

To examine how altering a single parameter affects the dynamic behavior of system (1), we obtained the following results.

By changing the parameter $a \geq 0.06$, with reminder the other parameter value as in data (11), then system (1) approaches asymptotically at FPDEP $S_1 = (\bar{x}_1, \bar{x}_2, 0,0)$, as shown in **Figure 4**. Additionally, when α_1 increases to a value in the range $\alpha_1 \geq 1.2$, with the other parameters kept the same as in data (11), system (1) asymptotically approaches the fixed point $S_1 = (\bar{x}_1, \bar{x}_2, 0,0)$, as shown in **Figure 5**.

Finally, the trajectory of system (1) asymptotically approaches the FPDEP $S_1 = (\bar{x}_1, \bar{x}_2, 0,0)$ when using the parameter values from data set (11) with d_2 varying in the range $d_2 \geq 0.05$, as illustrated in **Figure 6**.

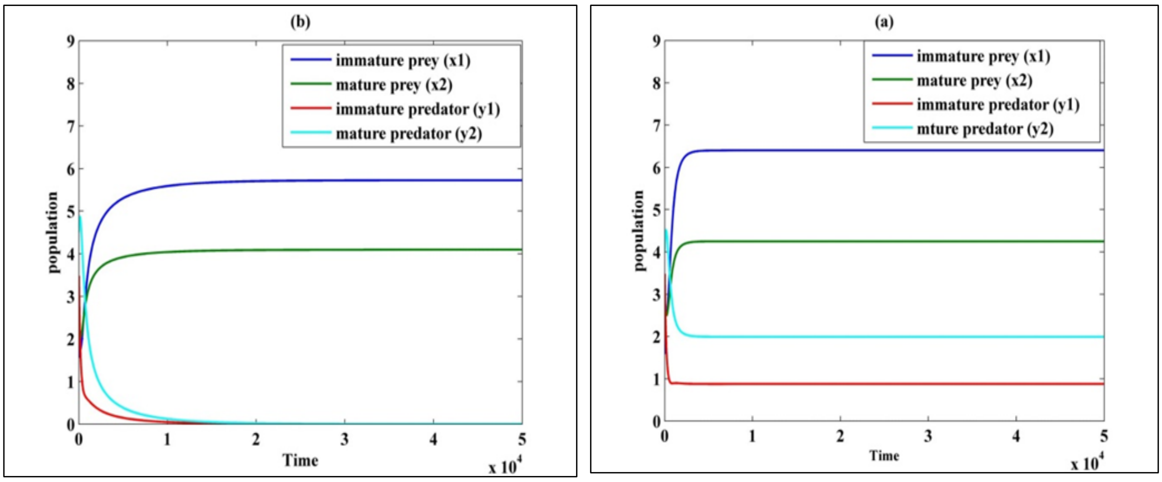

Figure 4. The solution to system (1) in data (11) over time, with different a values. (a) Globally asymptotically stable IEP for $a = 0.01$. (b) Globally asymptotically stable FPDEP for $a = 0.06$.

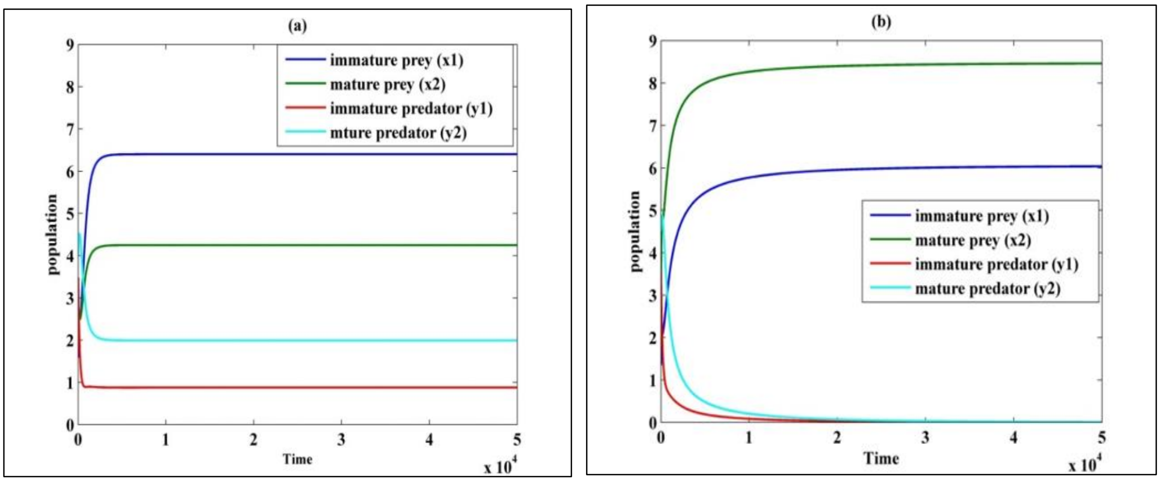

Figure 5. The solution to system (1) in data (11) over time, with different α_1 values. (a) Globally asymptotically stable IEP for $\alpha_1 = 0.3$. (b) Globally asymptotically stable FPDEP for $\alpha_1 = 1.2$.

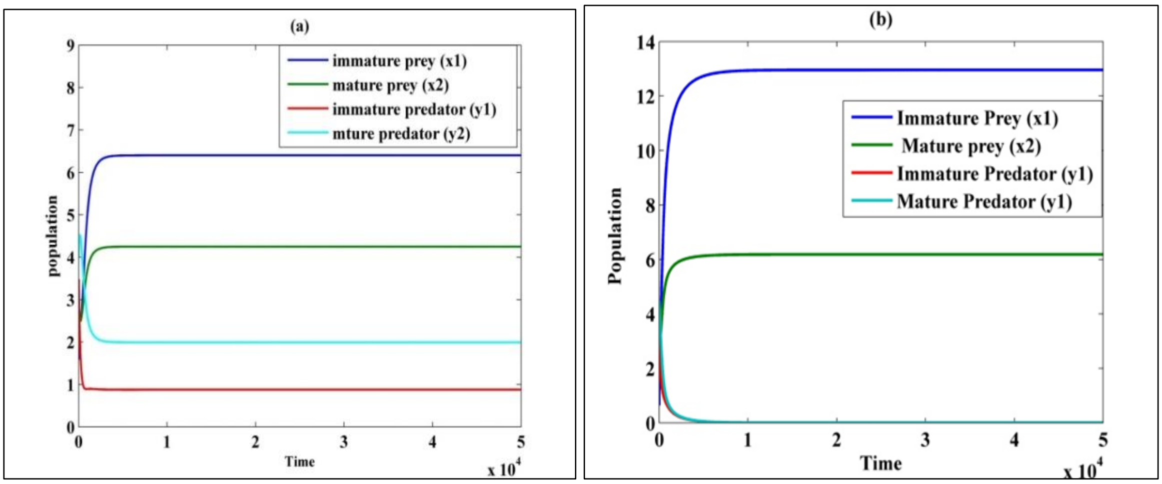

Figure 6. The solution to system (1) in data (11) over time, with different d_2 values. (a) Globally asymptotically stable IEP for $d_2 = 0.02$. (b) Globally asymptotically stable FPDEP for $d_2 = 0.05$.

4. Conclusion

In this paper, we developed a stage-structured prey-predator ecosystem model for both species, incorporating immature and mature stages. To construct a four-dimensional system, we made several assumptions: the immature predator is solely responsible for attacks, while only the mature species undergoes harvesting. We examined the boundedness of system (1) and defined existence conditions for all three equilibrium points. Local and global stability analyses were then conducted for these points. Finally, numerical simulations were used to evaluate the parameters influencing system dynamics and to verify the analytical findings. Using the hypothetical data in (11), which aligns with the outcomes of the numerical simulation, we obtained the following results:

System (1) has no periodic behavior; rather, the solution converges asymptotically to any of the equilibrium points.

By making immature prey more competitive such that a value greater than 0.06 $(a \geq 0.06)$, the IEP becomes destabilized, and system (1) asymptotically approaches the equilibrium point FPDEP.

Increasing the maturity rate of prey to a value greater than 1.2 $(\alpha_1 \geq 1.2)$ destabilizes the IEP, causing system (1) to asymptotically approach the fixed point FPDEP.

Increasing the $d_2 > 0.05$ destabilizes the IEP, and the system (1) approaches to FPDEP asymptotically.

As previously explained, system (1) demonstrates notable sensitivity to changes in specific parameters, making it highly controllable.

Acknowledgements

We would like to express our gratitude other referees for their valuable comments and suggestions that led to a truly significant improvement of the paper.

Conflict of Interest

The authors declare that there are no competing interests regarding the publication of this paper.

Funding

This work is not supported by any the Foundation.

Reference

1. Chauhan S, Bhatia SK, Sharma S. Harvesting of Stage Structured Fishery Model in The Presence of Toxicity. Electronic Journal of Mathematical Analysis and Applications. 2018;6(1):65–78. https://doi.org/10.21608/ejmaa.2018.312512.
2. Bhattacharjee D, Ray N, Das D, Sarmah HK. A discrete-time dynamical model of prey and stage-structured predator with juvenile hunting incorporating negative effects of prey refuge. Partial Differ Equations Appl Math. 2024;10:100710. https://doi.org/10.1016/j.padiff.2024.100710.
3. Mortoja SG, Panja P, Mondal SK. Dynamics of a predator-prey model with stage-structure on both species and anti-predator behavior. Informatics Med Unlocked 2018;10:50–7. https://doi.org/10.1016/j.imu.2017.12.004.
4. Pang Q, Gao Y. Stability analysis of a predator-prey model with stage structure. J Phys Conf Ser. 2022;2294(1).https://doi.org/10.1088/1742-6596/2294/1/012026.
5. Nasra ND, Putri AR, Syafwan M. Stability Analysis and Numerical Simulation of Prey-Predator System with Age Structure. Int J Progr Sci Technol. 2022;32(1):105–11. http://dx.doi.org/10.52155/ijpsat.v32.1.4202.
6. Bahlool DK. the Dynamics of a Stage-Structure Prey-Predator Model With Hunting Cooperation and Anti-Predator Behavior. Commun Math Biol Neurosci. 2023;2023:1–23. https://doi.org/10.28919/cmbn/8003 .
7. Palani V, Krishnan K. A Stage Structure Prey Predator Model Using Pentagonal Fuzzy Numbers and Functional Response. Baghdad Sci J. 2024; 21:3234–47.

https://doi.org/10.21123/bsj.2024.9945 .

8. Ibrahim HA, Naji RK. The complex dynamic in three species food webmodel involving stage structure and cannibalism. AIP Conf Proc. 2020; 2292. https://doi.org/10.1063/5.0030510.
9. Dubey B, Kumar A. Dynamics of prey–predator model with stage structure in prey including maturation and gestation delays. Nonlinear Dyn.2019; 96(4), 2653–2679. https://doi.org/10.1007/s11071-019-04951-5.
10. Ahmed LS, AL-Husseiny HF. Dynamical Behavior of an eco-epidemiological Model involving Disease in predator and stage structure in prey. Iraqi J Sci. 2019; 60(8):1766–82. https://doi.org/10.24996/ijs.2019.60.8.14.
11. Shi W, Huang Y, Wei C, Zhang S. A Stochastic Holling-Type II Predator-Prey Model with Stage Structure and Refuge for Prey. Adv Math Phys. 2021;2021.https://doi.org/10.1155/2021/9479012.
12. Lu W, Xia Y. Periodic solution of a stage-structured predator-prey model with Crowley-Martin type functional response. AIMS Math. 2022;7(5):8162–75. https://doi.org/10.3934/math.2024208.
13. Chen B, Chen J. Complex dynamic behaviors of a discrete predator-prey model with stage structure and harvesting. Int J Biomath. 2017;10(1). https://doi.org/10.1142/S1793524517500139.
14. Pandey S, Ghosh U, Das D, Chakraborty S, Sarkar A. Rich dynamics of a delay-induced stage structure prey–predator model with cooperative behaviour in both species and the impact of prey refuge. Math Comput Simul. 2024;216:49–76. https://doi.org/10.1016/j.matcom.2023.09.002.
15. Didiharyono D, Toaha S, Kusuma J, Kasbawati. Global Stability of Prey-Predator Model with CrowleyMartin type Functional Response and Stage Structure for Predator. Proc Int Conf Ind Eng Oper Manag. 2021;714–24. http://dx.doi.org/10.31219/osf.io/wv9gf.
16. Tripathi JP, Jana D, Tiwari V. A Beddington–DeAngelis type one-predator two-prey competitive system with help. Nonlinear Dyn. 2018;94(1):553–73.https://doi.org/10.1007/s11071-018-4377-9.
17. Slimani S, de Fitte PR, Boussaada I. Dynamics of a prey-predator system with modified Leslie-Gower and Holling type II schemes incorporating a prey refuge. Discret Contin Dyn Syst - Ser B. 2019;24(9):5003–39. https://doi.org/10.3934/dcdsb.2019042.
18. Raad SR. and Al-Husseiny HF. Influence of Infection Delay on the Covid-19 Pandemic with Vaccination Control: Modeling and Simulation. Ibn AL-Haitham J Pure Appl Sci. 2025; 38(1): 445–464. https://doi.org/10.30526/38.1.3638.
19. Antwi-Fordjour K, Parshad RD, Beauregard MA. Dynamics of a predator–prey model with generalized Holling type functional response and mutual interference. Math Biosci. 2020;326: 108407. https://doi.org/10.1016/j.mbs.2020.108407.
20. Yadav U, Nayak AK, Gakkhar S. Mathematical Scrutiny of Singular Predator-Prey Model with Stage-Structureq2q of Prey. Acta Appl Math. 2024;189(1):0–25. https://doi.org/10.21203/rs.3.rs-2317759/v1.
21. Kashem BE, Al-Husseiny HF. The dynamic of two prey–One predator food web model with fear and harvesting. Partial Differ Equations Appl Math. 2024;11(August):100875. https://doi.org/10.1016/j.padiff.2024.100875.
22. Mondal A, Pal AK, Samanta GP. Complex dynamics of two prey–one predator model together with fear effect and harvesting efforts in preys. J Comp Math Data Sci. 2023; 6:100071. https://doi.org/10.1016/j.jcmds.2022.100071.
23. Ritwick B Pritha D, Debasis M. Stability and permanent co-existence of a discrete-time two-prey one-predator system with harvesting effort. AIP Conf Proc. 2019; 2159 (1): 030002. https://doi.org/10.1063/1.5127467.
24. Waldani GE, Chalimatusadiah C. Hopf Bifurcation in a Modified Leslie-Gower Two Preys One Predator Model and Holling Type II Functional Response with Harvesting and Time-Delay. J Matem, Statistika dan Komputasi. 2023; 19, 520–536. https://doi.org/10.20956/j.v19i3.24851.

25. Oladele FJ, Yakubu AS, Kehinde OH. Numerical Analysis of a prey harvesting of One Predator – Two Prey Dynamics in Tropical Zone. 2020 Int Conf Math Comp Eng Comp Sci. (ICMCECS), Ayobo, Nigeria, 2020, 1-7, https://doi.org/ 10.1109/ICMCECS47690.2020.246993.
26. Ahmed MA, Bahlool DK. The influence of fear on the dynamics of a prey-predator scavenger model with quadratic harvesting. Commun Math Biol Neurosci. 2022. https://doi.org/ 10.28919/cmbn/7506.
27. Kumar A, Dubey B. Modeling the effect of fear in a prey-predator system with prey refuge and gestation delay. Int J Bifurc Chaos. 2019;29(14):1–25. https://doi.org/10.1142/S0218127419501955.
28. Belew B, Melese D. Modeling and analysis of predator-prey model with fear effect in prey and hunting cooperation among predators and harvesting. J Appl Math. 2022; 1(2022). https://doi.org/10.1155/2022/2776698.
29. He M, Li Z. Stability of a fear effect predator–prey model with mutual interference or group defense. J Biol Dyn . 2022;16(1):480–98. https://doi.org/10.1080/17513758.2022.2091800.
30. Al-Jubouri KQ, Naji RK. Stability and Hopf Bifurcation of a Delayed Prey- Predator System with Fear, Hunting Cooperative, and Allee Effect. Iraqi J Sci. 2024;65(7):3901–21. https://doi.org/10.24996/ijs.2024.65.7.29.
31. Horn RA, Johnson CR. Matrix Analysis. Cambridge: Cambridge University Press; 1985. https://doi.org/10.1017/CBO9780511810817.